Exercise 1

Let $a, b \in \mathbb{N}$ be relatively prime. The rational q, t-Catalan polynomial is defined as

$$c_{a,b}(q,t) = \sum q^{\mathrm{area}(\pi)} t^{\mathrm{area}(\zeta(\pi))},$$

where the sum is over the (a, b)-Dyck paths π, the area $\mathrm{area}(\pi)$ is the number of boxes between the diagonal and the path, and ζ is the rational zeta map. Do the following for the particular case $a = 3$ and $b = 5$:

(i) Compute $c_{a,b}(q,t)$

(ii) Verify that $c_{a,b}(q,t)$ is symmetric on q and t

(iii) Verify that

$$q^{\frac{(a-1)(b-1)}{2}} c_{a,b}(q, q^-1) = \frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ a \end{bmatrix}_q$$

Exercise 2

Let π be an (a, b)-Dyck path and η be the eta map defined in Lecture 6. Show that $\eta(\pi)$ is an (a, b)-Dyck path.

Exercise 3

Let π be an (a, b)-Dyck path and π^c be its conjugate. Show that $\eta(\pi) = \zeta(\pi^c)$.

Exercise 4

(i) Let $a = n + 1$ and $b = n$ (the classical Catalan case). Show that the partition bounded above $\eta(\pi)$ is the complement of the partition bounded by $\zeta(\pi)$.

(ii) What happens in the Fuss-Catalan case $a = mn + 1$ and $b = n$?