Exercise 1

Given $X \subseteq \mathbb{R}^2$ and a vector $d \in \mathbb{R}^2$, a point $p \in X$ is called *extremal* in direction d if for all $q \in X$ we have $\langle q, d\rangle \leq \langle p, d\rangle$.

(i) Let $X, Y \subseteq \mathbb{R}^2$ and $X \oplus Y$ be their Minkowski sum. Show that an extreme point in direction d on $X \oplus Y$ is the sum of extreme points in direction d on X and Y.

(ii) Let P and R be two convex polygons with vertex sets V and W, respectively. Show that

$$P \oplus R = \mathrm{ConvexHull}(V \oplus W).$$

Exercise 2

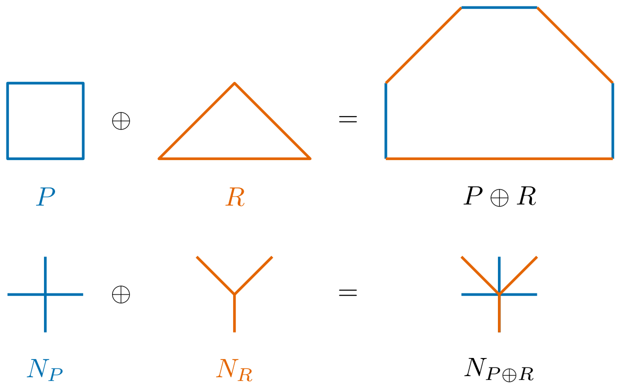
A convex polygon $P \in \mathbb{R}^2$ has three different kind of faces: its vertices, its edges, and the polygon itself. For $d \in \mathbb{R}^2$, we denote by $\mathrm{extremal}_d(P)$ the set of extremal points of P in direction d. Note that $\mathrm{extremal}_d(P)$ is a face of P (either a vertex, an edge orthogonal to d, or the polygon itself if $d = 0$).

(i) The *normal cone* of a face F of P is defined as

$$C_F := \{d \in \mathbb{R}^2 : F \subseteq \mathrm{extremal}_d(P)\}.$$

Show that if $x_1, \ldots, x_n \in C$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_{\geq 0}$ then $\sum_{i=1}^n \lambda_i x_i \in C$, that is C is a *cone*.

(ii) The *normal fan* of P is the collection of normal cones $N_P = \{C_F\}_{F \in \text{faces}(P)}$. Show that the normal fan of the Minkowski sum $P \oplus R$ of two convex polygons is the "overlay" of the normal fans of each.



Exercise 3

Let P and R be two convex polygons with n and m edges, respectively. Show that $P \oplus R$ is a convex polygon with at most $n + m$ edges.

Exercise 4

Show that the Minkowski sum of two convex polygons with n and m edges, respectively, can be computed in $O(n + m)$ time.