Discrete and Computational Geometry
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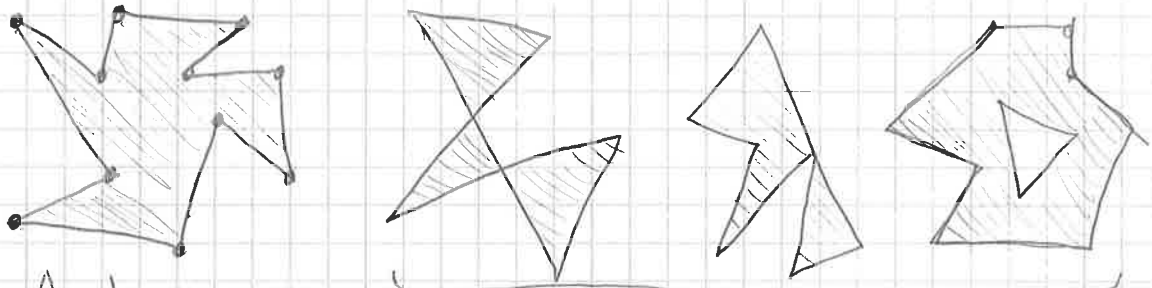
Last time : - Overview
- Inspiration : Art Gallery Problem

Today : - Polygons
- Triangulations
- Art Gallery Theorem

- Polygon

A polygon P is :
- a closed region of the plane
- bounded by finitely many line segments
- forming a closed curve that does not intersect itself.



A polygon with 9 vertices
9 edges

Not polygons.

line segments are called edges

Points where adjacent edges meet are called vertices

Theorem (Polygonal Jordan Curve)

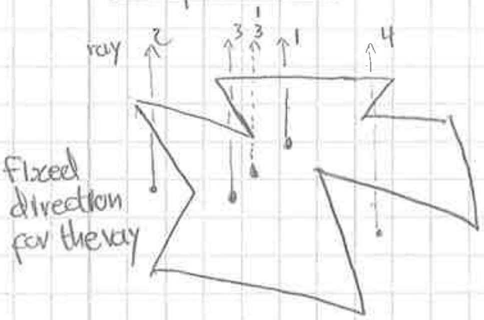
The boundary ∂P of a polygon P partitions the plane into two connected parts : the bounded interior and the unbounded exterior.

Remark This result works for simple closed curves (without intersections)

Notorious for being obvious and difficult!

For polygons : the proof is easier.

Proof sketch



Any point x not on ∂P is in one of the following two sets:

Exterior: ray through x in the fixed direction crosses ∂P an even number of times

Interior: ---- odd number of times.

All points on a line segment that does not interset ∂P must lie in the same set.

■

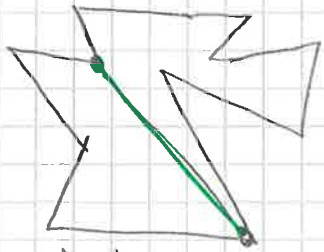
Note: this gives a simple algorithm to decide whether a given point is inside a polygon.

- Triangulations

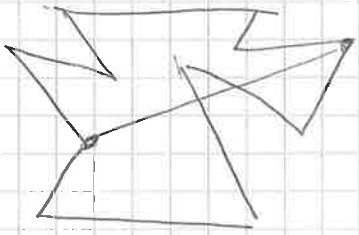
Now we are interested in decomposing a polygon into smaller simpler pieces.

A diagonal: line connecting two vertices of P and lying in the interior of P, not touchin ∂P except at its end points.

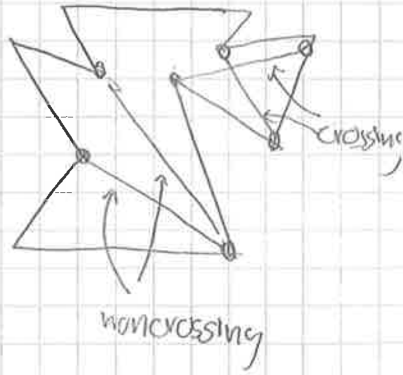
Two diagonal are noncrossing if they share no interior points.



A diagonal

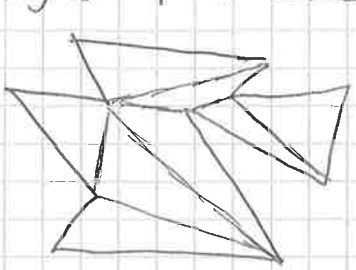


Not a diagonal



A triangulation of a polygon P is a decomposition of P into triangles by a maximal set of noncrossing diagonals.

↳ no further diagonal may be added.



Some natural questions

(1) • How many triangulations does a given polygon have?

(2) • How many triangles?
two triangulations of the same polygon with different number of triangles?

(3) • Can we always triangulate a polygon?

(4) • Can we always find a diagonal?

(4) **Lemma** Every polygon with more than three vertices has a diagonal.

Proof

Let v be the left most vertex of P
(If not unique rotate P slightly so that v is unique).

Let a,b be the two adjacent vertices to v.

Case 1

Case 2

We consider two cases:

Case 1: If the segment ab is inside $P \Rightarrow ab$ is a diagonal

Case 2: If not, there must be at least one point inside the triangle vab.

Sweep a diagonal parallel to ab from v until the first time that it touches a vertex x ~~me~~ P.

Then vx is a diagonal ∎

(3) $\boxed{\text{Theorem} \quad \text{Every polygon has a triangulation}}$

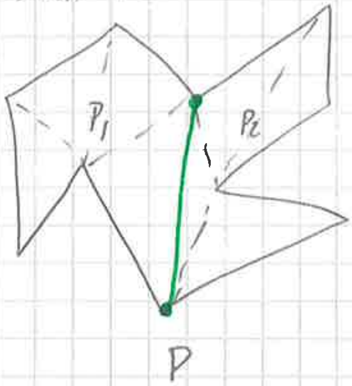
Proof by induction on the number n of vertices of P.

base case $n=3$  ✓

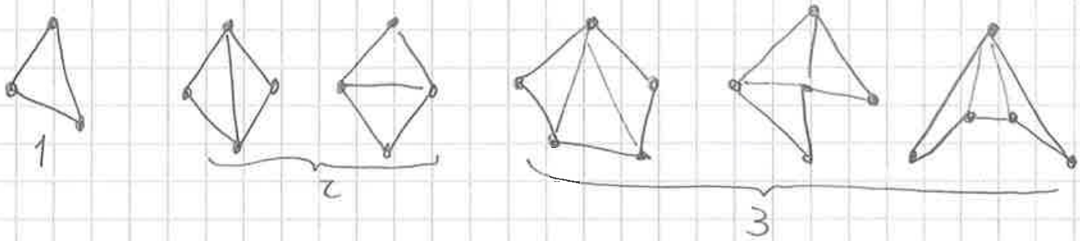
Let $n>3$.

By previous Lemma, there is a diagonal cutting P into two polygons P_1 and P_2.
Since P_1, P_2 have fewer vertices than P
they can by triangulated
by induction hypothesis.

Putting together the triangulations of P_1 and P_2 gives a triangulation of P.



We continue with question (2) How many triangles are in each triangulation?



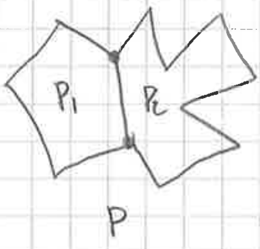
$n = \#$ vertices $\Rightarrow$ $\#$triangles $=$? $\quad n-2$?
$\#$diagonals $=$? $\quad n-3$?

$\boxed{\text{Theorem} \quad \text{Every triangulation of a polygon } P \text{ with } n \text{ vertices has } n-2 \text{ triangles and } n-3 \text{ diagonals.}}$

Proof By induction on n.

$n=3$:  $1=n-2$ triangle $\quad 0=n-3$ diagonals ✓

Assume the result is true for all polygons with $<n$ vertices.



Cut P by a diagonal into P_1 and P_2 with n_1 and n_2 vertices respectively. Then

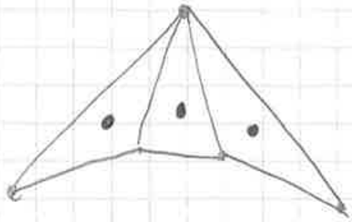
$$n_1 + n_2 = n+2$$

$\#$ triangles in P $=$ $\#$ trian in P_1 $+$ $\#$ triang P_2 $= (n_1-2)+(n_2-2) = n-2$

$\#$ diag in P $=$ $\#$diag P_1 $+$ $\#$diag P_2 $+ 1 = (n_1-3)+(n_2-3)+1 = n-3$.

- Art Gallery Problem

Art Gallery where floor is modelled by a polygon.



Problem:

(1) How many guards do we need to observe the entire gallery?

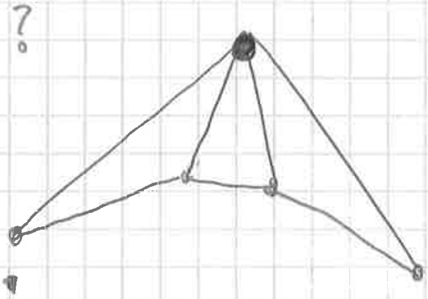
(2) How do we decide how to place them?

One option: triangulate the polygon
place one guard per triangle

If the polygon has n vertices, then $n-2$ guards (# of triangles) are sufficient to observe the entire gallery.

Question: Can we do better?

Place guards at vertices instead of triangles?



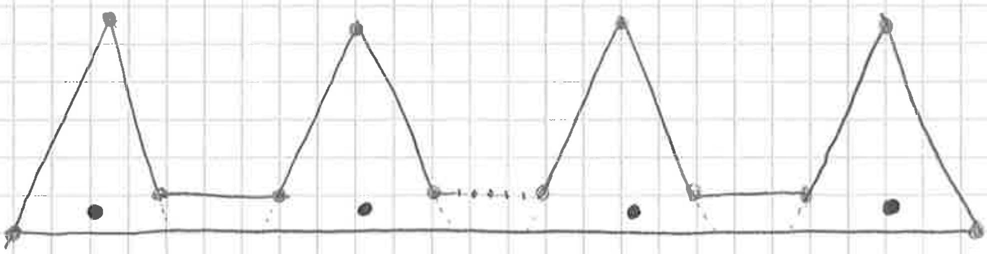
How many?

In our example: 1 is sufficient!

Theorem (Art Gallery Theorem)

To observe an entire polygon with n vertices, $\lfloor n/3 \rfloor$ guards are needed for some polygons, and sufficient for all of them.

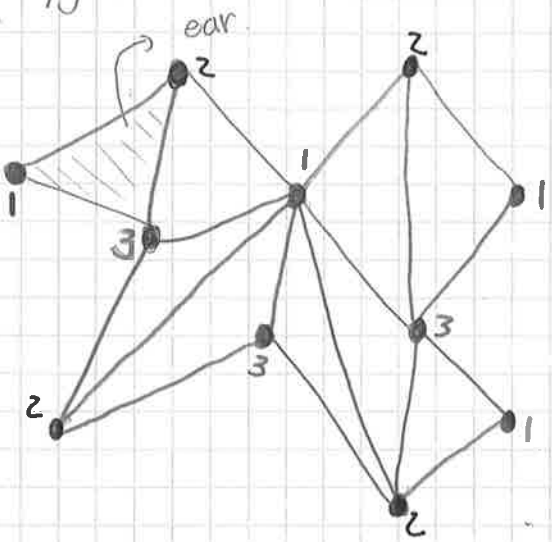
Example where $\lfloor n/3 \rfloor$ guards are needed:



at least one guard per triangle (top vertices need to be observed)

Before proving the theorem:
Consider a triangulation of a polygon. P

We say that the triangulation is <u>3-colorable</u> if each vertex can be assigned one of three colors so that any pair of vertices connected by an edge of P or a diagonal of the triangulation have different colors.



<u>Lemma</u> Every triangulation of a polygon P is 3-colorable.

<u>Proof</u> For $n=3$

✓

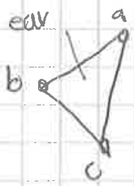
For $n>3$

<u>Claim</u>: Every triangulation of a polygon with more than three vertices has an <u>ear</u>

↳ three consecutive vertices abc of a triangulated polygon P form an ear if ac is a diagonal of P

In ..

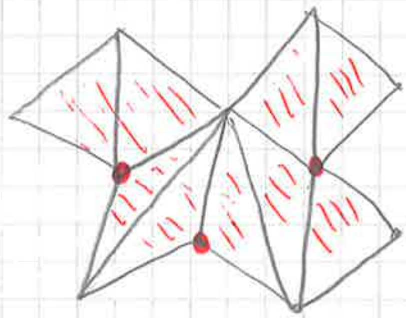
<u>Proof of claim</u>: Exercise.



Let abc be an ear of the triangulated P. with b being the ear tip

Removing the ear from P produces a polygon P' with $n-1$ vertices

By induction hypothesis the restricted triangulation of P' can be 3-colored. Since b is only connected to a and c, this remaining vertex can be colored with the missing color to obtain a 3-coloring of the full triangulation.



Proof of Art Gallery Theorem

For $n=3$ ✓

For $n>3$:

There exist a triangulation of P
This triangulation is 3-colorable!

Since there are n vertices, by the pigeonhole principle, the least frequently used color appears on at most $\lfloor n/3 \rfloor$ vertices.

Placing guards at these vertices, we are able to observe the entire gallery ∎

Question:

How do we decide how to place the $\lfloor n/3 \rfloor$ guards with an efficient algorithm?