Combinatorial reciprocity theorems via geometry
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Lecture 2 , 20.11.2020

Last lecture : - What is combinatorial reciprocity ?
- Example 1 : Graph colorings and acyclic orientation

In this lecture we will see a second instance of combinatorial reciprocities.

② Flows on graphs

Let $G=(V,E)$ be a graph and ρ be an orientation of G.
We say that a map $f: E \to \mathbb{Z}_n$ is a <u>$\mathbb{Z}_n$-flow</u> if for every $v \in V$

$$\sum_{e \to v} f(e) = \sum_{v \to e} f(e) ,$$

that is, what flows into the node v is precisely what flows out of v.

We say that a $\mathbb{Z}_n$-flow f is <u>nowhere zero</u> if $f(e) \neq 0$ for all $e \in E$.

We define $\boxed{\varphi_G(n) := |\{ f \text{ nowhere zero } \mathbb{Z}_n\text{-flow on } \rho G\}|}$

<u>Prop.</u> The flow counting function $\varphi_G(n)$ is independend of the orientation ρ of G.

<u>Proof</u> Let ρ and $\tilde{\rho}$ be two orientations of G which are equal to each other except in the orientation of one single edge $e \in E$.

f is a nowhere zero $\mathbb{Z}_n$ flow on ρG $\Leftrightarrow$ $\tilde{f}$ is a nowhere zero $\mathbb{Z}_n$-flow on $\tilde{\rho} G$.

where $\tilde{f}(e') := \begin{cases} f(e') & \text{if } e' \neq e \\ n - f(e) & \text{if } e' = e \end{cases}$

Therefore, the flow counting functions for the orientations ρ and ρ' coincide. Since all orientation are connected under changing the orientation of single edges, the result follows.

<u>Examples</u> Find the flow counting function of the following graphs.

i) 

G

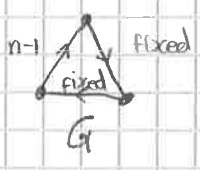
pick any orientation

$\varphi_G(n) = 0$ because $f(e)$ is forced to be zero for every edge.

We say that an edge $e \in E$ of G is a <u>bridge</u> is removing e increases the number of connected components of G.

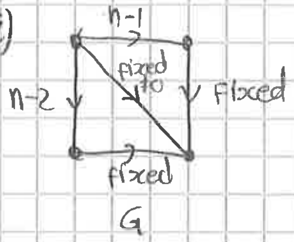
<u>Exercise</u> Show that if G has a bridge then $\varphi_G(n) = 0$.

ii)

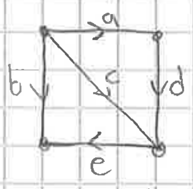


$$\varphi_G(n) = (n-1)$$

iii)



$$\varphi_G(n) = (n-1)(n-2)$$



a has $n-1$ possibilities : $a \neq 0$

b has $n-2$ possibilities : $b \neq 0$

$\underbrace{a+b \neq 0}$ c is forced $c \equiv -(a+b) \bmod (n)$

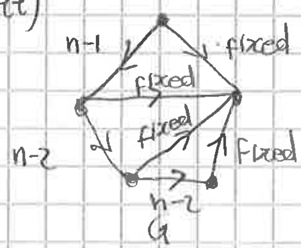
These two equations are different because $a \neq 0$

c is forced : $c = -(a+b)$

d is forced : $d = a$

e is forced : $e = b$

iii)



$$\varphi_G(n) = (n-1)(n-2)(n-2)$$

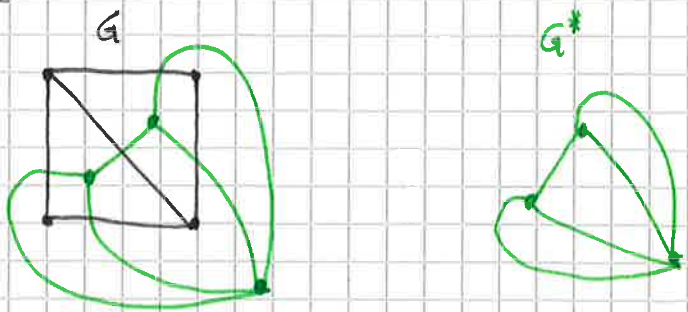
<u>Motivation : Flows vs proper colorings for planar graphs</u>

Given a planar graph G and a drawing of it in the plane without crossings, we can define a <u>dual graph</u> G^* of G as follows.
G subdivides the plane into several connected regions.

Vertices of G^* : these regions

Two regions share an edge e^* if an original edge e is contained in both their boundaries.

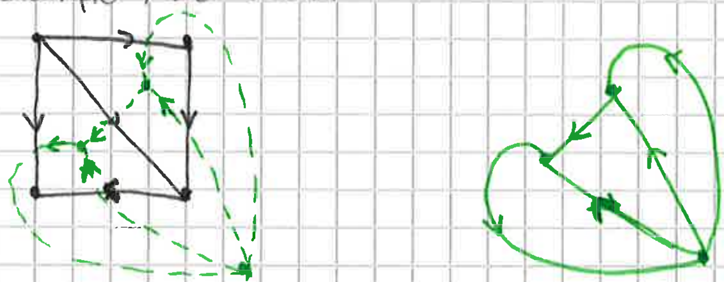
Example



G^* can be carefully drawn in the plane, and taking its dual recover the original graph: $(G^*)^* = G$.

Given an orientation of G, we can naturally obtain an orientation of G' by rotating the oriented edges of G clockwise:

In our example, we obtain



Given an n-coloring c of $H = G^*$ we can record the color gradient $t(uv) = c(v) - c(u)$ for each oriented edge $u \to v$ of H.

Conversely, if H is connected, we can recover the color of all nodes knowing the color of one single vertex v_0 & the values of t:

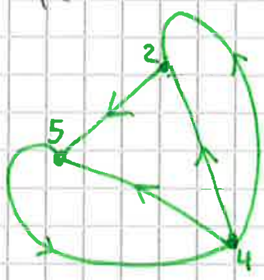
For v a vertex of H, choose a path $v_0 = p_0, p_1, \dots, p_k = v$ from v_0 to v, then.

$$c(p_i) = \begin{cases} c(p_{i-1}) + t(p_{i-1}p_i) & \text{if } p_{i-1} \to p_i \\ c(p_{i-1}) - t(p_{i-1}p_i) & \text{if } p_{i-1} \leftarrow p_i \end{cases}$$

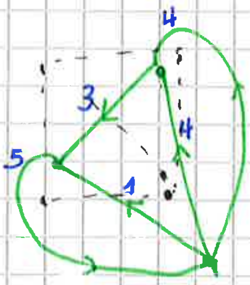
The color $c(v)$ is independet of the chosen path. Therefore, walking along a cycle, the sum of the values $t(e)$ of edges along their orientation minus those against their orientation has to be zero.

Now, via the correspondence between edges and dual edges, each node u of G corresponds to a region (node) of H which is bounded by a cycle. Adding and substracting the values of $t(e)$ along this cycle with respect to their orientation, corresponds to adding and substracting the values "$t(e^*)$" depending on whether e^* is pointing to or out of u.

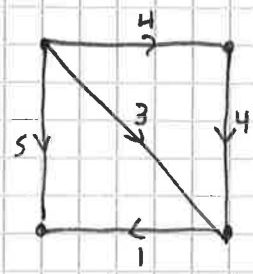
Example $n=6$



proper n-coloring



color gradients



corresponding nowhere zero $\mathbb{Z}_n$-flow.

Therefore, each proper n-coloring of $H = G^*$ induces a nowhere zero $\mathbb{Z}_n$-flow f on G. We have proven:

Prop: Let G be a connected planar graph with dual G^*. For every n-coloring c of G, the induced map f is a $\mathbb{Z}_n$-flow on G^*, and every such flow arises this way.
Moreover, the coloring c is proper if and only if f is nowhere zero.
[This map is $n^\ell \to 1$, where ℓ is the number of connected components of G]

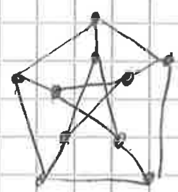
Corollary (Dual four-color theorem)
If G is a planar bridgeless graph, then $\varphi_G(4) > 0$.

↳ Tutte's approach to the 4-color theorem
studying also non-planar graphs (~~not~~ dual graphs but dual matroids)

Question: Has every graph a nowhere $\mathbb{Z}_4$-flow?

Tutte's sol: NO → Petersen graph.

Exercise
- Show that the Petersen graph has no nowhere zero $\mathbb{Z}_4$-flows.
- Can you find an explicit formula for $\varphi_G(n)$?



Proposition - Exercise

If G is a bridgeless connected graph, then $\varphi_G(n)$ agrees with a polynomial with integer coefficients of degree $|E|-|V|+1$ and leading coefficient 1.

Hint for a proof: Use a deletion-contraction argument:

- If $e \in E$ is a loop, then $\varphi_G(n) = (n-1)\,\varphi_{G\setminus e}(n)$
- If $e \in E$ is not a loop, then $\varphi_G(n) = \varphi_{G/e}(n) - \varphi_{G\setminus e}(n)$

Combinatorial reciprocity

We say that an orientation ρ of G is <u>totally cyclic</u> if every oriented edge is contained in an oriented cycle.

Define $\xi(G) := |E| - |V| + \#$ connected components of G.

Theorem (Breuer–Sanyal '10)
Let G be a bridgeless graph.

$$(-1)^{\xi(G)} \varphi_G(-n) = \left| \left\{ (f, \rho) : \begin{array}{l} f \text{ is a } \mathbb{Z}_n\text{-flow of } G \\ \rho \text{ totally cyclic reorientation of } G/\mathrm{supp}(f) \end{array} \right\} \right|$$

where $\mathrm{supp}(f) = \{ e \in E : f(e) \neq 0 \}$.
In particular,

$$(-1)^{\xi(G)} \varphi_G(-1) = \# \text{ totally cyclic orientations of } G.$$

We will prove this result later in the course using tools from geometry.