Course - Topics on combinatorics, algebra and geometry
13.10.2023
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Lecture 1

Today : • Three combinatorial sequences :
- factorial numbers $n!$
- catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$
- parking functions $(n+1)^{n-1}$

• Enumeration, combinatorial models and bijections.

• Factorial numbers

Sequence : $1, 2, 6, 24, 120, 720, \dots$

Formula : $n!$

Counts the number of permutations of $[n] = \{1, 2, \dots, n\}$.

Example : $n=3$ $n! = 3\cdot 2\cdot 1 = 6$

123 132 213 231 312 321

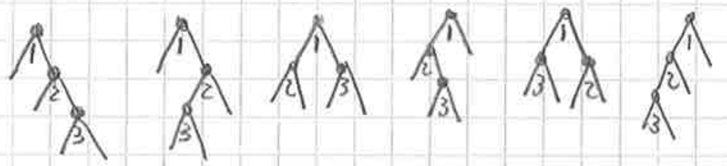
Another combinatorial model :

increasing rooted plane binary trees with n internal nodes :

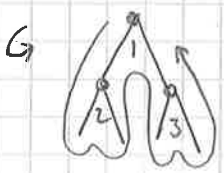
each internal node has exactly two children, left and right

the tree has one root on top.

internal nodes are labeled $1, 2, \dots, n$ increasing when walking away from the root.



Bijection • from increasing rooted binary trees to permutations



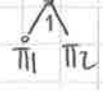
read labels of the corners in the order they appear

213

• From permutations to inc. r. binary trees :

$\pi = 213$ → 

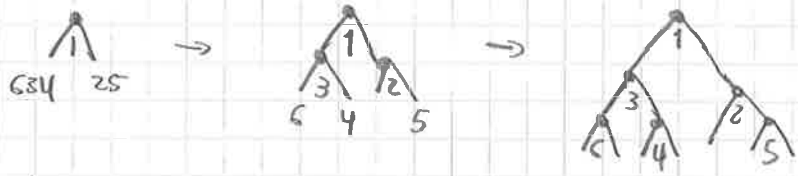
the minimal number of $\pi = \pi_1 \, 1 \, \pi_2$ splits the permutation in two



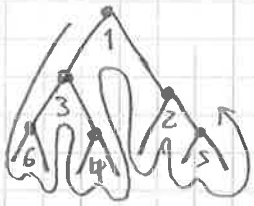
→ repeat the process

A bigger example:

- $\pi = 634125$



- Viceversa



$\rightarrow$ 634125

Catalan numbers

Sequence : 1, 2, 5, 14, 42, 132, ...

Formula : $C_n = \frac{1}{n+1}\binom{2n}{n}$

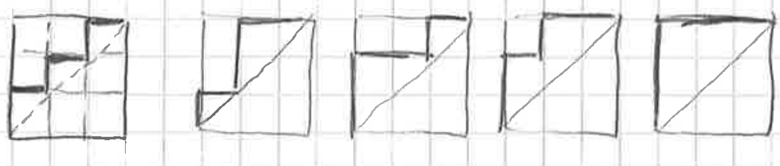
Count many different combinatorial objects.

Here is a small selection

1) Rooted plane binary trees with n internal nodes

$\boxed{n=3}$

2) Dyck paths inside an nxn rectangle. (lattice paths from (0,0) to (n,n) that stay weakly above the diagonal



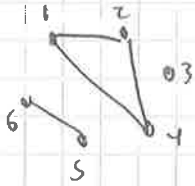
3) 312-avoiding permutations (Permutation with no ... c ... a ... b ... pattern)
a<b<c

123 132 213 231 321

4) Non-crossing partitions of $[n]$

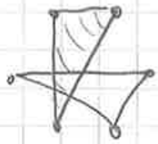
partition into disjoint blocks $[n] = \{B_1, \ldots, B_k\}$

such that there is no $a<b<c<d$ with $a,c \in B_i$ and $b,d \in B_j$ $i \neq j$



124, 3, 56

Non-crossing

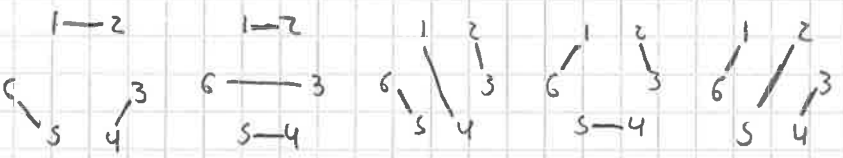


crossing.

$\boxed{n=3}$

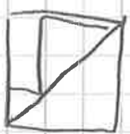
5) (Complete) non-crossing perfect matchings of $[2n]$

crossing. X



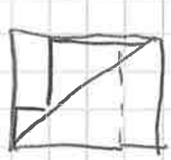
Enumeration formula:

Dyck paths from $(0,0)$ to (n,n) $\overset{\text{bijection}}{\longleftrightarrow}$ "extended" Dyck paths from $(0,0)$ to $(n+1,n)$



NENNEE

add one east step E



NENNEE(E)

- There are $\binom{2n+1}{n+1}$ lattice paths from $(0,0)$ to $(n+1,n)$

  from $2n+1$ steps choose $n+1$ which are E

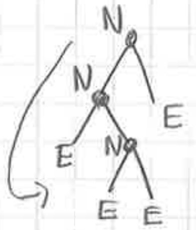
- We can rotate the last letters to the beginning. This creates orbits of $2n+1$ lattice paths. Among those there is a unique one that is above the diagonal

- Thus we have $\#$extended Dyck paths $= \frac{1}{2n+1}\binom{2n+1}{n+1} = \frac{1}{n+1}\binom{2n}{n}$ ∎

Bijections: from rooted planar binary trees to the other families.

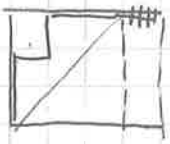
preorder:  rAB

1 → 2 To Dyck paths



label internal nodes N
leaves E

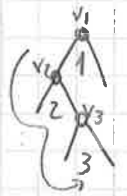
read labels of nodes → in preorder



NNENEEE

then remove last E

1 → 3 To 312-avoiding permutations.



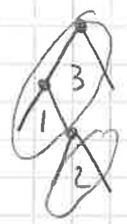
label internal nodes $1, \dots, n$
in preorder, and use
same label for their caverns

Then read caverns in preorder

read caverns → in preorder

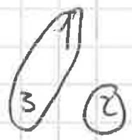
2 3 1

1 → 4 To non-crossing partitions.

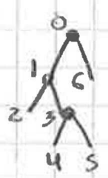


label caverns in preorder

Group labels of left descendants →



1 → 5 To complete non-crossing perfect matchings.



label nodes $0, 1, \dots, 2n$
in preorder (root = 0)

match the two children of → each internal node

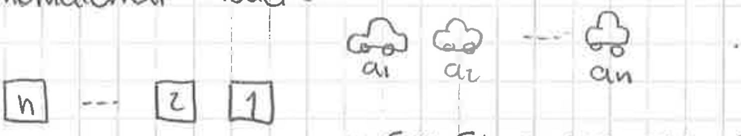
Exercise: show that all of these maps are bijections.

• Parking functions

Sequence : 1, 3, 16, 125, 1296, 16807, ...

Formula : $(n+1)^{n-1}$

Combinatorial model :



- Car C_i prefers space a_i
- If a_i is occupied, then C_i takes the next available space
- the tuple $(a_1, \ldots, a_n)$ is called a parking functio of length n is all cars can park

$\boxed{n=2}$	11	12	21	~~22~~				
$\boxed{n=3}$	111	112	121	211	113	131	311	122
	212	221	123	132	213	231	312	321

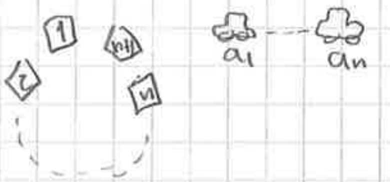
Let $b_1 \leq b_2 \leq \ldots \leq b_n$ be the increasing re-arrangement of $(a_1, \ldots, a_n)$

Fact: $(a_1, \ldots, a_n)$ is a parking function iff $b_i \leq i$

Exercise: • Prove this fact.
• Prove that parking functions are closed under the action of the symmetric group.

Enumeration: The number of parking functions of length n is $(n+1)^{n-1}$

Proof: Add extra space n+1 and arrange them in a circle. Allow n+1 as a preferred space.



In this escenario, all cars can park and there is one empty space

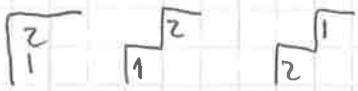
$(a_1, \ldots, a_n)$ is a parking function iff the empty space is n+1

Now $(a_1+j, \ldots, a_n+j)$ leads to a rotation of the final parking configuration and among the n+1 option in one orbit there is exactly one with n+1 being the empty space.

Since there are $(n+1)^n$ possibilities for $(a_1, \ldots, a_n)$

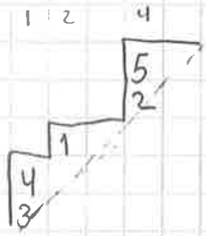
then the # parking functions $= (n+1)^n/(n+1) = (n+1)^{n-1}$ ∎

Another combinatorial model: labeled Dyck paths



label the north steps $1, 2, \dots, n$ such that the labels along columns increase

<u>Bijection</u>: from labeled Dyck paths to parking functions



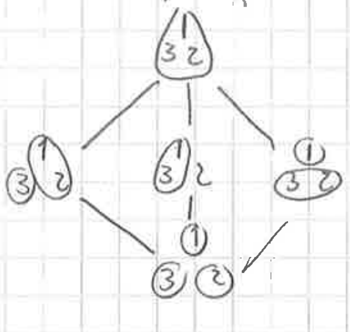
$\xrightarrow[\text{of label } i]{a_i = \text{column}}$ $(2, 4, 1, 1, 4)$

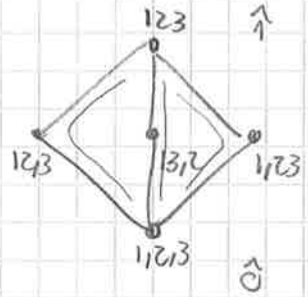
<u>Exercise</u>: show that this is a bijection.

Potential Final Projects :

Project A The lattice of non-crossing partitions NC_{n+1} of $[n+1]$

order partitions by refinement (gluing two blocks are covering relations)





① Fact, Kreweras 1972, Edelman 1980, ..

The number of maximal chains in NC_{n+1} is $(n+1)^{n-1}$

In our example: 3 maximal chains $= 3^1$

② Fact, Athanasiadis, Brady, Watt. ...

The reduced Euler characteristic of $NC_{n+1} \setminus \{\hat{0},\hat{1}\}$ is the catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ / Kreweras? Möbius function of NC_{n+1} $\mu(\hat{0},\hat{1})$

In our example:

$NC_3 \setminus \{\hat{0},\hat{1}\}$ $\cdot \quad \cdot \quad \cdot$

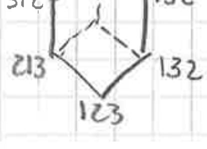
Its chain complex is just three disjoint vertices

Reduced Euler characteristic $= 3-1 = 2 = C_2$

A similar result known for Fuß-Catalan version, Armstrong Armstrong-Krattenthaler

③ Extensions in the context of signature Catalan combinatorics? Tom 'C

Project B Volume of permutahedra and parking functions, Postnikov 2005



volume $= 3 = (n+1)^{n-1}$ for $n=2$