Combinatorial reciprocity theorems via geometry
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Last lectures : • Towards Ehrhart polynomials in higher dimensions
• Preliminary 1 : polyhedral geometry

Today : • Preliminary 2 : generating functions

• Matrix powers and calculus of polynomials

This course develops around the following two questions:

Question 1 : When is a counting function a polynomial ?
Question 2 : What is the evaluation of this polynomial at negative integers?

Next : we will develop tools to study such polynomials.

We say that a matrix $A \in \mathbb{C}^{d\times d}$ is unipotent if

$$A = I + B$$

where I is the $d\times d$ identity matrix and $B^k = 0$ for some positive integer k.
(that is, B is nilpotent)

Example: The zeta function $\zeta : \Pi \times \Pi \to \mathbb{C}$ of a poset Π is unipotent when written in matrix form:

$$\zeta = \delta + \eta .$$

The entry $\zeta^n(\hat{0},\hat{1})$ gave rise to the zeta polynomial!

Proposition Let $A \in \mathbb{C}^{d\times d}$ be a unipotent matrix, fix indices $i,j \in [d]$, and consider the sequence $f(n) := (A^n)_{ij}$. Then $f(n)$ agrees with a polynomial in n!

Proof Suppose $A = I + B$, where $B^k = 0$. Then

$$f(n) = ((I+B)^n)_{ij} = \sum_{m=0}^{n} \binom{n}{m} (B^m)_{ij} = \sum_{m=0}^{k-1} \binom{n}{m} (B^m)_{ij}$$

which is a polynomial in n. ∎

The terms $\binom{n}{m}$ form a basis for the space of polynomials.
We will be switching bases in one way or another.

Here are the bases we will use:

(M)-basis : $\{ x^m \;:\; 0 \le m \le d \}$

(γ)-basis : $\{ x^m (1-x)^{d-m} \;:\; 0 \le m \le d \}$

(Δ)-basis : $\{ \binom{x}{m} \;:\; 0 \le m \le d \}$

(h^*)-basis : $\{ \binom{x+m}{d} \;:\; 0 \le m \le d \}$

<u>Proposition</u> The sets (M), (γ), (Δ), and (h^*) are bases for the vector space of polynomials of degree $\le d$:

$$\mathbb{C}[x]_{\le d} := \{ f \in \mathbb{C}[x] \;:\; \deg(f) \le d \}.$$

For example, for $k \geq 1$ we can write n^k as a linear combination of $\binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{k}$:

$$n = \binom{n}{1}$$

$$n^2 = \binom{n}{1} + 2\binom{n}{2} = n + n(n-1)$$

$$\vdots$$

$$n^k = s_1\binom{n}{1} + s_2\binom{n}{2} + \dots + s_k\binom{n}{k}$$

$\overbrace{\bullet\ \bullet\ \bullet\ \cdots\ \bullet}^{k}$ each of them n possibilities

where s_i = # surjective maps $[k] \twoheadrightarrow [i]$.

<u>Proof of proposition</u> : exercise.

Given a complex-valued sequence $(f(n))_{n \ge 0}$ we define the following linear operators:

$(If)(n) := f(n)$ (Identity operator)

$(\Delta f)(n) := f(n+1) - f(n)$ (difference operator)

$(Sf)(n) := f(n+1)$ (shift operator)

They are related by $S = I + \Delta$.

<u>Proposition</u> A sequence $f(n)$ is given by a polynomial of degree $\le d$ if and only if $(\Delta^m f)(0) = 0$ for all $m > d$.

<u>Proof</u> Note that $f(n) = (S^n f)(0)$.
If $(\Delta^m f)(0) = 0$ for all $m > d$, then

$$f(n) = (S^n f)(0) = ((I+\Delta)^n f)(0)$$

$$= \sum_{m=0}^{n} \binom{n}{m} (\Delta^m f)(0) = \sum_{m=0}^{d} \binom{n}{m} (\Delta^m f)(0)$$

which is a polynomial of degree $\le d$.

Conversely, if $f(n)$ is a polynomial of degree $\leq d$, then we can express it in terms of the (Δ)-basis.

$$f(n) = \sum_{m=0}^{d} \alpha_m \binom{n}{m},$$

for some $\alpha_1, \dots, \alpha_m \in \mathbb{C}$.

Now $$\boxed{\Delta \binom{n}{m} = \binom{n}{m-1}}$$ (Δ is some kind of derivative in this basis)

Hence, applying $\Delta^m f = 0$ for $m > d$. ∎

Note that the numbers $(\Delta^m f)(0)$ play a crucial role in the above proof. In fact, if we let

$$\boxed{f^{(m)} := (\Delta^m f)(0)}$$

then $$\boxed{f(n) = \sum_{m=0}^{d} f^{(m)} \binom{n}{m}} \qquad (*)$$

$f^{(m)}$ → coefficients of f expressed in the (Δ)-basis.

- Generating functions

The generating function of a sequence $f(n)$ is the formal power series:

$$\boxed{F(z) := \sum_{n \geq 0} f(n) z^n}$$

Example Let $m \in \mathbb{N}$ fixed and $f(n) = \binom{n}{m}$.

$$F(z) = \sum_{n\geq 0} \binom{n}{m} z^n = \sum_{n \geq m} \binom{n}{m} z^n = \frac{1}{m!} z^m \sum_{n \geq m} n(n-1)\cdots(n-m+1) z^{n-m}$$

$$= \frac{1}{m!} z^m \left(\frac{d}{dz}\right)^m (1 + z + z^2 + \dots)$$

$$= \frac{1}{m!} z^m \left(\frac{d}{dz}\right)^m \frac{1}{1-z}$$

$$= \frac{z^m}{(1-z)^{m+1}}$$

Using this and $(*)$ above we can derive the generating function of every polynomial $f(n)$.

Let $f(n) = \sum_{m=0}^{d} f^{(m)} \binom{n}{m}$ be a polynomial written in the (Δ)-basis.

Its generating function is:

$$F(z) = \sum_{n\geq 0} f(n) z^n = \sum_{m=0}^{d} f^{(m)} \sum_{n\geq 0} \binom{n}{m} z^n$$

$$\boxed{F(z) = \sum_{m=0}^{d} f^{(m)} \frac{z^m}{(1-z)^{m+1}}}$$

$$\boxed{F(z) = \frac{\sum_{m=0}^{d} f^{(m)} z^m (1-z)^{d-m}}{(1-z)^{d+1}}} \qquad (**)$$

This implies the following proposition.

Proposition A sequence $f(n)$ is given by a polynomial of degree $\leq d$ if and only if

$$\sum_{n\geq 0} f(n) z^n = \frac{h(z)}{(1-z)^{d+1}}$$

for some polynomial $h(z)$ of degree $\leq d$. Furthermore, $f(n)$ has degree d if and only if $h(1) \neq 0$.

Proof The first part follows from $(**)$ and by writing $f(n)$ in terms of the (Δ)-basis and $h(z)$ in terms of the (S)-basis. The second part follows from the fact that $f(n)$ has degree d $\Leftrightarrow f^{(d)} \neq 0 \Leftrightarrow h(1) \neq 0$. ∎

This proposition implies that the sequence $f(n)$ satisfies a recurrence relation. To see this we compute

$$h(z) = (1-z)^{d+1} \sum_{n\geq 0} f(n) z^n = \sum_{n\geq 0} f(n) \sum_{j=0}^{d+1} \binom{d+1}{j} (-1)^j z^{n+j}$$

Since $h(z)$ has degree $\leq d$, then the coefficient of z^n for $n \geq d$ on the right-hand side must be zero:

$$\sum_{j=0}^{d+1} f(n-j) \binom{d+1}{j} (-1)^j = 0$$

for all $n \geq d+1$.

On the other hand, we can find an explicit formula for the generating function of any sequence satisfying a recurrence relation.

Proposition Let $(f(n))_{n\geq 0}$ be a sequence of numbers. The following two statements are equivalent:

(1) $(f(n))_{n\geq 0}$ satisfies a recurrence relation

$$c_0 f(n+d) + c_1 f(n+d-1) + \dots + c_d f(n) = 0$$

for some $c_0, c_1, \dots, c_d \in \mathbb{C}$ with $c_0, c_d \neq 0$

(2) The generating function

$$F(z) = \sum_{n\geq 0} f(n) z^n = \frac{P(z)}{c_0 + c_1 z + \dots + c_d z^d}$$

for some polynomial $P(z)$ of degree $< d$.

Proof exercise

Example: Fibonacci numbers $f(0)=0$, $f(1)=1$ and

$$f(n+2) - f(n+1) - f(n) = 0$$

The proposition states that

$$F(z) = \sum_{n\geq 0} f(n) z^n = \frac{P(z)}{1-z-z^2}$$

for some polynomial $P(z)$ of degree < 2. This $P(z)$ is determined by the initial values of the sequence:

$$P(z) = (1-z-z^2)\,F(z) = \begin{array}{r} 0 + 1z + \cancel{1z^2} + \cancel{\dots} \\ - 0z - \cancel{1z^2} - \cancel{1z^3} - \cancel{\dots} \\ - \cancel{0z^2} - \cancel{1z^3} - \cancel{1z^4} - \cancel{\dots} \\ \hline 0 + 1z \end{array}$$

$\Rightarrow$

$$F(z) = \sum_{n\geq 0} f(n) z^n = \frac{z}{1-z-z^2}$$

The general case is not more complicated than this.

A generating function, such as in the above proposition, that can be expressed as a proper rational function

$$F(z) = \sum_{n\geq 0} f(n) z^n = \frac{P(z)}{q(z)}$$

of two polynomials with $\deg P(z) < \deg q(z)$ is called rational.
Note that in this case $(\gcd(P(z), q(z)) = 1)$ implies $q(0) \neq 0$.