Course - Topics on combinatorics, algebra and geometry
1.12.2023
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Lecture 6

Last time : rational q,t-Catalan combinatorics

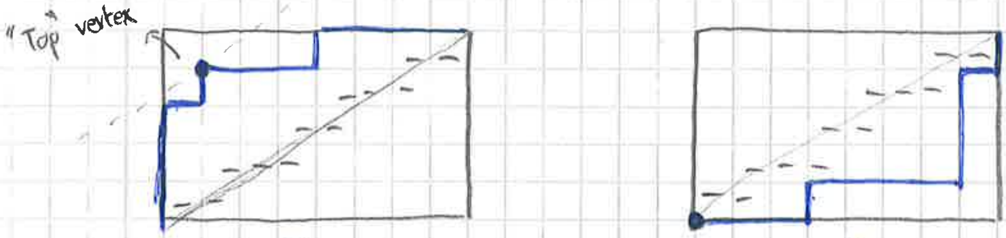
Today : rational zeta map.

- Rational Dyck paths

Let $a<b \in \mathbb{N}$ coprime. An (a,b)-Dyck path is a lattice path from $(0,0)$ to (b,a) using north (N) and east steps (E) that stays weakly above the main diagonal.

Given an (a,b)-Dyck path π, its conjugate π^c is the $180°$ rotation of the unique path below the diagonal obtained from π by cyclic shifts.

Example $a=5$, $b=8$



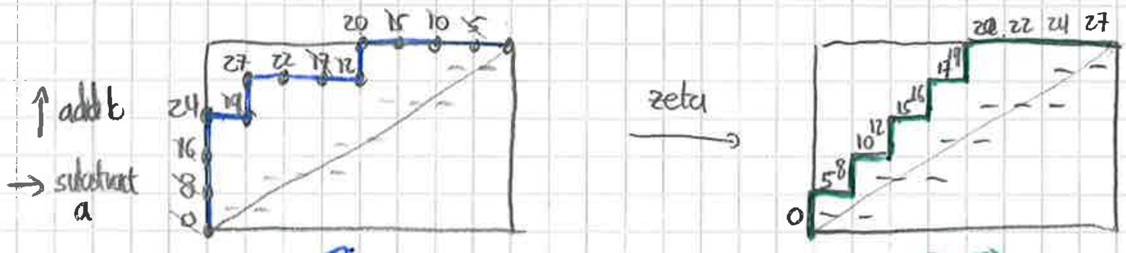
π π^c.

- The rational zeta map (Based on sweep maps of Armstrong - Loehr - Warrington '15)

The rational zeta map is a map

$$z : \begin{matrix}(a,b)\text{-Dyck} \\ \text{paths}\end{matrix} \longrightarrow \begin{matrix}(a,b)\text{-Dyck} \\ \text{paths}\end{matrix}$$
$$\pi \longrightarrow z(\pi)$$

The idea is to sweep the diagonal and take the N and E steps of π along the way.

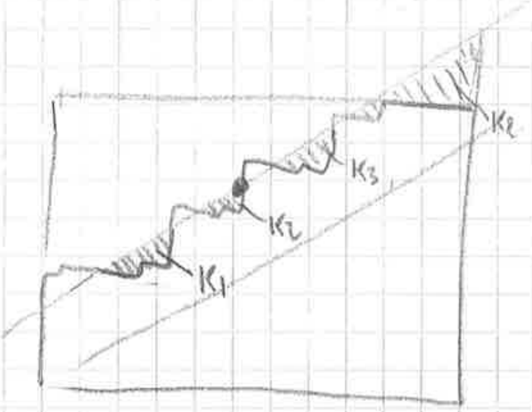


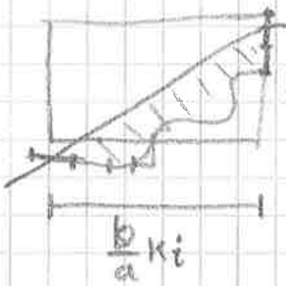
π $z(\pi)$

<u>Proposition</u> $Z(\pi)$ is an (a,b)-Dyck path ; i.e. it is weakly above the diagonal.

<u>Proof by picture</u>

We need to show that during the sweeping procedure, when we touch the starting point of the $(k+1)$th north step N, the number of east steps E that we have already touched is $\leq \frac{b}{a} k$





Consider the shaded pieces below the sweeped diagonal at that point.

Let $k_1, \dots, k_l$ the number of north steps in these pieces. So

$$k = k_1 + \dots + k_l .$$

Moreover, the number of starting points of east steps E in the k_i piece is at most

$$\frac{b}{a} k_i$$

Adding up, we get the result ∎

<u>Theorem</u> (Conjectured by Armstrong-Loehr-Warrington '15
Proved by Thomas-Williams '18)

The rational zeta map is a bijection on (a,b)-Dyck paths

Their conjecture/result is more general : works for "sweep maps"

The proof is algorithmic

Not known explicit description of the inverse is known!

Recall that in the classical Catalan case, $a = n+1, b = n$

The zeta map sends dinv → area → bounce.

The rational zeta map coincides with the classical zeta map in that case.
In general, one can define

$$\boxed{\operatorname{dinv}(\pi) = \operatorname{area}(Z(\pi))}$$

The rational Catalan polynomial is

$$\boxed{\begin{aligned} C_{a,b}(q,t) &= \sum_{\substack{\pi\ (a,b)\text{-Dyck} \\ \text{path}}} q^{area(\pi)}\, t^{dinv(\pi)} \\ &= \sum_{\pi} q^{area(\pi)}\, t^{area(\zeta(\pi))} \end{aligned}} \quad (*)$$

Interestingly: $\boxed{(1)\ C_{a,b}(q,t) \text{ is symmetric in } q \text{ and } t.}$ Not known combinatorial proof. (**)

(2) $$\boxed{q^{\frac{(a-1)(b-1)}{2}} C_{a,b}(q,q^{-1}) = \frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ a \end{bmatrix}_q} \quad (***)$$

Exercise Compute $C_{3,5}(q,t)$ using the zeta map definition (*) and verify that (**) (***) holds for this case.

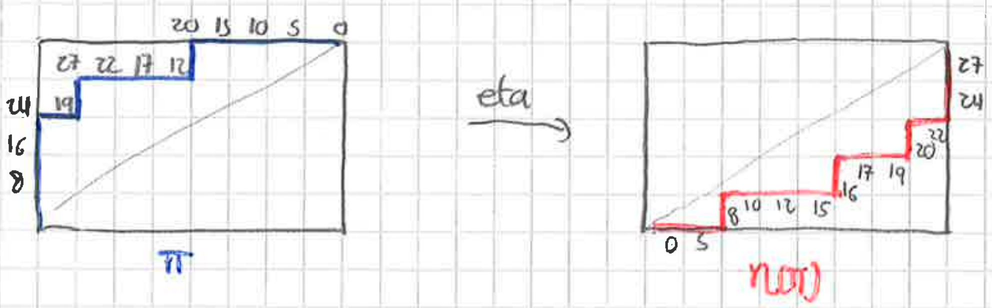
- The eta map

Similarly, we can define the eta map

$$\eta : (a,b)\text{-Dyck paths} \to (a,b)\text{-Dyck paths}$$

by sweeping the diagonal from top to bottom and recording the south S and west steps W that we encounter.

In our example



As before

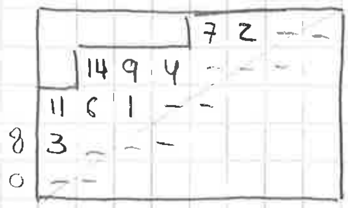
Prop η is a path below the diagonal.

Moreover $\boxed{\eta(\pi) = \zeta(\pi^c)}$

Exercise: show this

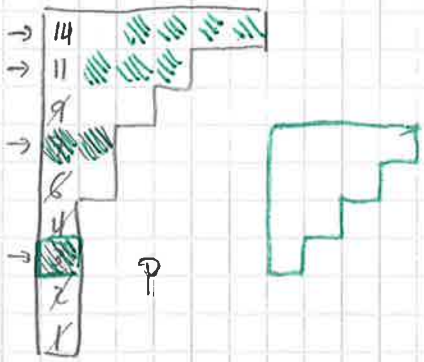
- Zeta and eta via (a,b)-cores

$a=5$
$b=8$



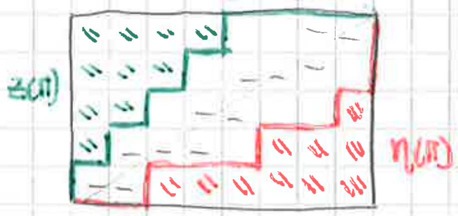
π

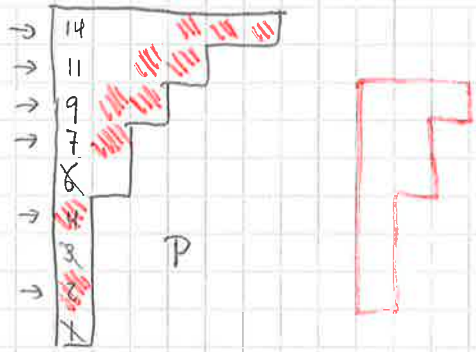
Inverse of Anderson's bijection



P

a-rows and b-boundary





P

b-rows and boundary

Let P be the (a,b)-core corresponding to an (a,b)-Dyck path π.

Let $\lambda(\pi) := \lambda(P)$ be the partition whose parts count the number of boxes in the a-rows of P that are in the b-boundary

Let $\mu(\pi) := \mu(P)$ be the partition whose parts counts the number of boxes in the b-rows of P that are in the a-boundary

Magic

- The partition bounded above $\zeta(\pi)$ equals $\lambda(\pi)$
- The partition bounded below $\eta(\pi)$ equals $\mu(\pi)^c$

Therefore

$\mathrm{dinv}(\pi) = \text{co-skewlength}(P)$

$$\text{coarea}(\zeta(\pi)) = \text{skew length}(P) =$$
$$\text{coarea}(\eta(\pi)) = \text{skew length}(P)$$

preserved by changing a & b.

Corollary The map sending $\zeta(\pi) \to \eta(\pi)$ is an area preserving involution

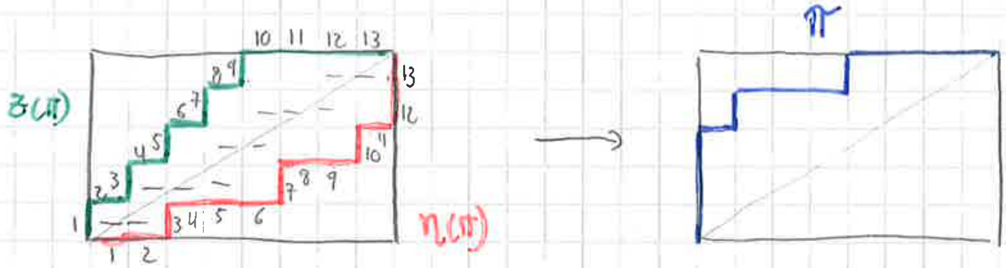
Potential Project

- Learn more about the rational zeta map and rational Catalan combinatorics,
- area preserving involutions
- generalizations of the zeta map in the context of Coxeter groups

and/or

- Zeta inverse from zeta and eta

Although no explicit description of the inverse of zeta is currently known, there is a nice way to reconstruct π from the pair $(\zeta(\pi), \eta(\pi))$



$\gamma =$ 1 3 7 12 9 13 11 8 5 10 6 4 2
N N N E N E E E N E E E E

Label steps in the two paths $1, 2, \dots, a+b$ as shown

Let $\gamma : [a+b] \to [a+b]$ be the permutation defined by

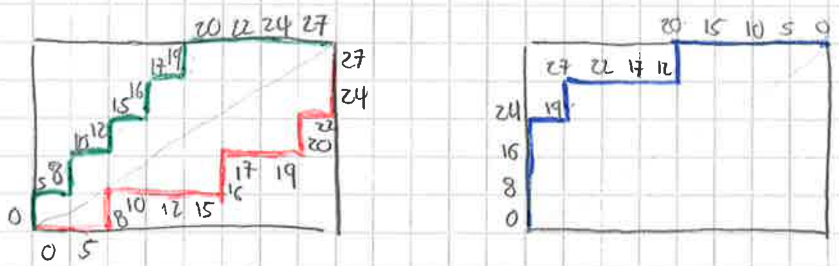
If i is a north step in $\zeta(\pi)$ $\to$ $\gamma(i)$ = label of the north step in $\eta(\pi)$ in the same row

If i is an east step in $\zeta(\pi)$ $\to$ $\gamma(i)$ = " " east " " " " " column

It turns out the γ is a cycle permutation!

Replacing the labels by the corresponding letter N or E in $\zeta(\pi)$ we recover the path π!

Reason



Put the originals labels back ("distance to the diagonal")

A label r in $\zeta(\pi)$ "sees" the label s in $\eta(\pi)$ if the lattice point labeled r in π is followed by s.

Open Problem

Find a description of the area preserving involution $\zeta(\pi) \to \eta(\pi)$

This would solve the problem of inverting the rational zeta map

• Application: The square case

Recall the zeta map in the classical square case $a = n+1$, $b = n$.

We already know z inverse using bounce paths.

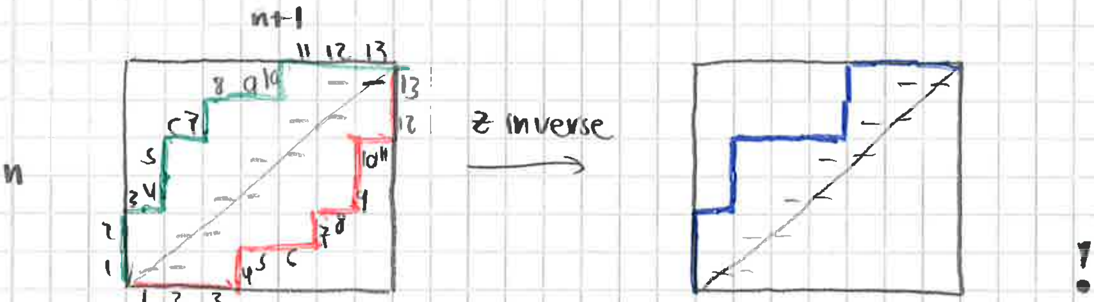
Alternative inverse: using area preserving involution.

Exercise: Show that the partition bounded above $\eta(\pi)$ is the complement of the partition bounded by $z(\pi)$

In our example.

The same
(just a coincidence)

The inverse



$\gamma =$