Combinatorial reciprocity theorems via geometry (1)
Cesar Ceballos
Lecture 3, 27.10.2020

Last lectures:
- Examples of combinatorial reciprocity
- Example 1: Graph colorings & acyclic orientations
- Example 2: Flows on graphs.

In this lecture: we will see a third example

(3) Order polynomials

A partially ordered set, or poset for short, is a set Π together with a binary relation $\preceq_\Pi$ that satisfies three properties

- reflexive: $a \preceq_\Pi a$
- transitive: $a \preceq_\Pi b \preceq_\Pi c$ implies $a \preceq_\Pi c$
- antisymmetric: $a \preceq_\Pi b$ and $b \preceq_\Pi a$ implies $a = b$.

for all $a,b,c \in \Pi$. We write just $\preceq$ if the poset is clear from the context.

We say that an element $a \in \Pi$ is covered by an element $b \in \Pi$ if

$$[a,b] := \{ z \in \Pi : a \preceq z \preceq b \} = \{a,b\}$$

That is, if $a \prec b$ and there is nothing else between a and b. In such a case we write $a \prec\!\cdot\, b$ and say that a is covered by b.

We can recover the full poset from its cover relations, by taking the transitive closure and adding in the reflexive relations.

The Hasse diagram of Π is a drawing of the graph of cover relations, where a node a is connected to a node b when a is covered by b ($a \prec\!\cdot\, b$), and a node a is drawn lower than a node b whenever $a \prec b$.

Examples

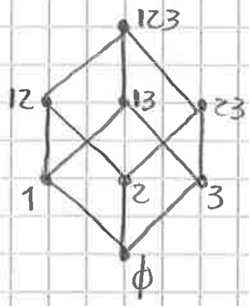
i) A chain $[d]$ on d elements $1,2,\dots,d$ with the order $\leq$ of the natural numbers

$d=3$

ii) <u>The Boolea lattice B_d</u> :

elements : subsets of $[d]$.
order : containment.

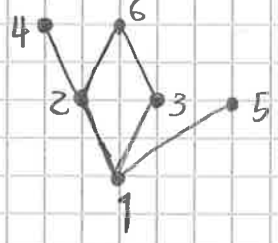
$d=3$



$23 \to$ subset $\{2,3\}$ for simplicity.

iii) The poset D_n whose element are $[n]$ ordered by divisibility

$n=6 \Rightarrow D_6$:



Given two posets Π and Π', we say that a map

$$\phi : \Pi \to \Pi'$$

is <u>order preserving</u> if

$$a \preceq_\Pi b \Rightarrow \phi(a) \preceq_{\Pi'} \phi(b)$$

ϕ is <u>strictly ordered preserving</u> if

$$a \prec_\Pi b \Rightarrow \phi(a) \prec_{\Pi'} \phi(b)$$

We are interested in counting such maps for a fixed poset Π and $\Pi' = [n]$ the n-chain.

We define the counting function

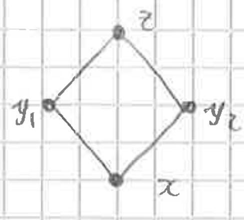
$$\boxed{\Omega^\circ_\Pi(n) := |\{\phi : \Pi \to [n] \text{ strictly order preserving}\}|}$$

<u>Examples</u>: i) Let Π be the d-chain $[d]$. A map $\phi : [d] \to [n]$ is strictly order preserving if and only if

$$1 \leq \phi(1) < \phi(2) < \cdots < \phi(d) \leq n$$

Therefore $\quad \Omega^\circ_{[d]}(n) = \binom{n}{d}$

ii) Let $\Pi = B_2$



The strictly order preserving maps $\phi : B_2 \to [n]$ correspond to tuples (x, y_1, y_2, z) satisfying

$$x < y_1, y_2 < z$$

There are three possible cases:

- $y_1 = y_2 \Rightarrow$ In this case we have $\binom{n}{3}$ possibilities
- $y_1 < y_2 \Rightarrow \binom{n}{4}$ possibilities
- $y_2 < y_1 \Rightarrow \binom{n}{4}$ possibilities

Thus

$$\Omega^{\circ}_{B_2}(n) = \binom{n}{3} + 2\binom{n}{4}$$

iii) Exercise: compute $\Omega^{\circ}_{D_6}(n)$

<u>Proposition</u> For a finite poset Π, the counting function $\Omega^{\circ}_{\Pi}(n)$ agrees with a polynomial of degree $|\Pi|$ with rational coefficients.

We call $\Omega^{\circ}_{\Pi}(n)$ the <u>strict order polynomial</u> of Π.

<u>Proof</u> Let $d = |\Pi|$ and $\phi : \Pi \to [n]$ a strictly order preserving map. This map factors uniquely as

$$\begin{tikzcd} \Pi \arrow[rr, "\phi"] \arrow[dr, two heads, "\sigma"'] & & {[n]} \\ & \phi(\Pi) \arrow[ur, hook, "i"'] & \end{tikzcd}$$

where $\sigma(a) := \phi(a)$ and $i(a) := a$.
If $|\phi(\Pi)| = r$ then the image $\phi(\Pi)$ is an r-chain inside $[n]$ and for $r = 1, 2, \ldots, d$ there are only finitely many surjections

$$\sigma : \Pi \to [r]$$

Denote the number of such surjections by s_r, for $r = 1, 2, \ldots, d$.
Therefore

$$\Omega^{\circ}_{\Pi}(n) = s_d \binom{n}{d} + s_{d-1}\binom{n}{d-1} + \ldots + s_1 \binom{n}{1} \qquad \blacksquare$$

Similarly, we define the counting function

$$\Omega_\Pi(n) = |\{\phi : \Pi \to [n] \text{ order preserving}\}|$$

A similar proof shows that this expression is also a polynomial in n (exercise). We call it the <u>order polynomial</u> of Π.

<u>Example</u>: Let Π be a d-chain $[d]$. A map $\phi : [d] \to [n]$ is order preserving is

$$1 \leq \phi(1) \leq \phi(2) \leq \dots \leq \phi(d) \leq n$$

This is equivalent to choose d values from $[n]$ with possible repetitions!

Therefore

$$\Omega_{[d]}(n) = \binom{n+d-1}{d}$$

As we have seen before this is equal to $(-1)^d$ times the polynomial $\binom{n}{d}$ evaluated at $-n$. Therefore

$$(-1)^d \Omega^\circ_{[d]}(n) = \Omega_{[d]}(n)$$

This combinatorial reciprocity holds for posets in general!

(*) <u>Theorem</u> Let Π be a finite poset. Then

$$(-1)^{|\Pi|} \Omega^\circ_\Pi(-n) = \Omega_\Pi(n)$$

We will prove this theorem later in the course.

<u>Back to graph colorings and acyclic orientations</u>

Every oriented acyclic graph ρG can be completed to a partial order $\Pi(\rho G)$ on the set of nodes of G: for two nodes u, v we say that

$$u \geq_\Pi v \quad \text{if there is a oriented path } u \to \dots \to v$$

from v to u. In fact, this order relation is a partial order if and only if the orientation is acyclic.

<u>Prop</u>: Let $\rho G = (V, E, \rho)$ be an acyclic oriented graph and $\Pi(\rho G)$ be the induced poset on V. A map $c : V \to [n]$ is (strictly) compatible with the orientation ρ of G if and only if c is a (strictly) order preserving map $\Pi(\rho G) \to [n]$.

As we have seen before.

$$\chi_G(n) = \#\text{ proper } n\text{-colorings of } G$$
$$= \#\text{ of strictly compatible pairs } (\rho, c) \text{ of an acyclic orientation } \rho \text{ and an } n\text{-coloring } c \text{ of } G$$

As a consequence:

<u>Corollary</u> The chromatic polynomial $\chi_G(n)$ is a sum of order polynomials:

$$\boxed{\chi_G(n) = \sum_{\rho} \Omega^{\circ}_{\Pi(\rho G)}(n)}$$

where the sum runs over all acyclic orientations of G.

Similarly, we defined

$$\tilde{\chi}_G(n) := \#\text{ of compatible pairs } (\rho, c) \text{ of an acyclic orientation } \rho \text{ and an } n\text{-coloring } c \text{ of } G$$

<u>Thus</u>

$$\boxed{\tilde{\chi}_G(n) = \sum_{\rho} \Omega_{\Pi(\rho G)}(n)}$$

where the sum runs over all acyclic orientations of G.

As a consequence of our combinatorial reciprocity theorem (*) for order polynomials, we recover the combinatorial reciprocity for graphs:

$$(-1)^{|V|}\chi_G(-n) = \tilde{\chi}_G(n)$$

Theorem (*) is a stronger refined result!