Combinatorial reciprocity theorems via geometry
Cesar Ceballos
Lecture 8, 15.12.2020

Last lecture: - Insight into Ehrhart polynomials in higher dimensions
- Introduction to polyhedral geometry

Today: Continue developing tools from polyhedral geometry

- Polyhedral geometry

Recall from last time: A polyhedron $Q \subseteq \mathbb{R}^d$ is given by

$$Q = \{ x \in \mathbb{R}^d : Ax \leq b \}$$

for some matrix $A = \begin{pmatrix} - a_1 - \\ \vdots \\ - a_k - \end{pmatrix} \in \mathbb{R}^{k \times d}$ and $b \in \mathbb{R}^k$

For $S \subseteq \mathbb{R}^d$ we define the affine hull $\operatorname{aff}(S)$ of S as the inclusion-minimal affine subspace of $\mathbb{R}^d$ containing S.

We define the dimension of a polyhedron Q as

$$\boxed{\dim Q := \dim \operatorname{aff}(Q)}$$

If Q is full dimensional we can describe its interior in terms of strict inequalities as

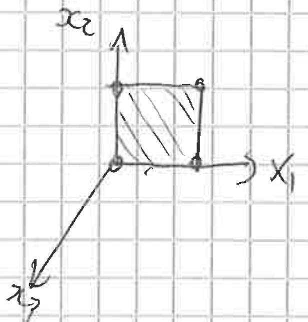
$$\{ x \in \mathbb{R}^d : \langle a_i, x \rangle < b_i \text{ for all } 1 \leq i \leq k \}$$

However, if Q is not full dimensional we need to be more careful.

Example

$$\left\{ x \in \mathbb{R}^3 : \begin{bmatrix} 1 & & \\ -1 & & \\ & 1 & \\ & -1 & \\ & & 1 \\ & & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$x_1 \leq 1$
$x_1 \geq 0$
$x_2 \leq 1$
$x_2 \geq 0$
$x_3 \leq 0$
$x_3 \geq 0$



Let $I := \{ i \in k : \langle a_i, x \rangle = b_i \text{ for all } x \in Q \}$

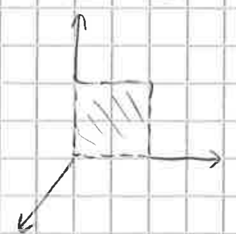
The relative interior Q° of Q is define as

$$\boxed{Q^\circ := \{ x \in Q : \langle a_i, x \rangle < b_i \text{ for all } i \notin I \}}$$

In our example

$\dim Q = 2$! and $Q^\circ =$

$x_1 < 1$
$x_1 > 0$
$x_2 < 1$
$x_2 > 0$



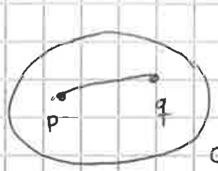
• Polytopes

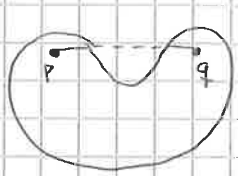
In this course we are dealing with convex polyhedra and convex polytopes.

A set $S \subseteq \mathbb{R}^d$ is convex if for every $p, q \in S$ the line segment

$$[p,q] := \{ (1-\lambda)p + \lambda q \; : \; 0 \le \lambda \le 1 \}$$

with end points p and q is contained in S.

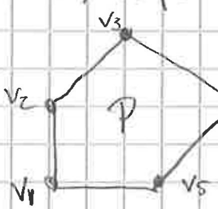


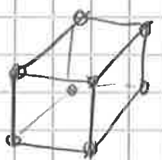


The convex hull $\mathrm{conv}(S)$ is the unique inclusion-minimal convex set containing S.

A (convex) polytope P is the convex hull $P = \mathrm{conv}(S)$ of a finite set $S \subseteq \mathbb{R}^d$.

P is a rational polytope or lattice polytope if we can choose S in $\mathbb{Q}^d$ or $\mathbb{Z}^d$, respectively.





We call $v \in S$ a vertex of $P = \mathrm{conv}(S)$ if $\mathrm{conv}(S \setminus \{v\}) \neq P$.

We denote by $V = \mathrm{vert}(P)$ the vertices of P. Note that $P = \mathrm{conv}(V)$.

A convex cone is a nonempty convex set $C \subseteq \mathbb{R}^d$ such that $\mu C \subseteq C$ for all $\mu \ge 0$.

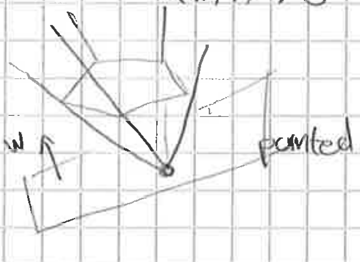
The conical hull $\mathrm{cone}(S)$ of a set S is the inclusion-minimal convex cone containing S.

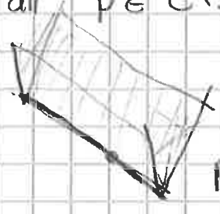
A convex cone C is finitely generated if $C = \mathrm{cone}(S)$ for some finite set S.
If $S = \{s_1, s_2, \dots, s_k\}$, we also write

$$\mathrm{cone}(S) = \mathbb{R}_{\ge 0} s_1 + \dots + \mathbb{R}_{\ge 0} s_k$$

A finitely generated convex cone C is pointed if there is some $w \in \mathbb{R}^d$ such that

$$\langle w, p \rangle > 0 \quad \text{for all} \quad p \in C \setminus \{0\}$$





One fundamental theorem of polyhedral geometry is the often call Minkowsky-Weyl theorem:

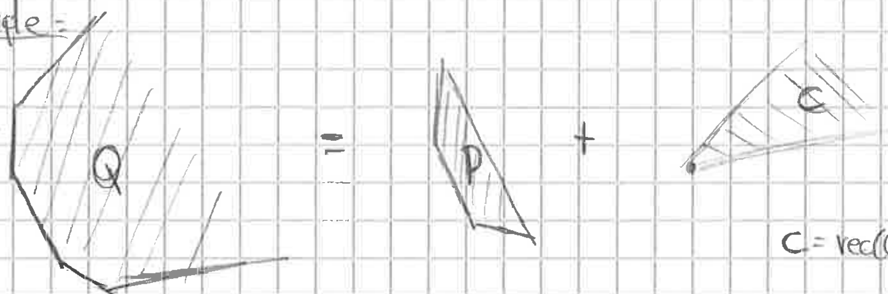
> **Theorem** A set $Q \subseteq \mathbb{R}^d$ is a polyhedron if and only if there exist a polytope P and a finitely generated cone C such that
>
> $$Q = P + C$$
>
> In particular, C is the recession cone of Q, and polytopes are precisely the bounded polyhedra.

Here

$$P+C := \{ p+c : p \in P \text{ and } c \in C \}$$

denotes the Minkowski sum of P and C.

Example:



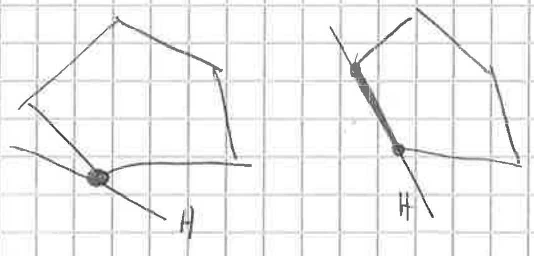
$C = \mathrm{rec}(Q)$

This result has remarkable non-trivial corollaries. For example:

- The image of a polyhedron under an affine linear transformation is also a polyhedron
- The intersection of a polytope with a polyhedron is also a polytope

We will omit the proof for time restrictions.

Faces of polyhedra



We say that a hyperplane

$$H = \{ x \in \mathbb{R}^d : \langle w, x \rangle = \delta \}$$

is admissible for a polyhedron Q if $Q \subseteq H^{\leq}$.

A face of Q is a subset of the form $F = Q \cap H$ for an admis. hyperplane H.

We also decree that $F = \emptyset$ and $F = Q$ are faces of Q. All other faces are called proper faces. Admissible hyperplanes that yield nonempty faces are called supporting hyperplanes.

Some faces have special names: 0-dim faces $\rightarrow$ vertices
bounded 1-dim faces $\rightarrow$ edges
1-codim faces $\rightarrow$ facets.

The dimension of $F = \emptyset$ is -1.

Here are some nice facts about faces of polyhedra:

<u>Proposition</u>. The faces of a polyhedron $Q \subset \mathbb{R}^d$ satisfy:

i) Every face of a face F of Q is also a face of Q.

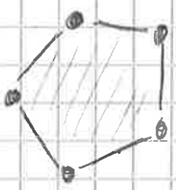
ii) If $F, F' \subseteq Q$ are two faces of Q, then $F \cap F'$ is a face of both F and F'.

iii) Every face F of Q is the intersection of all facets containing it.

I rely in our geometric intuition and will concentrate our efforts towards combinatorial reciprocities. Here is one result that will be important in this direction.

<u>Proposition</u>: Let $Q \subseteq \mathbb{R}^d$ be a polyhedron. For every point $p \in Q$ there is a unique face F of Q such that $p \in F^\circ$. Equivalently, Q is the disjoint union of the relative interior of its faces:

$$Q = \biguplus F^\circ .$$



<u>Proof</u> The inclusion $\supseteq$ is clear. We need to show that $\subseteq$ holds and that the union is disjoint.

Suppose Q is the intersection of <u>irredundant</u> halfspaces

$$Q = \bigcap_{j=1}^{m} H_j^{\leq}$$

(a half space is redundant if removing it does not alter the intersection, otherwise we say it is irredundant)

Let $p \in Q$. After possibly renumbering the half spaces we may assume.

$$p \in H_1^=, H_2^=, \dots, H_K^= \quad \text{and} \quad p \in H_{K+1}^<, \dots, H_m^<$$

for some $0 \leq K < m$.

Let

$$F := \bigcap_{j=1}^{K} H_j^= \cap \bigcap_{j=K+1}^{m} H^{\leq}$$

be a face of Q (intersection of the facets defined by $H_1, \dots, H_K$).

The relative interior of this face is

$$F^\circ = \bigcap_{j=1}^{K} H_j^= \cap \bigcap_{j=K+1}^{m} H^<$$

and therefore $p \in F^\circ$.

Uniqueness follows from the fact that $F_1^\circ \cap F_2^\circ = \emptyset$ for every two different faces of Q (exercise) ∎

Euler characteristic

Another important concept of interest for us the Euler characteristic of a polyhedron, a fundamental concept that relates its geometric combinatorial and topological structure.

More generally, we can define an Euler characteristic for polyconvex sets.

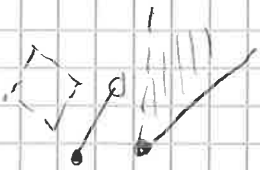
A set $S \subseteq \mathbb{R}^d$ is <u>polyconvex</u> if it is the union of finitely many relatively open polyhedra:

$$S = P_1^{\circ} \cup P_2^{\circ} \cup \dots \cup P_k^{\circ}$$

Where $P_1, \dots, P_k$ are polyhedra. For instance, as we have seen before, a polyhedron is polyconvex (it is the disjoint union of the relative interior of its faces). But this definition allows much more!

Polyconvex

Polyconvex



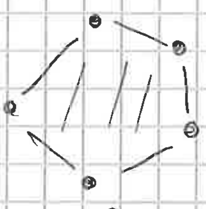
Polyconvex.

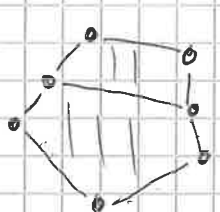
If $P_1^{\circ}, P_2^{\circ}, \dots, P_k^{\circ}$ are disjoint and nonempty, then we define the <u>Euler characteristic</u> of S as

$$\chi(S) := (-1)^{\dim P_1^{\circ}} + (-1)^{\dim P_2^{\circ}} + \dots + (-1)^{\dim P_k^{\circ}}$$

Note that S can be written in different ways as a disjoint union of relatively open polyhedra, and so, it is not clear that $\chi(S)$ is well defined.

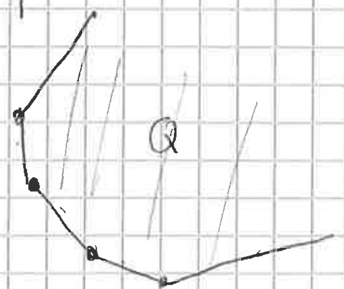
<u>Example</u>

$S =$ 

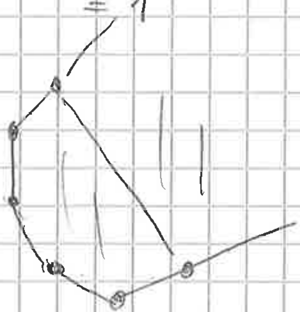
$S =$ 

$$\chi(S) = \underbrace{5(-1)^0}_{\text{vertices}} + \underbrace{5(-1)^1}_{\text{edges}} + \underbrace{1(-1)^2}_{\text{pentagon}} = 1$$

$\Rightarrow$ $\chi(S) = 7(-1)^0 + 8(-1)^1 + 2(-1)^2 = 1$ ✓



$$\chi(Q) = 4(-1)^0 + 5(-1)^1 + 1(-1)^2 = 0$$



$$\chi(Q) = 6(-1)^0 + 8(-1)^1 + 2(-1)^0 = 0$$

Proposition χ is well defined.

Proof Idea:
- Every polyconvex set can be obtained as the disjoint union of relatively open faces induced by some arrangement of hyperplanes
- The definition of χ using two different arrangements coincide (and equal χ for the refined arrangement of both).

Proposition If $Q \in \mathbb{R}^d$ is a polyhedron, then

$$\chi(Q) = \sum_{\emptyset < F \leq Q} (-1)^{\dim F} = \sum_{i=0}^{\dim Q} (-1)^i f_i(Q)$$

where the sum is over the nonempty faces of Q and f_i is the number of faces of dimension i of Q.

If Q is a polytope, this is equal to 1 (Euler-Poincaré formula)

What about unbounded polyhedra?

By Minkowsky-Weyl theorem we can write Q as

$$Q = P + C + L$$

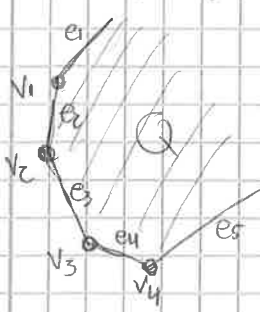
where C is a pointed cone and $L = \mathrm{lineal}(Q)$.

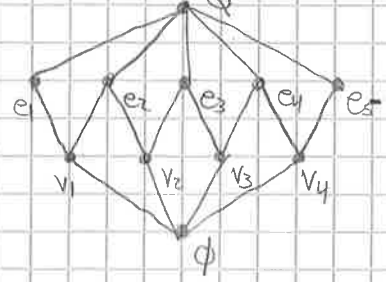
Theorem Let $Q = P + C + L$ where P is a polytope, C is a pointed cone, and $L = \mathrm{lineal}(Q)$. Then

$$\chi(Q) = \begin{cases} (-1)^{\dim L} & \text{if } C = \{0\} \\ 0 & \text{otherwise} \end{cases}$$

- **The poset of faces** (The face lattice).

Let $Q \subseteq \mathbb{R}^d$ be a polyhedron. The collection $\Phi(Q)$ of faces of Q (including $\emptyset$ and Q) is partially ordered by inclusion.





Combinatorial reciprocity for this poset implies an important result in the case of simplicial polytopes, known as the Dehn-Sommerville relations (Exercise)

The Möbius function can also be nicely described.

Theorem Let $Q \subseteq \mathbb{R}^d$ be a polyhedron with face lattice $\Phi = \Phi(Q)$. Then for $F \leq G \in \Phi$

$$\mu_\Phi(F,G) = \begin{cases} (-1)^{\dim G - \dim F} & \text{if } \emptyset < F \leq G, \\ (-1)^{\dim G + 1} \chi(G) & \text{if } \emptyset = F < G \end{cases}$$