Combinatorial reciprocity theorems via geometry

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Last lectures: Four examples of combinatorial reciprocity

In this lecture: We will give a combinatorial proof of the combinatorial reciprocity for order polynomials (Example 3)

- Order polynomials and order ideals

Let Π be a finite poset. The order polynomial is

$$\Omega_\Pi(n) := |\{\phi : \Pi \to [n] \text{ order preserving}\}|$$

The strict order polynomial is

$$\Omega^\circ_\Pi(n) := |\{\phi : \Pi \to [n] \text{ strict order preserving}\}|$$

Our objective is to prove the following combinatorial reciprocity:

Theorem Let Π be a finite poset. Then

$$(-1)^{|\Pi|}\Omega_\Pi(-n) = \Omega^\circ_\Pi(n)$$

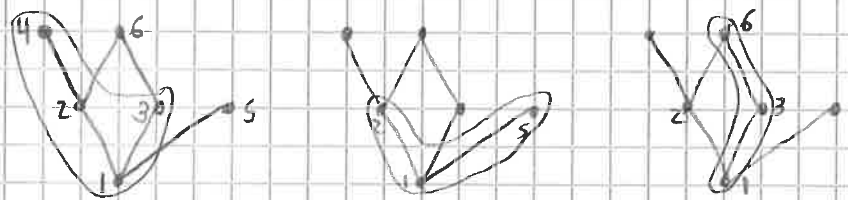
For this, we will relate these problems to counting order ideals, and will use zeta functions and Möbius functions.

A subset $I \subseteq \Pi$ is called an order ideal if

$$y \in I \text{ and } x \leq_\Pi y \implies x \in I$$

Example

D_6



$I = \{1,2,3,4\}$ is an order ideal

$I = \{1,2,5\}$ is an order ideal

$I' = \{1,3,6\}$ is NOT an order ideal

$6 \in I$ and $2 \leq_{D_6} 6$ but $2 \notin I$

Lemma

If $\phi : \Pi \to [n]$ is order preserving and $1 \leq j \leq n$, then $I_j = \phi^{-1}([j])$ is an order ideal.

Proof Let $y \in I_j$ and $x \leq_\Pi y$. Then

$$\phi(x) \leq \phi(y) \leq j$$

Therefore $x \in \phi^{-1}([j]) = I_j$

Proposition Order preserving maps $\phi : \Pi \to [n]$ are in bijection with multichains of order ideals

$$\emptyset = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n = \Pi$$

of length n. The map ϕ is strict order preserving if and only if $I_j \setminus I_{j-1}$ is an antichain for all $j = 1, 2, \ldots, n$.

Antichain : a set of pairwise uncomparable elements in Π.

Proof Letting $I_j = \phi^{-1}([j])$, the first part of the proposition follows from the previous Lemma.

For the second part, ϕ is strict order preserving iff it is order preserving and there are no two elements $x <_{\Pi} y$ such that $\phi(x) = \phi(y)$.

Thus, ϕ is strict order preserving iff it is order preserving and

$$\phi^{-1}(j) = I_j \setminus I_{j-1}$$

does not contain a pair of comparable elements for every j ∎

The collection $\mathcal{L}(\Pi)$ of order ideals of Π is itself a poset under set inclusion, which we call the lattice of order ideals or Birkhoff lattice of Π.

We showed that $\Omega_\Pi(n)$ counts the number of multichains

$$\emptyset = x_0 \leq x_1 \leq \cdots \leq x_n = \Pi$$

in $\mathcal{L}(\Pi)$. We will now count multichains in general posets.

- The incidence algebra

Let Π be a finite poset. The incidence algebra $I(\Pi)$ is a $\mathbb{C}$-vector space spanned by those functions

$$\alpha : \Pi \times \Pi \to \mathbb{C}$$

satisfying $\alpha(x, y) = 0$ whenever $x \not\leq y$.

We define the convolution product of $\alpha, \beta : \Pi \times \Pi \to \mathbb{C}$ as

$$\boxed{(\alpha * \beta)(r, t) := \sum_{r \leq s \leq t} \alpha(r, s)\beta(s, t)}$$

We also define $\boxed{\delta(x, y) := \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}}$

This gives $I(\Pi)$ the structure of an associative $\mathbb{C}$-algebra with unit δ.

The <u>zeta function</u> $\zeta \in I(\Pi)$ is defined as

$$\boxed{\zeta(x,y) = \begin{cases} 1 & \text{if } x \preceq y \\ 0 & \text{otherwise} \end{cases}}$$

<u>Proposition</u> Let Π be a finite poset and $x,y \in \Pi$. Then $\zeta^n(x,y)$ equals the number of multichains

$$x = x_0 \preceq x_1 \preceq \dots \preceq x_n = y$$

of length n.

<u>Proof</u> We proceed by induction:

- $n=1$: $\zeta(x,y) = 1$ iff $x = x_0 \preceq x_1 = y$. ✓
- Assume $\zeta^{n-1}(x,y)$ is the number of multichains of length $n-1$ for all x,y.

$$\zeta^n(x,z) = (\zeta^{n-1} * \zeta)(x,z) = \sum_{x \preceq y \preceq z} \zeta^{n-1}(x,y)\, \zeta(y,z)$$

This counts the number of multichains of length $n-1$ ending at y that can be extended to z. ∎

> **Corollary** For a finite poset Π, let ζ be the zeta function of $\mathcal{L}(\Pi)$, the lattice of order ideals of Π. Then
>
> $$\Omega_\Pi(n) = \zeta^n(\emptyset, \Pi)$$

We can use this to give an alternative proof that $\Omega_\Pi(n)$ agrees with a polynomial. For this, we use the following lemma.

Let $\eta \in I(\Pi)$ be defined as

$$\boxed{\eta(x,y) = \begin{cases} 1 & \text{if } x \prec y \\ 0 & \text{otherwise} \end{cases}}$$

<u>Lemma</u> The following hold:

(1) $\zeta = \delta + \eta$

(2) $\zeta^n(x,y) = (\delta + \eta)^n(x,y) = \sum_{k=0}^{n} \binom{n}{k} \eta^k(x,y)$

(3) $\eta^k \equiv 0$ for $k > |\Pi|$

Proof (1) follows by definition.

(2) For this note that

$$(\delta+\eta)^n(x,y) = \sum (\alpha_1 * \alpha_2 * \dots * \alpha_n)(x,y)$$

where α_i is either δ or η. If exactly k α_i's are equal to η then $\alpha_1 * \dots * \alpha_n = \eta^k$. Therefore,

$$(\delta+\eta)^n(x,y) = \sum_{k=0}^{n} \binom{n}{k} \eta^k(x,y)$$

(3) Note that $\eta^k(x,y) = \sum_{x=x_0 \le x_1 \le \dots \le x_k = y} \eta(x_0,x_1)\,\eta(x_1,x_2) \dots \eta(x_{k-1},x_k)$

and this is equal to the number of multichains.

$$x = x_0 < x_1 < x_2 < \dots < x_k = y.$$

where strict inequality is required (because $\eta(x_{j-1},x_j)=1$ iff $x_{j-1} < x_j$)

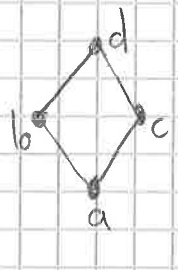
Since the number of such strict multichains is zero if $k > |\Pi|$, then, $\eta^k \equiv 0$ for $k > |\Pi|$. ∎

We say that a poset Π has a minimum $\hat{0}$ and a maximum $\hat{1}$ if $\hat{0} \le x \le \hat{1}$ for all $x \in \Pi$.

Proposition Let Π be a finite poset with minimum $\hat{0}$, maximum $\hat{1}$, and zeta function ζ. Then $\zeta^n(\hat{0},\hat{1})$ agrees with a polynomial in n.

Proof This follows from (2) and (3) in the previous Lemma. ∎

Example

$\Pi =$ 

$$\zeta = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 1 & 1 & 1 \\ b & & 1 & 0 & 1 \\ c & & & 1 & 1 \\ d & & & & 1 \end{array}$$

in matrix form $\zeta(x,y)$ in row x and column y.

$$\zeta^2 = \begin{pmatrix} 1 & 2 & 2 & 4 \\ & 1 & 0 & 2 \\ & & 1 & 2 \\ & & & 1 \end{pmatrix} \qquad \zeta^3 = \begin{pmatrix} 1 & 3 & 3 & 9 \\ & 1 & 0 & 3 \\ & & 1 & 3 \\ & & & 1 \end{pmatrix}$$

$\zeta^2(a,d) = 4$ counts the number of multichains of length 2:

$$a = x_0 \le x_1 \le x_2 = d$$

where x_1 is either equal to $a, b, c,$ or d.

$\zeta^3(a,d) = 9$

- Exercise: Show that $\zeta^n(a,d) = n^2$. More generally, Let B_d be the Boolean lattice on d elements and $Z_{B_d}(n) = \zeta^n_{B_d}(\phi, [d])$. Show that $Z_{B_d}(n) = n^d$.

As we have seen in this example, the zeta function can be represented in matrix form by choosing an order $p_1, p_2, \dots, p_d$ of the elements in Π.

We say that this ordering is a <u>linear extension</u> of Π if

$$p_i \leq p_j \quad \text{implies} \quad i \leq j.$$

This allows us to identify the incident algebra $I(\Pi)$ with a subalgebra of upper triangular $(d \times d)$-matrices, by setting

$$\alpha := \left(\alpha(p_i, p_j)\right)_{1 \leq i,j \leq d}$$

$$\begin{pmatrix} * & \times & \times & \cdots \\ & * & & \vdots \\ & & \ddots & \\ & & & * \\ & & & & * \end{pmatrix}$$

This linear algebra perspective affords a simple criterion for when α is invertible! (by inverting the matrix).

<u>Proposition</u> An element $\alpha \in I(\Pi)$ is invertible if and only if $\alpha(x,x) \neq 0$ for all $x \in \Pi$.

<u>Proof</u> Exercise.

Inverting the zeta function will be a key step in our proof of the combinatorial reciprocity for order polynomials.

In our example

$$\zeta = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix} \end{array} \qquad \zeta^{-1} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ & 1 & 0 & -1 \\ & & 1 & -1 \\ & & & 1 \end{pmatrix} \in I(\Pi)$$