Discrete and Computational Geometry

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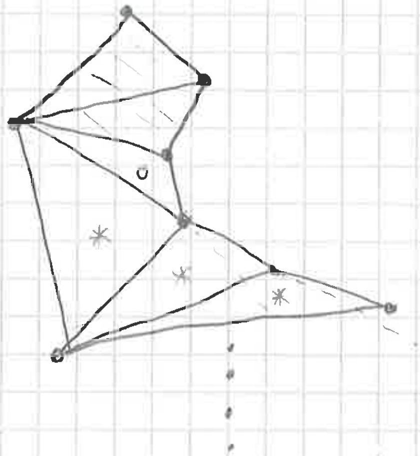
Last time : Art Gallery Algorithm

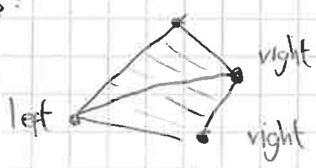
Today : Finish algorithm
Triangulations in 2D : further properties
convex polygons.

Triangulating a strictly y-monotone polygon

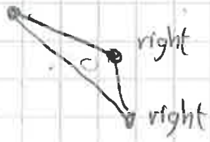


Triangulate step by step
from top to bottom

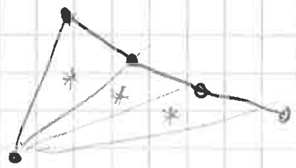
Cases :



change side
→ triangulate

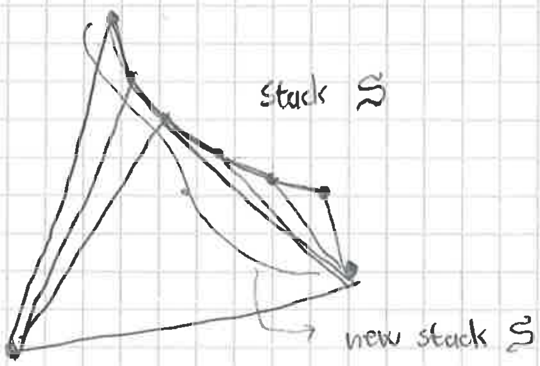


same side
triangulate if possible



If not stack until possible

Other possibility .



Algorithm

Input : strictly y-monotone polygon P stored in a doubly-connected edge list .

Output : A triangulation of P stored in a double-connected edge list D

Time:

linear time — 1. Merge the left and right chain vertices into one sequence, $u_1, \dots, u_n$, sorted on decreasing y-coordinate.
(If two vertices have the same y-coordinate the the left comes first)

2 constant — 2. Initialize an empty stack S, and push u_1 and u_2 onto it.

n-3 times — 3. For $j \leftarrow 3$ to $n-1$

4. **do if** u_j and the vertex on top of S are on different chains

\#pushes ≤ 2

\#pops ≤ \#pushes

O(n)

5. **then** Pop all vertices of S
6. Insert into D a diagonal from u_j to each popped vertex, except last.
7. Push u_{j-1} and u_j to S
8. **else** Pop one vertex from S.
9. Pop the other vertices from S as long as the diagonals from u_j to them are inside P. Insert these diagonals to D. Push the last vertex that has been popped back into S.
10. Push u_j into S

at most linear — 11. Add diagonals from u_n to all stack vertices except first and last.

How much does this algorithm take?

Answer: linear time.

Theorem A strictly y-monotone polygon with n-vertices can be triangulated in linear time $O(n)$

Back to the general case:

Theorem Let P be a polygon with n vertices. A set of positions of $\lfloor \frac{n}{3} \rfloor$ guards to observe the entire polygone can be computed in $O(n \log n)$ time

Proof sketch

- Decompose P into monotone pieces in $O(n \log n)$ time
  See Book: Computational Geometry. de Berg, Cheong, van Kreveld, Overmars
- Triangulate monotone pieces : $O(n)$
- 3-coloring of vertices : $O(n)$
- place guards in less frequently used color.

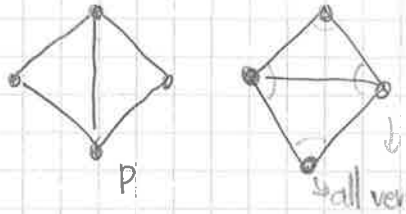
Total time: $O(n \log n)$ ∎

Remark There is a quite complicated linear time algorithm by Chazelle '90 '91

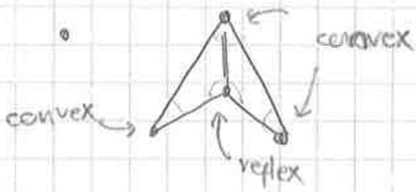
Next, we are interested in the following question.

Question: How many triangulations does a given polygon P with n vertices have?

Examples:

- 

  2 triangulations

- 

  1 triangulation.

Answer depends on how the vertices are placed!

A vertex of P is called <u>reflex</u> if its angle is $> \pi$

" " " " " " <u>convex</u> " " " " $< \pi$

A polygon is called a <u>convex polygon</u> if all its vertices are convex.

<u>Exercise</u> Show that for a polygon P a diagonal exists between any two nonadjacent vertices if and only if P is a convex polygon.

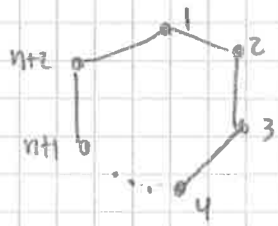
<u>Theorem</u>:

The number of triangulations of a convex polygon with $n+2$ vertices is the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Example: $n=2 \Rightarrow C_2 = \frac{1}{3}\binom{4}{2} = 2 \Rightarrow$ 2 triangulations of a convex $(2+2)$-gon (4-gon)

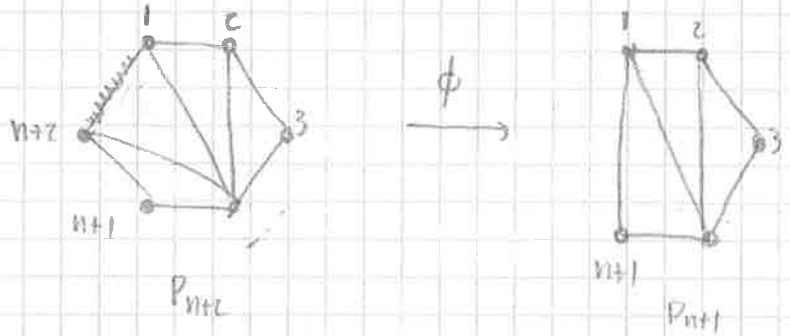
<u>Proof</u> Let P_{n+2} be a convex polygon with $n+2$ vertices labeled from 1 to $n+2$ clockwise



Let T_{n+2} be the set of triangulations of P_{n+2} and

$$t_{n+2} = |T_{n+2}|$$

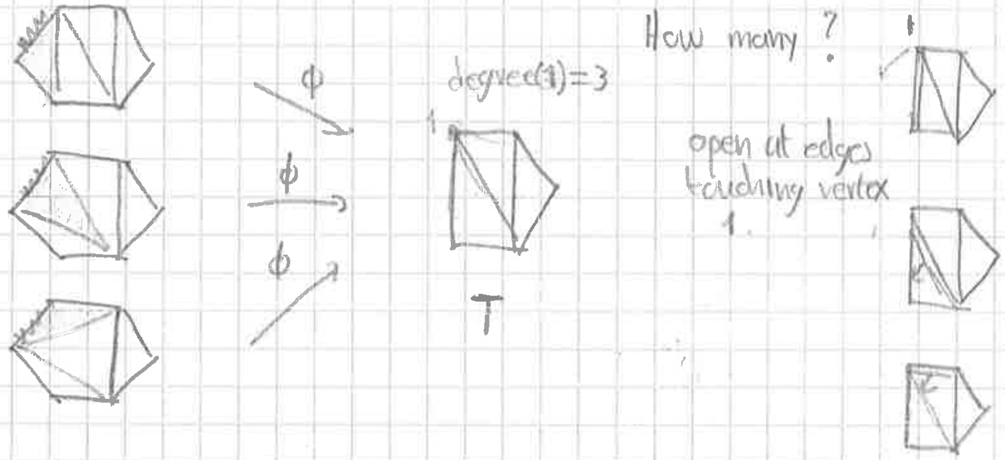
We want to show that $t_{n+2} = C_n = \frac{1}{n+1}\binom{2n}{n}$.



Let ϕ be the map

$$\phi : \mathcal{T}_{n+2} \to \mathcal{T}_{n+1}$$

given by contracting the edge $\{1, n+2\}$ of P_{n+2}.
Note that different triangulations can be mapped to the same triangulation T of P_{n+1}.



<u>Answer</u> :

$$|\phi^{-1}(T)| = \text{degree of vertex 1 in } T$$

Therefore :

$$t_{n+2} = \sum_{T \in \mathcal{T}_{n+1}} (\text{degree of vertex 1 in } T)$$

The same formula is true for all vertices of T. Summing over all vertices of T:

$$(n+1)\, t_{n+2} = \sum_{i=1}^{n+1} \sum_{T \in \mathcal{T}_{n+1}} (\text{degree of vertex } i \text{ in } T)$$

$$= \sum_{T \in \mathcal{T}_{n+1}} \sum_{i=1}^{n+1} (\text{degree of vertex } i \text{ in } T)$$

$$= 2(2n-1)\, t_{n+1}$$

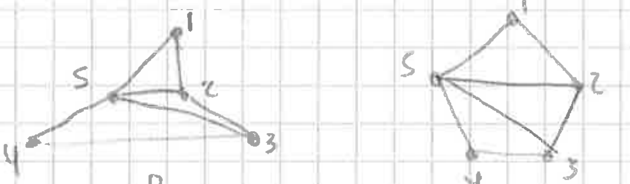
Thus:

$$
\begin{aligned}
t_{n+2} &= \frac{2(2n-1)}{n+1}\, t_{n+1} \\
&= \frac{\overset{n}{2}(2n-1)}{n+1} \cdot \frac{\overset{n-1}{2}(2n-3)}{n} \cdot \ldots \cdot \frac{\overset{1}{2}(1)}{2} \cdot \boxed{\frac{n!}{n!}} \\
&= \frac{2n(2n-1)}{n+1} \cdot \frac{(2n-2)(2n-3)}{n} \cdot \ldots \cdot \frac{2(1)}{2} \cdot \frac{1}{n!} \\
&= \frac{(2n)!}{(n+1)!\, n!} \\
&= \frac{1}{n+1}\binom{2n}{n} \qquad \blacksquare
\end{aligned}
$$

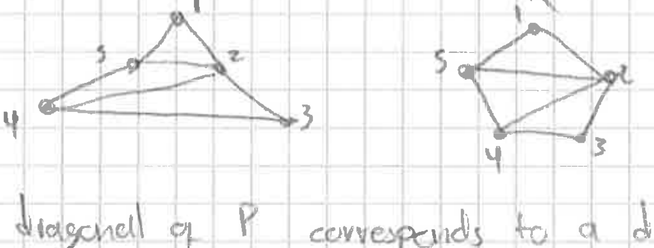
Our next result shows that convex polygons achieve the maximum possible number of triangulations.

Theorem Let P be a polygon with $n+2$ vertices. The number of triangulations of P is between 1 and C_n.

Proof Let Q be a convex polygon with $n+2$ vertices. Label vertices of P and Q from 1 up to $n+2$ in clockwise order.



P Q



Every diagonal of P corresponds to a diagonal of Q

If two diagonals of P do not cross $\Rightarrow$ neither do they cross in Q

So, every triangulation of P determines a triangulation of Q

Therefore, P can have no more triangulations than Q, i.e. C_n $\blacksquare$