Course - Topics on combinatorics, algebra and geometry
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## Lecture 4

Last time : Diagonal harmonics $DH_n$
$q,t$-polynomials :

$\mathrm{Hilb}_{DH_n}(q,t)$ counts parking functions

$\mathrm{Hilb}_{Alt_n}(q,t)$ counts Catalan objects.

Today : Combinatorial model for these polynomials
$q,t$-Catalan combinatorics
area, dinv and bounce statistics
zeta map

### $q,t$-Catalan Combinatorics

Recall that $\boxed{\dim(Alt_n) = \frac{1}{n+1}\binom{2n}{n}}$ the $n$th Catalan number

$Alt_n$ can be decomposed into homogeneous components

$$Alt_n = \bigoplus_{i,j} Alt_n^{i,j}$$

degree $i$ in $x$ variables
degree $j$ in $y$ variables

We define the $q,t$-Catalan number as

$$\boxed{C_n(q,t) := \mathrm{Hilb}_{Alt_n}(q,t)}$$

Examples $n=2$ $\qquad Alt_2 = \mathrm{sp}\{\overset{q}{x_1 - x_2}, \overset{t}{y_1 - y_2}\}$

$$\boxed{C_2(q,t) = q+t}$$

$n=3$

$$\boxed{C_3(q,t) = q^3 + q^2t + qt + qt^2 + t^3}$$

We also know that

$$\boxed{C_n(q,1) = \sum_{\pi \in Dyck(n)} q^{area(\pi)}}$$

Therefore

$$\boxed{C_n(q,t) = \sum_{\pi \in Dyck(n)} q^{area(\pi)} t^{?}} \to$$ what is this counting?

It was a very challenging problem to find a combinatorial description of the second missing statist.

After intensive study of tables of $C_n(q,t)$ ;

Haglund discovered the <u>bounce</u> statistic. . Shortly after:

Haiman discovered <u>dinv</u>.

Surprisingly, these statistics were different, but gave rise to the same $q,t$-Catalan polynomial.

$$\boxed{C_n(q,t) = \sum_{\pi \in \mathrm{Dyck}(n)} q^{\mathrm{area}(\pi)} t^{\mathrm{dinv}(\pi)} \overset{(*)}{=} \sum_{\pi \in \mathrm{Dyck}(n)} q^{\mathrm{bounce}(\pi)} t^{\mathrm{area}(\pi)}}$$

conjectured at that point
now a theorem

Then, Haglund uncovered a bijection on Dyck paths, called the zeta-map, which sends

$$(\mathrm{area}, \mathrm{dinv}) \longrightarrow (\mathrm{bounce}, \mathrm{area})$$

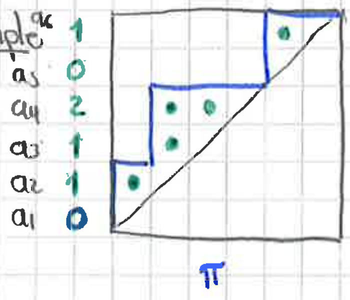
explaining this phenomenon.(*) The zeta map appeared before in the work of Andrews, Krattenthaler, Orsina and Papi.

- <u>The dinv statistic</u>

Let $\pi \in \mathrm{Dyck}(n)$ be a Dyck path of size $n$.

The area vector $a(\pi) = (a_1, \dots, a_n)$ is the vector whose $i$th entry $a_i$ counts the number of boxes in row $i$ below $\pi$ above the diagonal.

<u>Example</u>




$a(\pi) = (0,1,1,2,0,1)$

$= (a_1, a_2, a_3, a_4, a_5, a_6)$

$\boxed{\begin{array}{l} \mathrm{dinv}(\pi) = 7 \\ \mathrm{area}(\pi) = 5 \end{array}}$

The <u>dinv statistic</u> (# of diagonal inversions) is defined as

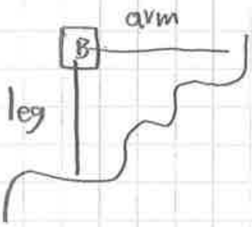
$$\boxed{\mathrm{dinv}(\pi) = \left| \left\{ (i,j) : \begin{array}{c} i<j \text{ and} \\ a_i = a_j \text{ or } a_i = a_j + 1 \end{array} \right\} \right|}$$

In our example, the diagonal inversions are: $(1,5), (2,3), (2,6), (3,6)$ and $(2,5), (3,5), (4,6)$

So $\mathrm{dinv}(\pi) = 7$

Alternatively, $\mathrm{dinv}(\pi)$ counts certain boxes above $\pi$.

If $B$ is a box above $\pi$ we define $\mathrm{arm}(B)$ and $\mathrm{leg}(B)$ as follows



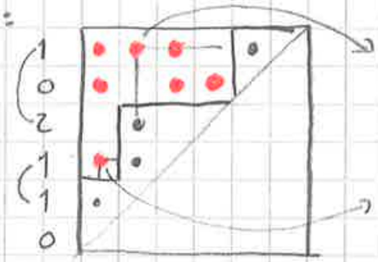

$\mathrm{arm}(B) = \#$ boxes between $B$ and $\pi$ in the same row

$\mathrm{leg}(B) = \#$ boxes between $B$ and $\pi$ in the same column

Then,

$$\boxed{\mathrm{dinv}(\pi) = \left| \left\{ B \text{ box above } \pi : \begin{array}{l} \mathrm{arm}(B) = \mathrm{leg}(B) \text{ or} \\ \mathrm{arm}(B) = \mathrm{leg}(B)+1 \end{array} \right\} \right|}$$

In our example:




Magically:

$$\boxed{C_n(q,t) = \sum_{\pi \in \mathrm{Dyck}(n)} q^{\mathrm{area}(\pi)} t^{\mathrm{dinv}(\pi)}}$$

Example $n=2$ $\qquad C_2(q,t) = q+t$

| | | |
|---|---|---|
| area | 1 | 0 |
| dinv | 0 | 1 |
| | $q$ | $t$ |

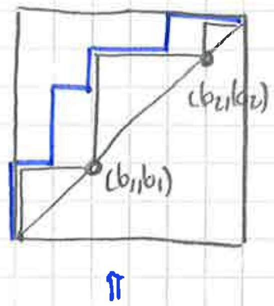
$n=3$ $\qquad C_3(q,t) = q^3 + q^2t + qt + qt^2 + t^3$

| | | | | | |
|---|---|---|---|---|---|
| area | 3 | 2 | 1 | 1 | 0 |
| dinv | 0 | 1 | 1 | 2 | 3 |
| | $q^3$ | $q^2t$ | $qt$ | $qt^2$ | $t^3$ |

## The bounce statistic

Let $\pi \in Dyck(n)$

The bounce path of $\pi$ is constructed as follows




- start at $(0,0)$ and move up until hiting the path, then move east to the diagonal
- Then move up until hiting the path again, and move east to the diagonal
- repeat the process until reaching $(n,n)$

Let $(b_1,b_1),\ldots,(b_k,b_k)$ with $0<b_1<b_2<\ldots<b_k<n$, be the diagonal points of the bounce path of $\pi$ in the interior of the square.

We define the bounce statist as

$$\boxed{\textbf{bounce} = \sum (n-b_i)}$$


Equivalently, labeled the diagonal points with numbers $0,1,2,\ldots,n$ from top to bottom

bounce = sum of the labels of the interior diagonal points of the bounce path.

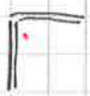
In our example

$$\boxed{\begin{array}{l} \text{bounce} = 1+4 = 5 \\ \text{area} = 7 \end{array}}$$

Magically

$$\boxed{C_n(q,t) = \sum_{\pi \in Dyck(n)} q^{\text{bounce}(\pi)} t^{\text{area}(\pi)}}$$

Example $n=2$ $\quad C_2(q,t) = q + t$

| | | |
|---|---|---|
| bounce | 1 | 0 |
| area | 0 | 1 |

Example n=3 $\quad C_3(q,t) = q^3 + q^2t + qt + qt^2 + t^3$

| | | | | | |
|---|---|---|---|---|---|
| bounce | 3 | 2 | 1 | 1 | 0 |
| area | 0 | 1 | 1 | 2 | 3 |
| | $q^3$ | $q^2t$ | $qt$ | $qt^2$ | $t^3$ |

The fact that

$$C_n(q,t) = \sum q^{\text{area}(\pi)} t^{\text{dinv}(\pi)} \overset{(*)}{=} \sum q^{\text{bounce}(\pi)} t^{\text{area}(\pi)}$$

took many years to be proved ; it follows from stronger results of Haiman and other formulations of $C_n(q,t)$ conjectured and proved by others ( long story ) . See Haglunds book for references .

As a corollary:
$$\frac{1}{[n+1]_q}\begin{bmatrix}2n\\n\end{bmatrix}_q = \sum q^{\text{area}(\pi)+\text{codinv}(\pi)}$$
where $\text{codinv} = \binom{n}{2} - \text{dinv}$

Ex: Verify this for n=3

The second equality (*) was proved by Haglund using the zeta map.

The polynomial $C_n(q,t)$ is symmetric in $q,t$ , and it is still a big open problem to find a combinatorial proof :

a bijection interchanging area $\leftrightarrow$ dinv or area $\leftrightarrow$ bounce .

Tentative Project C : The $n!$ and $(n+1)^{n-1}$ conjectures / theorems .

$\to$ to learn more about these topics .

- The zeta map

The zeta map is a bijection $\quad z : \text{Dyck}(n) \to \text{Dyck}(n)$

that send the pair of statistics (area, dinv) to (bounce, area) :

$$\boxed{\begin{aligned} \text{area}(\pi) &= \text{bounce}(z(\pi)) \\ \text{dinv}(\pi) &= \text{area}(z(\pi)) \end{aligned}}$$

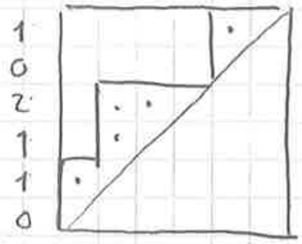
It is described as follows .

Let $\pi \in \text{Dyck}(n)$ and $a(\pi) = (a_1, \dots, a_n)$ be its area vector .

In order to describe $z(\pi)$ , we first construct its bounce path .

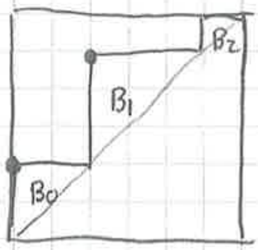
It consists of blocks $B_0, B_1, \dots, B_k$ where $k = \max\{a_i\text{'s}\}$ and the "size" of $B_i$ is the number of entries of $a(\pi)$ equal to $i$ .

<u>Example</u>




$\pi$

$a(\pi) = 0\ 1\ 1\ 2\ 0\ 1$




bounce path of $z(\pi)$

→ one 2

→ three 1's

→ two 0's

The next step is to construct $z(\pi)$, whose bounce path has blocks $B_0, \dots, B_k$

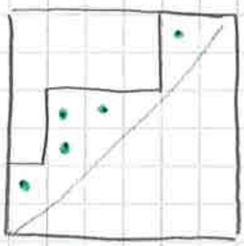
For $i = 0, 1, \dots, k-1$ we construct a path between the pick of $B_i$ and the pick of $B_{i+1}$ by replacing the entries of $a(\pi)$ as follows:

$i \to E$ (east step)

$i+1 \to N$

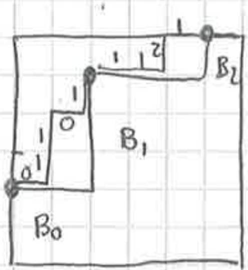
ignore all other entries

Example



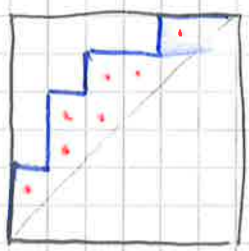
area = 5

dinv = 7




$B_0 B_1$ : 01101

$B_1 B_2$ : 1121



$z(\pi)$ bounce = 5

area = 7

$z(\pi)$ is the result of connecting all these paths.

**<u>Theorem</u>** (Haglund)

The zeta map is a bijection $z : \mathrm{Dyck}(n) \to \mathrm{Dyck}(n)$ satisfying

$$\mathrm{area}(\pi) = \mathrm{bounce}(z(\pi))$$
$$\mathrm{dinv}(\pi) = \mathrm{area}(z(\pi))$$

<u>Proof</u>
- $z(\pi)$ is a Dyck path by construction
- Since the first entry $i+1$ in $a(\pi)$ is preceded by $i$ then the bounce path of $z(\pi)$ consists of the blocks $B_0, \dots, B_k$
- The relative order of the entries $i$ and $i+1$ in $a(\pi)$ is enough to reconstruct $a(\pi)$, because $i+2$ can not be preceded by preceded by something $\leq i$
- Thus, $z$ is a bijection.
- $\mathrm{bounce}(z(\pi)) = \sum i \cdot \mathrm{size}(B_i) = \mathrm{area}(\pi)$
- $\mathrm{dinv}(\pi)$ has two kind of inversions $\begin{cases} (i,j) & : a_i = a_j \\ (i,j) & : a_i + 1 = a_j \end{cases}$ $i<j$

The first corresp. to boxes below bounce path of $z(\pi)$. The second to the other boxes below $z(\pi)$

Thus $\mathrm{dinv}(\pi) = \mathrm{area}(z(\pi))$