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Topics in Combinatorics, Algebra and Geometry
Exercise Sheet 08

Winter Semester
2023-2024

Exercise 1

A poset P is said to be *graded of rank* n if every maximal chain of P has the same length n. In this case, the *rank function* $\rho : P \to \{0, 1, \dots, n\}$ is the unique function satisfying $\rho(s) = 0$ if s is a minimal element of P and $\rho(t) = \rho(s) + 1$ if t covers s in P ($s \lessdot t$).
Let P be a finite graded poset of rank n with $\hat{0}$. The *characteristic polynomial* $\chi_P(t)$ of P is defined as

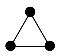
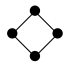
$$\chi_P(x) = \sum_{t \in P} \mu(\hat{0}, t) x^{n-\rho(t)}.$$

(i) Let B_n be the Boolean poset of subsets of $[n]$ ordered by inclusion. Show that $\chi_{B_n}(x) = (x-1)^n$.

Exercise 2

Let G be a simple graph (without loops or double edges) with vertex set V and edge set $E \subseteq \binom{V}{2}$. A *proper n-coloring* of G is a function $f : V \to [n]$ such that $f(a) \neq f(b)$ if $\{a, b\} \in E$. Let $\chi_G(n)$ be the number of proper n-coloring of G. The function is $\chi_G : \mathbb{N} \to \mathbb{N}$ is called the *chromatic polynomial* of G.

(i) Compute the chromatic polynomial χ_G for the following graphs:

$G =$  $G =$ 

Exercise 3

Let G be a simple graph with vertex set V. A set $A \subseteq V$ is *connected* if the induced subgraph on A is connected. Let L_G be the poset of all partitions π of V ordered by refinement, such that every block of V is connected.

(i) Show that the chromatic polynomial of G can be computed as

$$\chi_G(n) = \sum_{\pi \in L_G} \mu(\hat{0}, \pi) n^{\#\pi},$$

where $\#\pi$ is the number of blocks of π and μ is the Möbius functions of L_G.

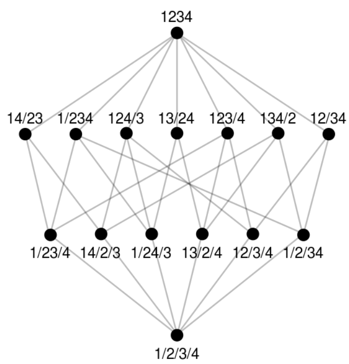
(ii) Show that the chromatic polynomial $\chi_G(n)$ and the characteristic polynomial $\chi_{L_G}(n)$ are related by

$$\chi_G(n) = n^c \chi_{L_G}(n),$$

where c is the number of connected components of G.

Exercise 4

Let P_n be the lattice of partitions of $[n]$ ordered by refinement.



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(i) Show that the characteristic polynomial of P_n is $\chi_{P_n}(x) = (x-1)(x-2)\dots(x-n+1)$.

(ii) Show that $\mu_{P_n}(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!$