Exercise 1

Divine-and-Conquer Algorithm for computing convex hulls in 2D:
Let P be a point set in the plane with no three points collinear and no two points on the same vertical line. Sort the points according to their x-coordinate. Divide the points into two (nearly) equal groups, A and B, where A contains the left $\lceil \frac{n}{2} \rceil$ points and B the right $\lfloor \frac{n}{2} \rfloor$ points. Compute the convex hull of A and B recursively (using the algorithm). Finally, merge $\operatorname{conv}(A)$ and $\operatorname{conv}(B)$ to obtain $\operatorname{conv}(P)$.

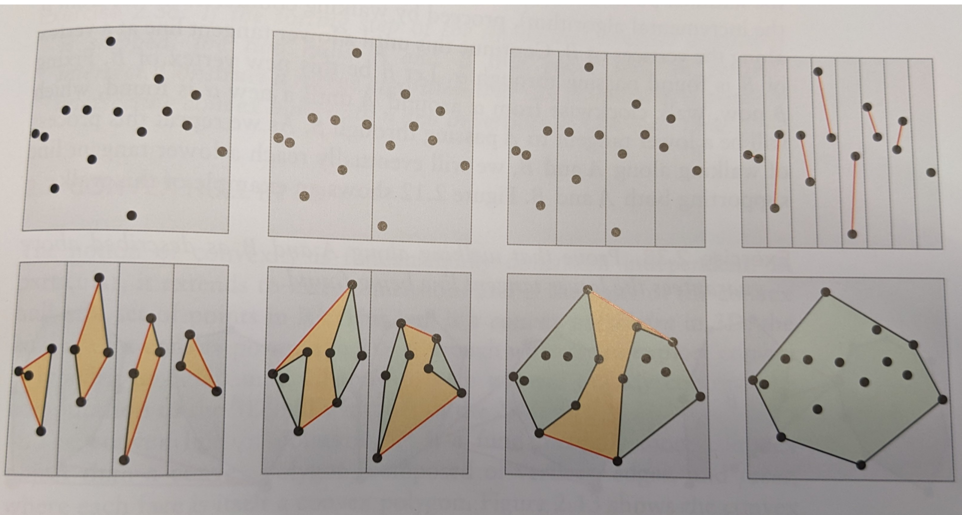
Figure 2.10. The divide-and-conquer algorithm in action.

(i) Show that the merging step can be computed in linear time $O(n)$.

(ii) Let $T(n)$ be the time complexity of the divine-and-conquer hull algorithm for n points. Show that $T(n) = 2T(n/2) + O(n)$, a recurrence relation in computer science, whose solution is $T(n) = O(n \log n)$.

Exercise 2

Let P be a point set in the plane. Show that T is a Delaunay triangulation of P if and only if T is a legal triangulation of P.

Exercise 3

Let P be a point set in the plane, and $\overline{P}$ be the set of corresponding lifted points in the parabola $z = x^2 + y^2$. That is, a point $(a, b) \in \mathbb{R}^2$ is lifted to the point $(a, b, a^2 + b^2) \in \mathbb{R}^3$. Consider the union of the tangent planes $T_{\overline{p}}$ of the parabola at the lifted points $\overline{p} \in \overline{P}$, and the projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\pi(x, y, z) = (x, y)$.

(i) Show that the intersection of two tangent planes $T_{\overline{p}_i} \cap T_{\overline{p}_j}$ projects to the bisector line ℓ_{ij} between the points p_i and p_j.

(ii) Show that the intersection of three tangent planes $T_{\overline{p}_i} \cap T_{\overline{p}_j} \cap T_{\overline{p}_k}$ projects to the center of the circumcircle through the points p_i, p_j, p_k.

(iii) Show that the tangent planes and intersections that you can see from the top ($z = +\infty$), project to the Voronoi diagram $\operatorname{Vor}(P)$ in the xy-plane.

Exercise 4

Consider the paraboloid in $\mathbb{R}^{n+1}$ defined by the equation $y = x_1^2 + \cdots + x_n^2$, and its tangent hyperplane E at the point $(a_1, \ldots, a_n, a_1^2 + \cdots + a_n^2)$. The projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is defined by $\pi(x_1, \ldots, x_n, y) = (x_1, \ldots, x_n)$. Show that the intersection of the paraboloid with a translation of E projects to a sphere with center $(a_1, \ldots, a_n)$.