Course - Topics on combinatorics, algebra and geometry
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Lecture 2

Today : q- analogs
the space of harmonics.

- **q- analogs**

Given $n \in \mathbb{N}$, define $[n]_q = 1+q+q^2+\dots+q^{n-1}$

Evaluating $[n]_q$ at $q=1$ we recover the number n.

Using this we can define q-analogs of various combinatorial sequences

- factorial numbers
- Catalan numbers
- Parking functions.

• The q-factorial number is

$$[n]_q\,! := [1]_q\,[2]_q \dots [n]_q$$

Example $n=3$:

$$[3]_q\,! = [1]_q\,[2]_q\,[3]_q = 1\cdot(1+q)\cdot(1+q+q^2)$$
$$= 1+2q+2q^2+q^3$$

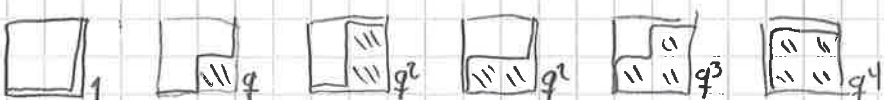
• For $n \geq k \in \mathbb{N}$, the q-binomial number is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q!\,[k]_q!}$$ Is it a polynomial ?

Example $n=4,\ k=2$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[4]_q!}{[2]_q!\,[2]_q!} = \frac{[4]_q[3]_q}{[2]_q[1]_q} = \frac{\overbrace{\cancel{(1+q+q^2+q^3)}}^{1+q^2}(1+q+q^2)}{\cancel{(1+q)}}$$

$$= (1+q^2)(1+q+q^2) = 1+q+2q^2+q^3+q^4$$



Hint: $[m+n]_q = [n]_q + q^n[m]_q$

Exercise Show that $\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \sum q^{\text{area}(\pi)}$ where the sum runs over all lattice paths from $(0,0)$ to (m,n).

• Two q-analogs of the Catalan numbers and parking functions

First q-analogs:

• The $\overset{\text{first}}{q}$-Catalan number is

$$\frac{1}{[n+1]_q}\begin{bmatrix}2n\\ n\end{bmatrix}_q$$

Is it a polynomial? If so, what is it counting?

Example $n=2$

$$\frac{1}{[3]_q}\begin{bmatrix}4\\ 2\end{bmatrix}_q = \frac{[4]_q}{[2]_q[1]_q} = \frac{1+q+q^2+q^3}{1+q} = \boxed{1+q^2}$$

Counting Dyck paths? which weigths?

Example $n=3$

$$\frac{1}{[4]_q}\begin{bmatrix}6\\ 3\end{bmatrix}_q = \frac{\overset{1+q^3}{\cancel{[6]_q}}[5]_q}{[3]_q[2]_q} = \frac{\overset{1-q+q^2}{(1+q^3)}(1+q+q^2+q^3+q^4)}{\cancel{(1+q)}}$$

$$= \boxed{1+q^2+q^3+q^4+q^6}$$

What are the statistics/exponents?

• The $\overset{\text{first}}{q}$-parking function number is

$$[n+1]_q^{\,n-1}$$

Example $n=2$

$$\boxed{[3]_q^{\,1} = 1+q+q^2}$$

Counting weighted parking functions?

Example $n=3$

$$\boxed{[4]_q^{\,2} = (1+q+q^2+q^3)^2 = 1+2q+3q^2+4q^3+3q^4+2q^5+q^6}$$

Second q-analogs

- The <u>second q-Catalan number</u> is

$$C_n(q) := \sum_{\pi \in \mathrm{Dyck}(n)} q^{\mathrm{area}(\pi)}$$

$\mathrm{area}(\pi) :=$ # boxes between the main diagonal and π.

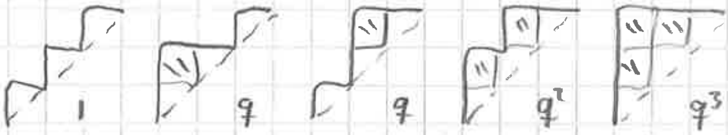
<u>Example</u> $n=2$

$1 \qquad q$

$$\boxed{C_2(q) = 1+q}$$

(different to what we got before)

$n=3$



$1 \qquad q \qquad q \qquad q^2 \qquad q^3$

$$\boxed{C_3(q) = 1+2q+q^2+q^3}$$

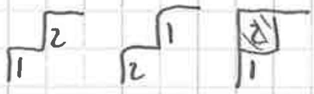
- The <u>second q-parking function number</u> is

$$\mathrm{Park}_n(q) := \sum_{P \in \mathrm{Park}(n)} q^{\mathrm{area}(P)}$$

$P \qquad \mathrm{area}(P) = 2$

$\mathrm{area}(P) :=$ area of the Dyck path supporting P

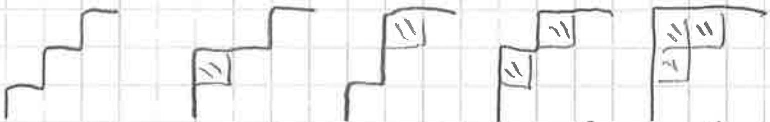
Example $n=2$



$1 \qquad 1 \qquad q$

$$\boxed{\mathrm{Park}_2(q) = 2+q}$$

$n=3$



$6 \qquad 3q \qquad 3q \qquad 3q^2 \qquad q^3$

$$\boxed{\begin{aligned}\mathrm{Park}_3(q) &= 6+3q+3q+3q^2+q^3 \\ &= 6+6q+3q^2+q^3\end{aligned}}$$

<u>Goal of this (and coming) lecture</u>: introduce q,t-analogs that generalize the previous q-analogs using diagonal harmonics.

Ingredients: polynomial rings
representation theory of the symmetric group.

- The space of harmonics

Let $\mathbb{Q}[x_1, x_2, \dots, x_n]$ be the ring of polynomial in n variables $x_1, \dots, x_n$.
A polynomial $f \in \mathbb{Q}[x_1, \dots, x_n]$ is called symmetric if

$$f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

for every permutation σ of $[n]$

Example: $n=2$ $\quad f(x_1,x_2) = x_1 + x_2$ $\quad f(x_1,x_2) = x_1^2 + x_2$

We denote by I the ideal generated by symmetric polynomial with no constant term.

The space of harmonics is

$$\boxed{H_n := \{ h \in \mathbb{Q}[x_1, \dots, x_n] : (\partial f)(h) = 0 \quad \forall f \in I \}}$$

Where ∂f is the partial differential operator

$$\partial f := f\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$$

Example

$$f(x_1, x_2) = x_1 + x_2 \implies \partial f = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$$

Since I is generated by power-sum symmetric functions

$$p_k(x_1, \dots, x_n) = x_1^k + \dots + x_n^k$$

We alternatively have

$$\boxed{H_n = \left\{ h \in \mathbb{Q}[x_1, \dots, x_n] : \sum_{i=1}^{n} \frac{\partial^k}{(\partial x_i)^k} h = 0 \quad \text{for } k \geq 1 \right\}}$$

Example $n=2$

$$H_2 = \left\{ h \in \mathbb{Q}[x_1, x_2] : \left(\frac{\partial^k}{(\partial x_1)^k} + \frac{\partial^k}{(\partial x_2)^k}\right) h = 0 \quad \text{for } k \geq 1 \right\}$$

For instece, $\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) h = 0$

From this we can deduce that $\deg h \leq 1$ and
One can check that

$$H_2 = \mathrm{sp}\{1, x_1 - x_2\}$$
$$\dim H_2 = 2$$

In general:

Exercise Show that

i) $H_n = sp\{$ Vandermonde determinant $\prod_{1\le i<j\le n}(x_i - x_j)$ and all its partial derivatives of all orders $\}$

ii) $\boxed{\dim H_n = n!}$

iii) Explicitely compute H_3.

Moreover, H_n can be decomposed into homogeneous components.

$$\boxed{H_n = \bigoplus_{i=0}^{\binom{n}{2}} H_n^i}$$

Where H_n^i is the span of homogeneous polynomials of degree i in H_n

Example $n=2$

$$\begin{aligned} H_2 &= sp\{1\} \oplus sp\{x_1 - x_2\} \\ &= H_2^0 \oplus H_2^1 \end{aligned}$$

The q-Hilbert series of H_n is defined by

$$\mathrm{Hilb}_{H_n}(q) = \sum_{i=0}^{\binom{n}{2}} \dim(H_n^i)\, q^i$$

Example $n=2$

$$\mathrm{Hilb}_{H_2}(q) = 1 + q$$

Exercise Show that

$$\boxed{\mathrm{Hilb}_{H_n}(q) = [n]_q!}$$

Remark The symmetric group acts on $\mathbb{Q}[x_1, \dots, x_n]$ by permuting indices. $(\sigma \cdot h)(x_1, \dots, x_n) = h(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ Furthermore, since

$$(\partial(\sigma \cdot f))(\sigma \cdot h) = \sigma \cdot ((\partial f)(h))$$

We have that H_n is closed under this action.

Exercise (optional): Show that the action of the symmetric group on H_n gives the regular representation.