Combinatorial reciprocity theorems via geometry
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Last lecture : Example 3 - Order polynomials

Before proving the combinatorial reciprocity for order polynomials we will take a little detour through geometry.

~~(4) Ehrhart polynomials in the plane~~

As we have seen in the previous lecture, the strict order polynomial of a d-chain $\Pi = [d]$ is equal to:

$$\boxed{\Omega^{\circ}_{[d]} = \binom{n}{d}}$$

This counts the number of integer solutions to the system of strict inequalities

$$0 < x_1 < x_2 < \dots < x_d < n+1$$

Equivalently, this counts the number of points in $\mathbb{R}^d$ with integer coordinates inside the set

$$(n+1)\,\Delta^{\circ}_d = \{ x \in \mathbb{R}^d : 0 < x_1 < x_2 < \dots < x_d < n+1 \}$$

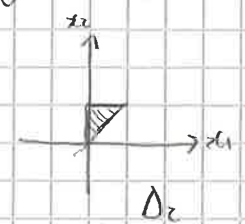
where

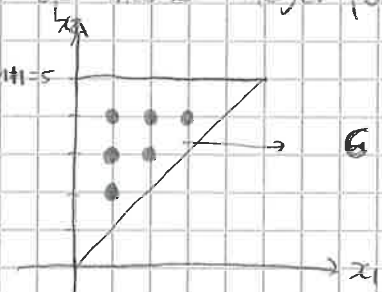
$$\Delta^{\circ}_d = \{ x \in \mathbb{R}^d : 0 < x_1 < x_2 < \dots < x_d < 1 \}$$

is a the interior of d-dimensional simplex in $\mathbb{R}^d$

Example For $d=2$ and $n=4$ we have $\binom{n}{d} = \binom{4}{2} = 6$

which counts the number of ~~interior~~ integer points strictly inside the dilated triangle $(4+1)\Delta_2$:





Similarly, the order polynomial of a d-chain is

$$\boxed{\Omega_{[d]}(n) = \binom{n+d-1}{d}},$$

which counts the number of integer solutions to the system of inequalities

$$0 \le x_1 \le x_2 \le \dots \le x_n \le n-1$$

Using 0 & n-1 instead of 1 & n for convenience.

Therefore, $\Omega_{[d]}(n)$ counts the number of integer points inside

$$(n-1)\Delta_d := \{x\in\mathbb{R}^d : 0\le x_1\le x_2\le \dots \le x_d\le n-1\}$$

where

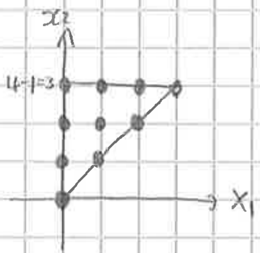
$$\Delta_d := \{x\in\mathbb{R}^d : 0\le x_1\le x_2\le \dots \le x_d\le 1\}$$

is the closure of $\mathring{\Delta}_d$.

In our example for $d=2$ and $n=4$:

$$\Omega_{[2]}(4) = \binom{4+2-1}{2} = 10$$

This counts the number of integer points in $(4-1)\Delta_2$:

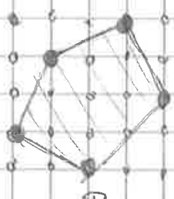


For a bounded set $S\subseteq\mathbb{R}^d$ we define

$$E(S) := |S\cap\mathbb{Z}^d|$$

to be the number of integer lattice points in S.

A <u>convex lattice polygon</u> $P\subseteq\mathbb{R}^2$ is the smallest convex set containing a finite set of non colinear integer points in the plane.



P

We define the <u>Ehrhart counting functions</u>:

$$\boxed{\begin{aligned} ehr_{P^\circ}(n) &:= E(nP^\circ) = |nP^\circ\cap\mathbb{Z}^2| \\ ehr_P(n) &:= E(nP) = |nP\cap\mathbb{Z}^2| \end{aligned}}$$

$\longrightarrow$ # integer lattice points strictly inside nP

$\longrightarrow$ # integer lattice points inside nP

In our example, $P=\Delta_2$ is a triangle and

$$ehr_{\Delta_2^\circ}(n+1) = \binom{n}{2} = \Omega^\circ_{[2]}(n) \qquad ehr_{\Delta_2}(n-1) = \binom{n+2-1}{2} = \Omega_{[2]}(n)$$

Applying the combinatorial reciprocity of order polynomials to a d-chain for $d=2$, we get

$$(-1)^{2}\,\Omega^\circ_{[2]}(-n) = \Omega_{[2]}(n)$$

Therefore

$$(-1)^2\,\mathrm{ehr}_{\Delta_2^\circ}\underbrace{(-n+1)}_{= -(n-1)} = \mathrm{ehr}_{\Delta_2}(n-1)$$

which we can restate as.

$$(-1)^2\,\mathrm{ehr}_{\Delta_2^\circ}(-n) = \mathrm{ehr}_{\Delta_2}(n)$$

For general d, we get

$$\boxed{(-1)^d\,\mathrm{ehr}_{\Delta_d^\circ}(-n) = \mathrm{ehr}_{\Delta_d}(n)}$$

We will see that this combinatorial reciprocity holds for any d-dimensional convex "polytope" P!!

Today: we will prove the 2-dimensional version of this result.

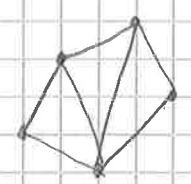
Theorem Let $P \subseteq \mathbb{R}^2$ be a lattice polygon. Then $\mathrm{ehr}_P(n)$ agrees with a polynomial of degree 2 with rational coefficients, and

$$(-1)^2\,\mathrm{ehr}_P(-n) = \mathrm{ehr}_{P^\circ}(n)$$

counts the number of integer points in nP°.

Strategy of the proof:

1) Decompose the polygon into triangular pieces.



2) Prove polynomiality and combinatorial reciprocity for the simpler pieces in the triangulation

3) combine them to prove the general result.

It is clear that each polygon can be triangulated by adding diagonals. The resulting triangulation $\mathcal{T}$ consists of triangles, edges, and vertices. We call these the *faces* of the triangulation.
The counting lattice points map E satisfies

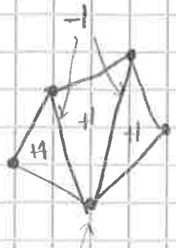
$$E(S \cup T) = E(S) + E(T) - E(S \cap T)$$

This is called a valuation.

Therefore

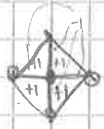
$$\mathrm{ehr}_P(n) = \sum_{F \in \mathcal{T}} \mu(F)\,\mathrm{ehr}_F(n)$$

for some coefficients $\mu(F)$.



counted +3 -2 = 1 time

$$\mu(F) = \begin{cases} 1 & \text{if } F \text{ is a triangle} \\ -1 & \text{if } F \text{ is an interior edge} \\ 0 & \text{if } F \text{ is a boundary edge} \\ 0 & \text{if } F \text{ is a boundary vertex} \\ 1 & \text{if } F \text{ is an interior vertex} \end{cases}$$

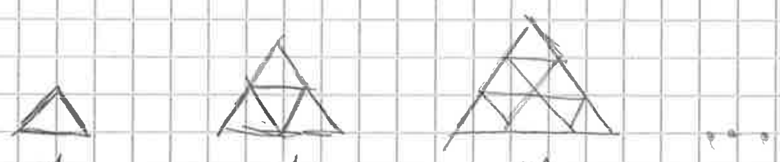


In summary

$$\boxed{\mu(F) = \begin{cases} (-1)^{\dim(F)} & \text{if } F \text{ is interior} \\ 0 & \text{otherwise} \end{cases}}$$

We need to show that $ehr_F(n)$ is a polynomial for every F.

- If F is a vertex, then $ehr_F(n) = 1$ ✓
- If F is an edge, exercise.
- Now let $F = \Delta$ be a tringle in the triangulation.



Δ 2Δ 3Δ ...

We triangulate $n\Delta$ as shown using copies of Δ and its reflection ∇ with respect to the origin.

There are three kind of edges: / — \
And only one kind of vertex •

We count the number of copies $t(Q,n)$ of each tile Q in the interior of our triangulation

$$t(\Delta,n) = \binom{n+1}{2} \qquad t(\nabla,n) = \binom{n}{2}$$

$$t(/,n) = t(-,n) = t(\backslash,n) = t(\nabla,n) = \binom{n}{2} \quad \text{interior edges}$$

$$t(\bullet,n) = \binom{n-1}{2} \quad \text{interior vertices}$$

Thus

$$ehr_\Delta(n) = \binom{n+1}{2} E(\Delta) + \binom{n}{2} E(\nabla) \qquad (*)$$
$$- \binom{n}{2} \left[E(/) + E(-) + E(\backslash) \right] + \binom{n-1}{2} E(\bullet)$$

This proves that $ehr_\Delta(n)$ is a polynomial of degree 2.
And so $ehr_P(n)$ is a polynomial.

In order to prove the combinatorial reciprocity $(-1)^2 ehr_P(-n) = ehr_{P^\circ}(n)$
we claim that it suffices to prove it for $n=1$:

Assume $(-1)^2 ehr_Q(-1) = ehr_{Q^\circ}(1)$ for any polygon Q
Then taking $Q = nP$ implies

$$(-1)^2 ehr_Q(-1) = (-1)^2 ehr_P(n) = ehr_{Q^\circ}(1) = ehr_{P^\circ}(n)$$

~~For edges:~~
Exercise: Show that for every edge e of the triangulation

$$(-1)^1 ehr_e(-n) = ehr_{e^\circ}(n)$$

For triangles: evaluating $n=-1$ in $(*)$ we get

$$ehr_\Delta(-1) = 0 \times E(\Delta) + 1 \times E(\nabla) - 1\,[E(/) + E(-) + E(\backslash)] + 3E(\cdot)$$

which equals the number of interior lattice points of ∇.

$$\Rightarrow \quad ehr_\Delta(-1) = ehr_{\nabla^\circ}(1) = ehr_{\Delta^\circ}(1)$$

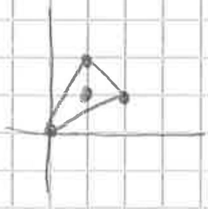
In the general case

$$ehr_P(-1) = \sum_{F \in \mathcal{T}} \mu(F)\, ehr_F(-1)$$

$$= \sum_{\substack{F \in \mathcal{T} \\ \text{interior}}} (-1)^{2-\dim(F)} (-1)^{\dim(F)} ehr_{F^\circ}(1)$$

$$= \sum_{\substack{F \in \mathcal{T} \\ \text{interior}}} E(F^\circ)$$

$$= ehr_{P^\circ}(1) \qquad \blacksquare$$

<u>Example</u> Let $P = \Delta =$  $\mathrm{ehr}_P(n) = ?$ $\mathrm{ehr}_{P^\circ}(n) = ?$

Since $P = \Delta$ is a triangle, we can proceed as in the proof of the theorem:

$$\mathrm{ehr}_P(n) = \binom{n+1}{2} E(\Delta) + \binom{n}{2} E(\nabla) - \binom{n}{2}\left[E(/) + E(-) + E(\backslash)\right] + \binom{n-1}{2} E(\cdot)$$

Here

$$E(\Delta) = 4 = E(\nabla) \qquad E(/) = E(-) = E(\backslash) = 2 \qquad E(\cdot) = 1$$

Therefore

$$\mathrm{ehr}_P(n) = 4\binom{n+1}{2} + 4\binom{n}{2} - 6\binom{n}{2} + \binom{n-1}{2}$$

$$= 2(n+1)n - n(n-1) + \frac{(n-1)(n-2)}{2}$$

$$\boxed{\mathrm{ehr}_P(n) = n(n+3) + \frac{(n-1)(n-2)}{2}} = \frac{3}{2}n^2 + \frac{3}{2}n + 1$$

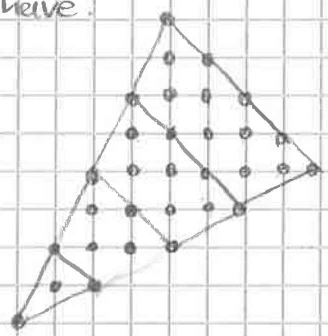
And

$$\mathrm{ehr}_{P^\circ}(n) = \mathrm{ehr}_P(-n)$$

$$\boxed{\mathrm{ehr}_{P^\circ}(n) = n(n-3) + \frac{(n+1)(n+2)}{2}} = \frac{3}{2}n^2 - \frac{3}{2}n + 1$$

For small values of n we have.

n	$\mathrm{ehr}_P(n)$	$\mathrm{ehr}_{P^\circ}(n)$
1	4	1
2	10	4
3	19	10
4	31	19



4	10	19	31
1	4	10	19

There is a simple way to find the Ehrhart polynomial of any 2-dimensional polygon, which follows from Pick's theorem.

Let $P \subseteq \mathbb{R}$ be a lattice polygon whit area A, I interior lattice points, and B boundary lattice points.

<u>Exercise</u> Show that:
- $A = I + B/2 - 1$
- $\mathrm{ehr}_{P^\circ}(n) = An^2 - Bn/2 + 1$
- $\mathrm{ehr}_P(n) = An^2 + Bn/2 + 1$