Exercise 1

Show that the number of (a,b)-cores is finite if and only if a and b are relatively prime.

Exercise 2

Let a, b be relatively prime. Show that:

(i) The number of (a,b)-Dyck paths is equal to $\frac{1}{a+b}\binom{a+b}{a}$.

(ii) The expression $\frac{1}{[a+b]_q}\begin{bmatrix} a+b \\ a \end{bmatrix}_q$ is a polynomial in q.

Hint: use $[a]_q\begin{bmatrix} a+b \\ a \end{bmatrix}_q = [a+b]_q\begin{bmatrix} a+b-1 \\ a-1 \end{bmatrix}_q$ and that $[a]_q$ and $[a+b]_q$ have no roots in common.

Exercise 3

Show that conjugation on (a,b)-cores corresponds to conjugation on (a,b)-Dyck paths under Anderson's bijection.

Exercise 4

Verify the following equalities of $a = 3$ and $b = 5$ (compute explicitly). The sums run over all (a,b)-cores λ.

(i) $\frac{1}{[a+b]_q}\begin{bmatrix} a+b \\ a \end{bmatrix}_q = \sum_\lambda q^{\ell(\lambda)+sl(\lambda)}$

(ii) $\sum_\lambda q^{\ell(\lambda)}t^{sl'(\lambda)} = \sum_\lambda t^{\ell(\lambda)}q^{sl'(\lambda)}$

Is there a simple proof of (i)?
Finding a combinatorial proof of the q,t-symmetry (ii) is an open problem.