Combinatorial reciprocity theorems via geometry

Cesar Ceballos

Lecture 6, 24.11.2020

Last lecture: We started a combinatorial proof of order polynomial reciprocity

Today : - We will finish that proof

- Recall from last time

Let Π be a finite poset and $\Omega_\Pi(n)$, $\Omega^\circ_\Pi(n)$ its order and strict order polynomials respectively.

Want to prove:

Theorem $(-1)^{|\Pi|}\Omega_\Pi(-n) = \Omega^\circ_\Pi(n)$

We showed:

Proposition $\Omega_\Pi(n)$ is the number of multichains of order ideals

$$\emptyset = I_0 \subseteq I_1 \subseteq \dots \subseteq I_n = \Pi$$

$\Omega^\circ_\Pi(n)$ is the number of such multichains satisfying that $I_j \setminus I_{j-1}$ is an antichain for all $j = 1, 2, \dots, n$

Proposition

$$\Omega_\Pi(n) = \zeta^n(\emptyset, \Pi)$$

where $\zeta = \zeta_{\mathcal{L}(\Pi)}$ is the zeta function of $\mathcal{L}(\Pi)$, the lattice of order ideals of Π.

Recall

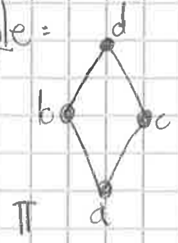
$$\zeta(x,y) = \begin{cases} 1 & \text{if } x \le y \\ 0 & \text{otherwise} \end{cases} \qquad \delta(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \qquad \eta(x,y) = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{otherwise} \end{cases}$$

We will show that evaluating the order polynomial at negative numbers is related to inverting the zeta function:

$$\Omega_\Pi(-n) = \zeta^{-n}(\emptyset, \Pi)$$

So, we need to give a description of the inverse of ζ.

Example:



$$\zeta = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 1 & 1 & 1 \\ b & & 1 & 0 & 1 \\ c & & & 1 & 1 \\ d & & & & 1 \end{array} \qquad \zeta^{-1} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & -1 & -1 & 1 \\ b & & 1 & 0 & -1 \\ c & & & 1 & -1 \\ d & & & & 1 \end{array} \in I(\Pi)$$

We call $\mu = \zeta^{-1}$ the Möbius function.

This function satisfies $(\mu * \zeta) = (\zeta * \mu) = \delta$.
Evaluating this at (x,z) with $x \preceq z$ we get

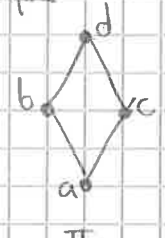
$$\sum_{x \preceq y \preceq z} \mu(x,y) = \sum_{x \preceq y \preceq z} \mu(y,z) = \begin{cases} 1 & \text{if } x = z \\ 0 & \text{otherwise} \end{cases}$$

Therefore, we can compute the Möbius function recursively as follows:

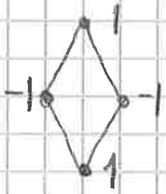
$$\mu(x,z) = -\sum_{x \preceq y \prec z} \mu(x,y) = -\sum_{x \prec y \preceq z} \mu(y,z) \quad \text{for } x \prec z$$

and $\mu(x,x) = 1$

Example



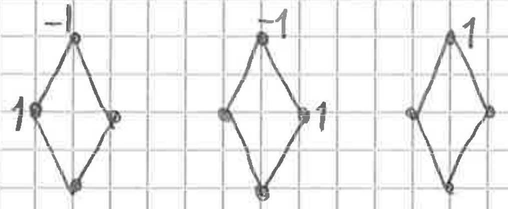
Π



$\mu(a,z)$

Sum of the labels between a and other node must be zero.

Fill the labels from bottom to top starting with label 1.

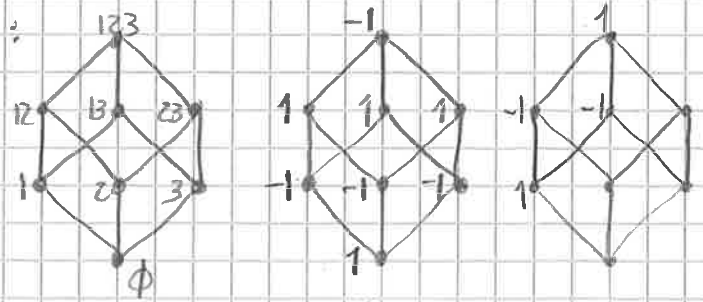


$\mu(b,z)$ $\quad$ $\mu(c,z)$ $\quad$ $\mu(d,z)$

Compare this with the matrix representation of $\zeta^{-1} = \mu$ above!

Example:

$\Pi = B_3$



$\mu(\phi,z)$ $\quad$ $\mu(1,z)$ $\quad$ ...

Exercise Let $\Pi = B_d$. Show that $\mu_{B_d}(A,B) = \begin{cases} (-1)^{|B \setminus A|} & \text{for } A \subseteq B \subseteq [d] \\ 0 & \text{otherwise} \end{cases}$

$$\boxed{\begin{array}{l}\underline{\text{Proposition}} \quad \Omega_\Pi(-n) = \zeta_{\mathcal{L}}^{-n}(\emptyset,\Pi) = \mu_{\mathcal{L}}^{n}(\emptyset,\Pi) \\ \text{where } \mathcal{L} = \mathcal{L}(\Pi) \text{ is the lattice of order ideals of } \Pi.\end{array}}$$

This is strongly suggested by our notation but requires a proof.

Proof Recall that

$$\zeta = \delta + \eta$$

and that we proved

$$\boxed{\zeta^n = \sum_{k=0}^{n} \binom{n}{k} \eta^k = \sum_{k=0}^{d} \binom{n}{k} \eta^k} \qquad (*)$$

where d is the size of the poset in which ζ is computed. ($\eta^k \equiv 0$ when $k > d$)

We can do a similar computation for the inverse of zeta:

$$\zeta^{-1} = (\delta + \eta)^{-1} = \delta - \eta + \eta^2 - \eta^3 + \dots + (-1)^d \eta^d$$

Taking powers of ζ^{-1}:

$$\zeta^{-n} = \sum_{k=0}^{d} (-1)^k a_k \eta^k$$

where a_k counts the number of compositions $(\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_i \in \mathbb{N}_{\geq 0}$ such that

$$\alpha_1 + \alpha_2 + \dots + \alpha_n = k$$

(You take the term η^{α_i} from the ith factor in ζ^{-n})

Exercise: show that $a_k = \binom{n+k-1}{k}$

As a consequence, we get

$$\boxed{\zeta^{-n} = \sum_{k=0}^{d} (-1)^k \binom{n+k-1}{k} \eta^k}$$

This is exactly the evaluation of the polynomial $(*)$ at $-n$ using the combinatorial reciprocity of binomial coefficients (subsets vs multisubset of cardinality k of $[n]$)

Thus, since $\Omega_\Pi(n) = \zeta_{\mathcal{L}}^{n}(\emptyset,\Pi)$

$$\Rightarrow \quad \Omega_\Pi(-n) = \zeta_{\mathcal{L}}^{-n}(\emptyset,\Pi) = \mu_{\mathcal{L}}^{n}(\emptyset,\Pi)$$

∎

Expanding $\mu_{\mathcal{L}}^n = \overbrace{\mu_{\mathcal{L}} * \cdots * \mu_{\mathcal{L}}}^{n}$ we get:

$$\mu_{\mathcal{L}}^n(\emptyset, \Pi) = \sum \mu_{\mathcal{L}}(I_0, I_1)\, \mu_{\mathcal{L}}(I_1, I_2) \cdots \mu_{\mathcal{L}}(I_{n-1}, I_n)$$

Where the sum runs over all multichains of order ideals

$$\emptyset = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = \Pi$$

of length n.

Our next goal: understand the evaluation $\mu_{\mathcal{L}}(K, M)$ where $K \subseteq M \subseteq \Pi$ are order ideals. This evaluation depend only on the interval $[K, M] := \{L \in \mathcal{L} : K \subseteq L \subseteq M\}$.

<u>Theorem</u> Let Π be a finite poset and $K \subseteq M$ order ideals in $\mathcal{L} = \mathcal{L}(\Pi)$. Then

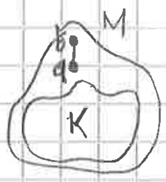
$$\mu_{\mathcal{L}}(K, M) = \begin{cases} (-1)^{|M \setminus K|} & \text{if } M \setminus K \text{ is an antichain} \\ 0 & \text{otherwise} \end{cases}$$

<u>Proof</u> We start with the simple case. Assume $M \setminus K$ is an antichain. In this case $K \cup A$ is an order ideal for all $A \subseteq M \setminus K$. In other words, the interval $[K, M]$ is isomorphic to the Boolean lattice B_r where $r = |M \setminus K|$.

By the exercise about computing the Möbius function of Boolean lattices, we conclude the first part of our result.

For the second part, we proceed by induction.

Base case: $M \setminus K$ consists of two comparable elements:



Π

$$\Longrightarrow \quad [K, M] \text{ in } \mathcal{L}(\Pi) \;\cong\; \begin{array}{ll} 0 & M = K \cup \{a, b\} \\ -1 & K \cup \{a\} \\ 1 & K \end{array}$$

$$\mu_{\mathcal{L}}(K, M) = 0 \qquad \checkmark$$

For the general case we use:

$$\mu_{\mathcal{L}}(K, M) = -\sum \mu_{\mathcal{L}}(K, L)$$

where the sum is over all order ideal L such that $K \subseteq L \subsetneq M$. By induction hypothesis $\mu_{\mathcal{L}}(K, L)$ is zero unless $L \setminus K$ is an antichain. Therefore

$$\mu_{\mathcal{L}}(K, M) = -\sum \left\{ (-1)^{|L \setminus K|} : \begin{array}{l} K \subseteq L \subsetneq M \text{ order ideal} \\ L \setminus K \text{ is an antichain} \end{array} \right\}$$

Now let $m \in M \setminus K$ be a minimal element. The order ideals in the sums are partition in two parts of equal size: those containing m and those that do not:

if $m \notin L$ then $L \cup \{m\}$ is also an order ideal.

if $m \in L$ then $L \setminus \{m\}$ is also an order ideal.

Since positive and negative term cancel each other we conclude that $\mu_{\mathcal{L}}(K,M) = 0$ ∎

Proof of order polynomial reciprocity

$$\Omega_\Pi(-n) = \zeta_{\mathcal{L}}^{-n}(\phi,\Pi) = \mu_{\mathcal{L}}^{n}(\phi,\Pi)$$

$$= \sum \mu(I_0,I_1)\mu(I_1,I_2)\dots\mu(I_{n-1},I_n)$$

where $\phi = I_0 \subseteq I_1 \subseteq \dots \subseteq I_n = \Pi$ is a multichain of order ideals

$$= (-1)^{|\Pi|} \# \text{ such multichains such that } I_j \setminus I_{j-1} \text{ is an antichain for } j=1,\dots,n$$

$$\Rightarrow (-1)^{|\Pi|}\Omega_\Pi(-n) = \bar{\Omega}_\Pi(n)$$ ∎