Exercise 1

For $m, n \in \mathbb{N}$, show that the q-binomial number is

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \sum q^{\text{area}(\pi)},$$

where the sum runs over all lattice paths using north and east step from $(0,0)$ to (m,n), and $\text{area}(\pi)$ is the number of boxes below the path. Hint: show first that

$$[m+n]_q = [n]_q + q^n [m]_q.$$

Exercise 2

Show that the first q-analog of the Catalan number

$$\frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

is a polynomial in q.

Exercise 3

The space of harmonics is defined as

$$H_n := \left\{ h \in \mathbb{Q}[x_1, \ldots, x_n] : \sum_{i=1}^{n} \frac{\partial^k}{(\partial x_i)^k} h = 0, \text{ for } k \geq 1 \right\}.$$

(i) Explicitly compute H_1, H_2 and H_3.

(ii) Show that the Vandermonde determinant $\prod_{1 \leq i < j \leq n} (x_i - x_j)$ belongs to H_n.
Hint: Use part (i) of Exercise 4, and the fact that every non-zero antisymmetric polynomial is divisible by the Vandermonde determinant (the "smallest" non-zero antisymmetric polynomial).

(iii) Show that H_n is the span of the Vandermonde determinant $\prod_{1 \leq i < j \leq n} (x_i - x_j)$ and all its partial derivatives of all orders.

(iv) Show that
$$\dim(H_n) = n!$$

Exercise 4 (Optional)

The symmetric group $\mathfrak{S}_n$ acts on a polynomial $h \in \mathbb{Q}[x_1, \ldots, x_n]$ by

$$(\sigma \cdot h)(x_1, \ldots, x_n) = h(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

For $f \in Q[x_1, \ldots, x_n]$, we denote by ∂f the partial differential operator $\partial f = f(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$.

(i) Show that
$$(\partial(\sigma \cdot f))\,(\sigma \cdot h) = \sigma \cdot (\partial(f)(h))\,.$$

(ii) Show that H_n is closed under the action of the symmetric group.

(iii) Show that $\mathfrak{S}_n$-action on H_n is isomorphic to the regular representation of $\mathfrak{S}_n$.