Course: Topics in combinatorics, algebra and geometry
26.1.2024
Cesar Ceballos

Lecture 10

Last time: Started with geometry: Polytopes
- Permutahedron.

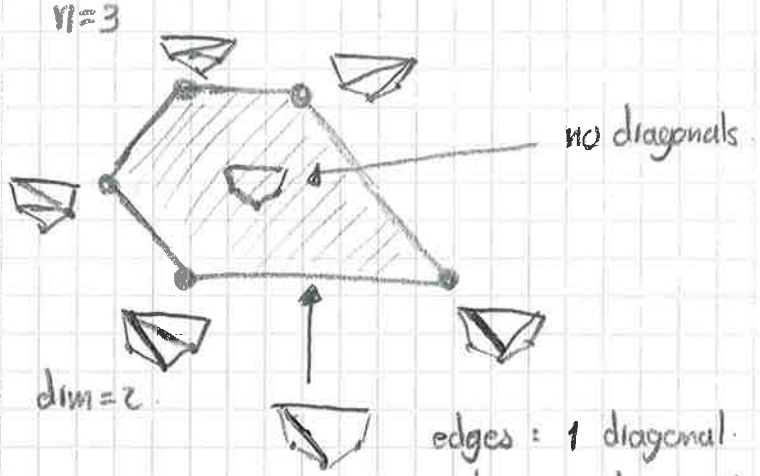
Today: Continue with polytopes:
- Associahedron
- Pitman-Stanley Polytope. (If time allows)

- The associahedron

Let $n \in \mathbb{N}$. The associahedron $\mathrm{Asso}(n)$ is a convex polytope whose

vertices : triangulation of a convex $(n+2)$-gon.
facets : diagonals. " " "
faces : subdivisions

Example $n = 2$ $n=3$



In general:

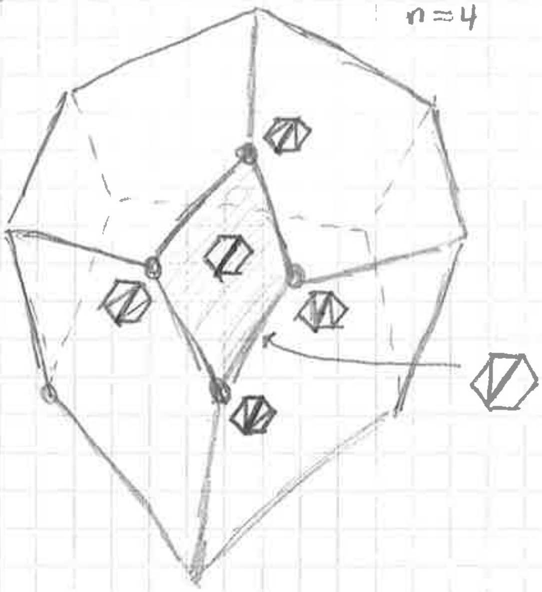
$$\boxed{\begin{array}{l} \text{dimension } \mathrm{Asso}(n) = n-1 \\ k\text{-dimensional face : subdivisions using } n-1-k \text{ diagonals} \end{array}}$$

Note that the previous definition of $\mathrm{Asso}(n)$ is purely combinatorial and it is apriori not clear how to obtain a geometric realization of it.

The associahedron has a long interesting hystory:

- Tamari '51 : 3D figure

n=4



- First polytopal geometric realizations :

  Haiman '84
  Lee '89
  ⋮

- Since then many different constructions emerged.

- The associahedron from the permutahedron by removing some facets (Shnider-Sternberg '93) Hohlweg-Lange '07 ....)

We can obtain Asso(n) as the set of points $(x_1, \dots, x_n) \in \mathbb{R}^n$ satisfying

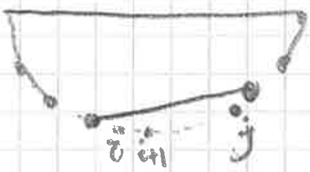
$$\boxed{x_1 + \dots + x_n = 1+2+\dots+n}$$

and one inequality for each diagonal of a convex $(n+2)$-gon :

n=3

$x_1 \geq 1$ $\qquad$ $x_2 \geq 1$ $\qquad$ $x_3 \geq 1$ $\qquad$ $x_1+x_2 \geq 1+2$ $\qquad$ $x_2+x_3 \geq 1+3$

In general



$$x_i + x_{i+1} + \dots + x_j \geq 1+2+\dots+|I|$$

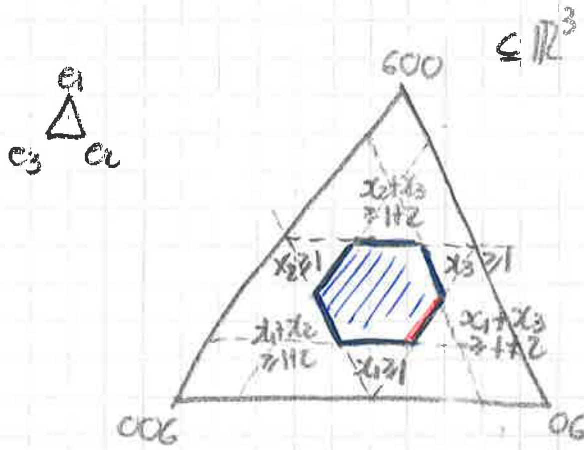
$I = [i,j]$ interval between i and j
$:= \{i, i+1, \dots, j\}$

For every interval $I \subseteq [n]$:

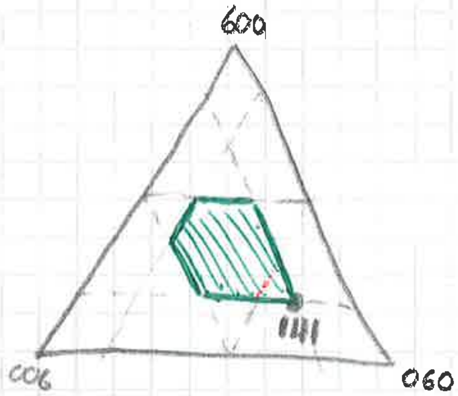
$$\boxed{\sum_{k \in I} x_k \geq 1 + \dots + |I|}$$

This is a subset of inequalities of the defining inequalities of the permutahedron!

$\subseteq \mathbb{R}^3$



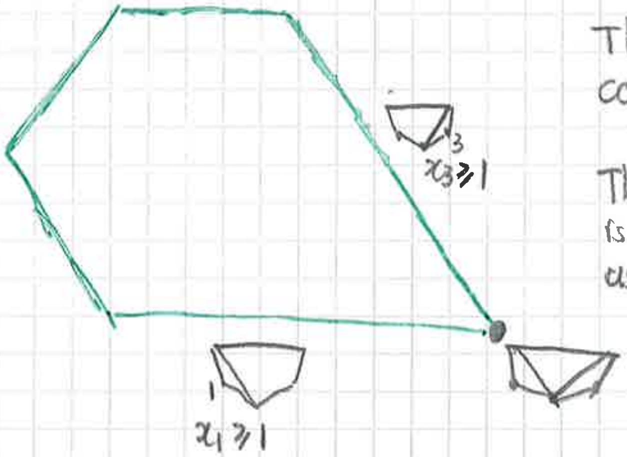
Perm (3)



Asso (3).

Keep all inequalities except for $x_1+x_3 \geqslant 1+2$.

$\{1,3\}$ not an interval



The facets (inequalities) correspond to diagonals

The intersection of several facets is labelled by the subdivision using the corresponding diagonals.

- <u>Loday's beautiful coordinates description of the vertices</u> (Loday '04)

 has coordinate $(1,4,1)$. 

Given a binary tree with n internal nodes label the nodes by the product.

(# leaves on its left) × (# leaves on its right)

$(1,2,3)$ $(2,1,3)$ $(3,1,2)$ $(3,2,1)$ $(1,4,1)$

Then read labels in in-order

A r B

• <u>Asso(n) as a Minkowski sum</u> (Postnikov '09)

For a subset $A \subseteq [n]$ define

$$\Delta_A = \mathrm{conv}\{ e_a : a \in A \}.$$

where $e_1, \dots, e_n \in \mathbb{R}^n$ are the standard basis vectors.

The <u>Minkowski sum</u> of two polytopes P and Q is

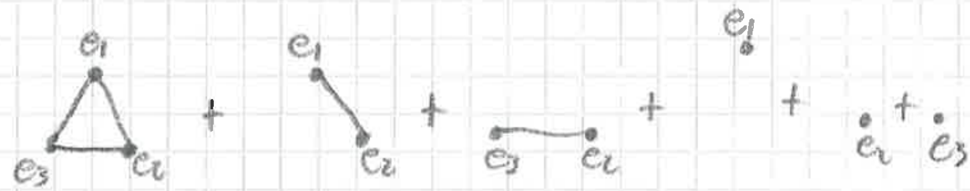
$$P + Q = \{ p + q : p \in P \text{ and } q \in Q \}.$$

The Loday associahedron can be obtained as the Minkowski sum:

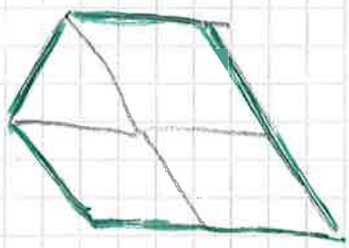
$$\boxed{\mathrm{Asso}(n) = \sum_{I \text{ interval of } [n]} \Delta_I}$$

<u>Example:</u>

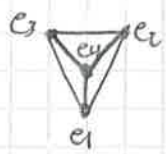
$\mathrm{Asso}(3) =$

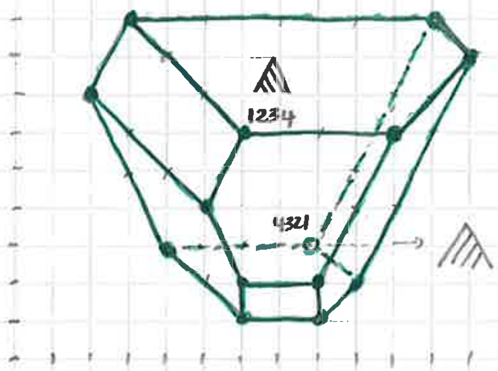


$=$

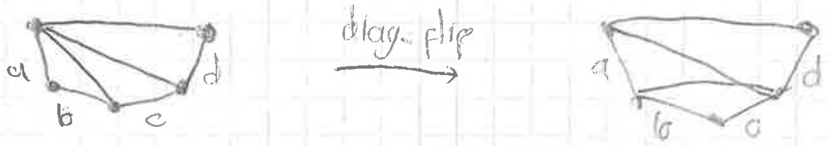


$\mathrm{Asso}(4) = \Delta_{1234} + \Delta_{123} + \Delta_{234} + \Delta_{12} + \dots + \Delta_{34} + \Delta_1 + \dots + \Delta_4$





- <u>Why associahedron ?</u>

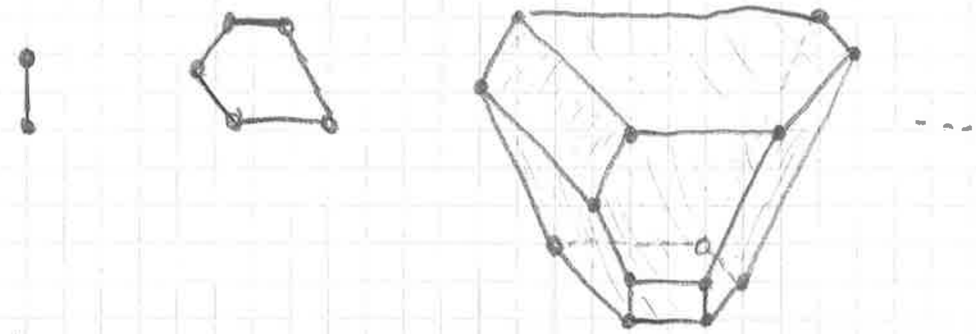


$((ab)\,c\,)\,d \xrightarrow{\text{Application of the associative rule.}} (\,a(bc)\,)\,d$

Vertices of Asso(n) : different ways to multiply n letter.

edges : Applications of the associative rule.

- Many nice applications / properties.

- 

Asso(1)   Asso(2)   Asso(3)   Asso(4)   ...

Asso(1)	Asso(2)	Asso(3)	Asso(4)
1 vertex.	1 edge 2 vertices	1 pentagon 5 edges 5 vertices	1 solid 6 hexagons + 4 squares 21 edges 14 vertices.

- Inverting power series under composition (Lagrange inversion)
(see Ardila–Aguiar '23)

The number of faces of the associahedron Asso(n) has nice applications in the context of formal power series.

If
$$C(x) = x + c_1x^2 + c_2x^3 + \dots$$
$$D(x) = x + d_1x^2 + d_2x^3 + \dots$$
and
$$(D \circ C)(x) = x \qquad \text{(compositional inverses)}$$
then
$$d_1 = -c_1$$
$$d_2 = -c_2 + 2c_1^2$$
$$d_3 = -c_3 + 5c_2c_1 - 5c_1^3$$
$$d_4 = -c_4 + (6c_3c_1 + 3c_2c_2) - 21c_2c_1^2 + 14c_1^4$$

- Potential project:
study the Pitman-Stanley polytope and its volume related to parking functions (normalized volume)