Exercise 1

Let B_n be the Boolean poset of subsets of $[n]$ ordered by inclusion. Show that for $S \subseteq T$ in B_n we have

$$\mu(S,T) = (-1)^{|S \smallsetminus T|}.$$

Exercise 2

Let D_n be the poset of divisors of n ordered by divisibility. Show that if $s,t \in D_n$ such that s divides t then

$$\mu(s,t) = \begin{cases} (-1)^t & \text{if } s/t \text{ is a product of } k \text{ distinct primes} \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 3

Let P be a finite poset with a minimal element $\hat{0}$ and a maximal element $\hat{1}$. Let c_k be the number of chains $\hat{0} = t_0 < t_1 < \cdots < t_k = \hat{1}$ of length k between $\hat{0}$ and $\hat{1}$. (In particular, $c_0 = 0$ if $\hat{0} \neq \hat{1}$ and $c_1 = 1$.) Show that

$$\mu_P(\hat{0},\hat{1}) = c_0 - c_1 + c_2 - c_3 + \ldots.$$

Hint: prove and use the following expression for μ:

$$\mu = 1 - (\zeta - 1) + (\zeta - 1)^2 - (\zeta - 1)^3 + \ldots,$$

where $1 = \delta$ is the identify element of the incident algebra.

Exercise 4

For $n \geq 2$, define $Z(P,n)$ to be the number of multi-chains $t_1 \leq t_2 \leq \cdots \leq t_{n-1}$ in P. Show that:

(i) $Z(P,n)$ is a polynomial in n. (This is called the *zeta polynomial* of P.)

(ii) If P has a $\hat{0}$ and a $\hat{1}$, then $Z(P,n) = \zeta^n(\hat{0},\hat{1})$. In particular,

$$Z(P,-1) = \mu(\hat{0},\hat{1}),\ \ Z(P,0) = 0 \text{ if } \hat{0} \neq \hat{1},\ \text{ and } Z(P,1) = 1.$$