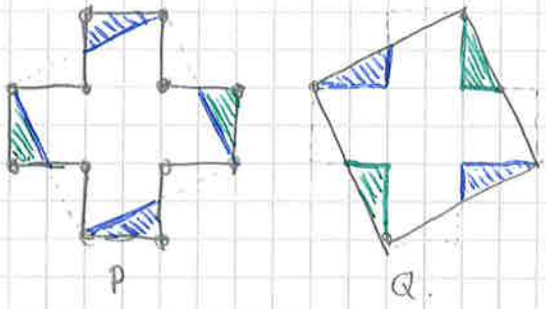
Discreate and Computational Geometry
Cesar Ceballos.

Today: Sciessors congruece

- Sciessors congruence

Question: Are any two polygons of the same area sciessors congruent?

YES!



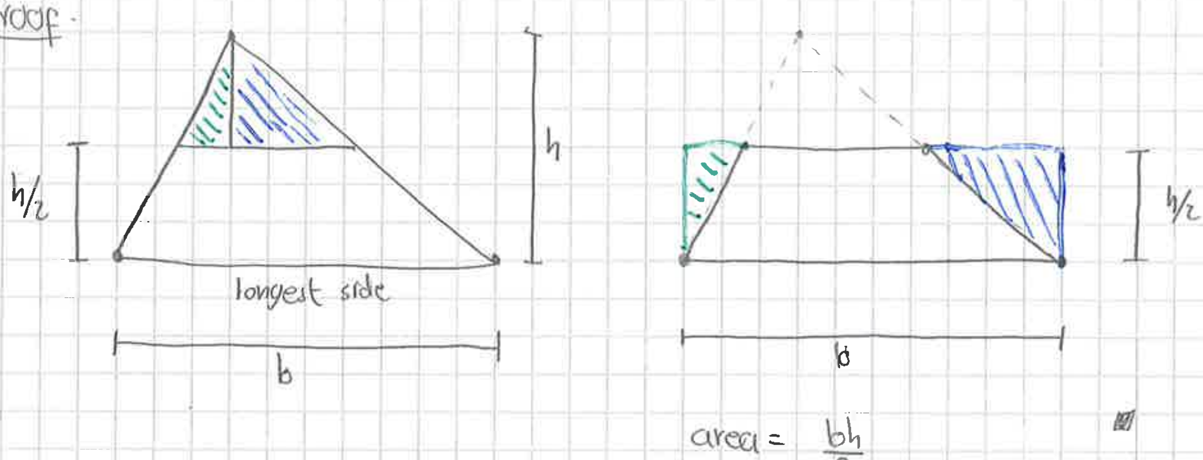
dissection of P cuts P into a finite number of smaller polygons $P_1, \dots, P_n$

Note: vertices of P_i are not required to be vertrces of P.

Two polygons P and Q are scissors congruent if P can be cut into polygons $P_1, \dots, P_n$ which can be reassembled by rotations and translations to obtain Q.

Lemma 1: Every triangle is sciessors congruent with some rectangle.

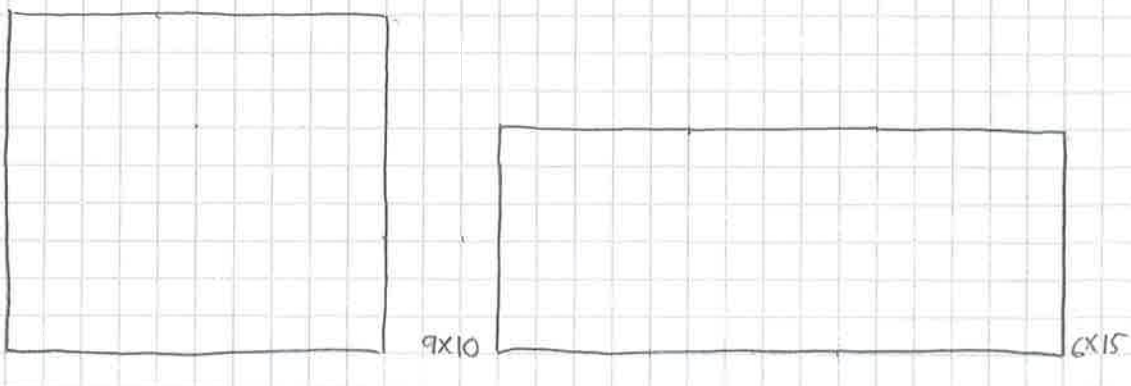
Proof.



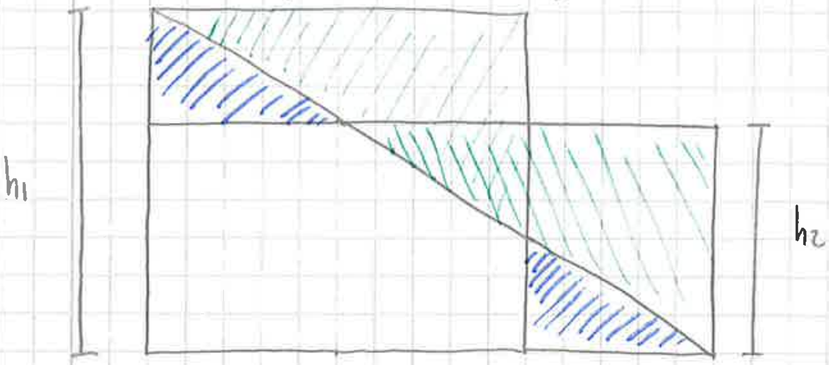
area = $\frac{bh}{2}$ ☒

Lemma 2: Any two rectangles of the same area are sciessors congruent.

Proof.

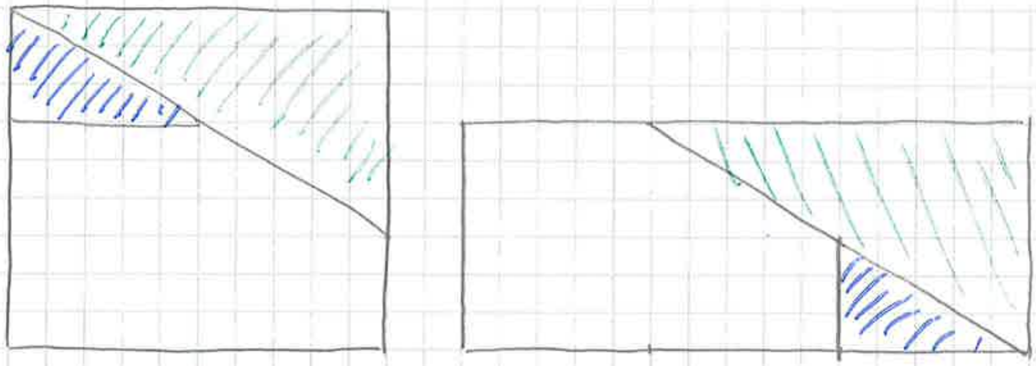


overlap the two rectangles



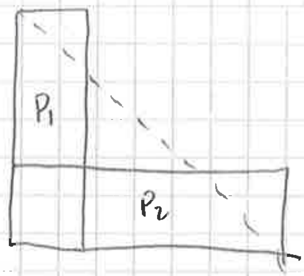
This works when the diagonal is inside of the overlapped figure

$2h_2 \geq h_1$

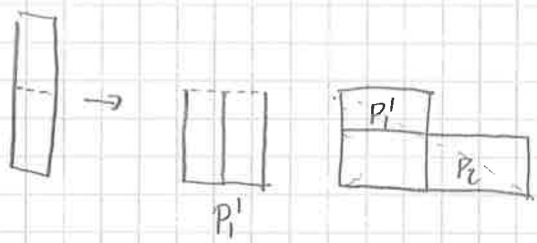


✓

In $h_1 > 2h_2$



Cut first polygon at height $h_1/2$



✓

scissors congruent. and repeat again if necessary ∎

Exercise Show that if $P_1 \sim P_2$ and $P_2 \sim P_3 \Rightarrow P_1 \sim P_3$. (Transitivity)

Theorem Any two polygons of the same area are scissors congruent

Proof: Let P and Q two polygons of the same area A.

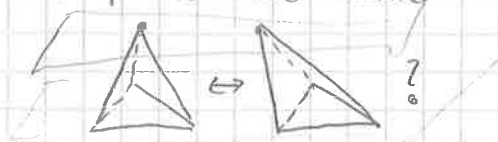
We show that P (and Q) is scissors congruent to a $1 \times A$ rectagle.

- Triangulate P into triangles $T_1, \dots, T_K$ of areas $A_1, \dots, A_K$
- T_i is scissors congruent to a rectangle R_i by Lemma 1.
- R_i is scissors congruent to a $1 \times A_i$ rectangle be Lemma 2.
- Putting all these rectangles together form an $1 \times A$ rectangle.

Since Q is also scissors congruent to a $1 \times A$ rectangle,
By transitivity (exercise), then P and Q are scissors congruent ∎

• Scissors congruence in 3D

Question: Are any two polyhedra of the same volume scissors congruent? NO



↳ related to

Hilbert's Third Problem
International Congress of Mathematics, Paris 1900

↳ Solved in the negative by Hilbert's student Max Dehn

Dehn's proof: not easy to understand
Hadwiger: elegant techniques to solve the problem.

See "Proofs from THE BOOK", Aigner and Ziegler Chapter 7.

Tool: $\mathbb{Q}$-linear functions.
For a finite set of real numbers

$$M = \{m_1, \dots, m_k\} \subseteq \mathbb{R}$$

Let

$$V(M) := \{\sum_{i=1}^{k} q_i m_i : q_i \in \mathbb{Q}\} \subseteq \mathbb{R}$$

A $\mathbb{Q}$-linear function is a linear map

$$f: V(M) \to \mathbb{Q}$$

Note:
- f is determined by fixing its images on a basis of $V(M)$
- for $M' \subseteq M$, a $\mathbb{Q}$-linear map $f: V(M) \to \mathbb{Q}$ induces a $\mathbb{Q}$-linear map $V(M') \to \mathbb{Q}$ by restriction
- Any $\mathbb{Q}$-linear map $V(M') \to \mathbb{Q}$ can be extended to $V(M) \to \mathbb{Q}$

Dehn invariants

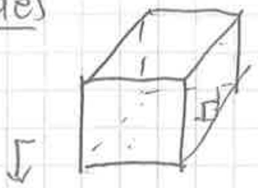
Given a 3-dimensional polyhedron P, the dihedral angle $\alpha(e)$ of an edge e of P is.

edge e

$\alpha = \alpha(e)$ dihedral angle
= angle between the two incident faces.

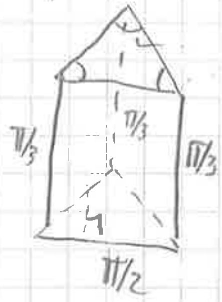
Examples

Cube.



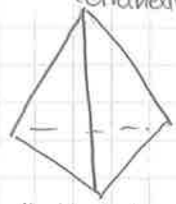
all dihedral angles are $\pi/2$

prism over an equilateral triangle



Dihedral angles: $\pi/3$, $\pi/2$

regular tetrahedron



all dihedral angles: $\arccos 1/3$ (Exercise).
$\to$ not a $\mathbb{Q}$-multiple of π !

Let

$$\boxed{M_P = \{\alpha(e) : e \text{ an edge of } P\} \cup \{\pi\}}$$

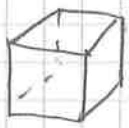
be the set of dihedral angles of P union π.

For $M \supseteq M_P$ and any $\mathbb{Q}$-linear function $f: V(M) \to \mathbb{Q}$ satisfying $\boxed{f(\pi) = 0}$
we define the <u>Dehn invariant</u> of P (with respect to f) as

$$\boxed{D_f(P) := \sum_{e \text{ edge of } P} l(e) \cdot f(\alpha(e))}$$

where $l(e)$ is the length of e.

<u>Examples continued.</u>



Cube C.

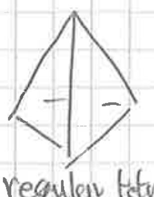
$M_C = \{\pi/2, \pi\}$ and $f(\pi) = 0 \xrightarrow{\text{by } \mathbb{Q}\text{-linearity}} f(\pi/2) = 0$

$\Rightarrow \quad D_f(P) = \sum_{e \text{ edge of } C} l(e) \cdot 0 = 0$

Prism P_Δ

$M_{P_\Delta} = \{\pi/2, \pi/3, \pi\}$ and $f(\pi) = 0 \longrightarrow f(\pi/2) = f(\pi/3) = 0$

$\Rightarrow \quad D_f(P_\Delta) = \sum_{e \text{ edge of } P_\Delta} l(e) \cdot 0 = 0.$



regular tetrahedron T

$M_T = \{\arccos 1/3, \pi\}$ and $f(\pi) = 0$, $f(\arccos 1/3) = \underset{\text{any number.}}{1}$

$\Rightarrow \quad D_f(T) = \sum_{e \text{ edge of } T} l(e) \cdot 1 = 6\, l(e)$

Dehn-Hadwiger Theorem

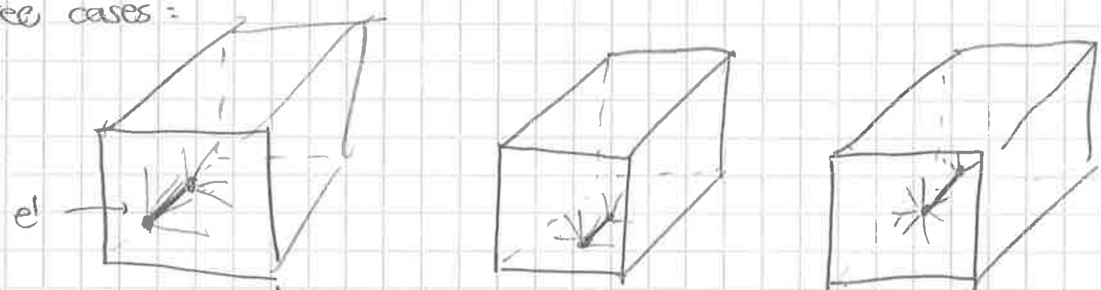
Let $P_1, P_2, \dots, P_n$ be a dissection of P, and let $M \subset \mathbb{R}$ be a set containing all dihedral angles of $P_1, \dots, P_n$, and π. Then, for any $\mathbb{Q}$-linear function

$$f : V(M) \to \mathbb{Q} \qquad \text{with } f(\pi) = 0$$

We have
$$D_f(P) = D_f(P_1) + \dots + D_f(P_n)$$

Proof. Let e' be an edge appearing in at least one P_i.

Three cases:



(a) on edge (b) on face (c) interior.

Contribution in the sum over all P_i's containg e' is:

(a) $\ell(e') \cdot f(\alpha(e'))$

(b) $\ell(e') \cdot f(\pi) = 0$

(c) $\ell(e') \cdot f(2\pi) = 0$

Total contribution:

$$D_f(P_1) + \dots + D_f(P_n) = \sum_{e \text{ edge } P} \ell(e) \cdot f(\alpha(e)) = D_f(P)$$

∎

Corollary Let P, Q be two polyhedra and $M \subseteq \mathbb{R}$ be a set containing all dihedral angles of P and Q and π, and let $f : V(M) \to \mathbb{Q}$ be a $\mathbb{Q}$-linear function with $f(\pi)=0$.

If $D_f(P) \neq D_f(Q)$ then P and Q are not scissors congruent.

Proof If P is scissors congruent to Q using $P_1, \dots, P_n$ and $Q_1, \dots, Q_n$ — P_i's congruent to Q_i's.

Let $M' \supseteq M$ be a set containing all dihedral angles of $P_1, \dots, P_n$

Extend f to $V(M') \to \mathbb{Q}$.

Note $D_f(P_i) = D_f(Q_i)$.

Then

$$\begin{aligned} D_f(P) &= D_f(P_1) + \dots + D_f(P_n) \\ &= D_f(Q_1) + \dots + D_f(Q_n) \\ &= D_f(Q). \end{aligned}$$

∎

Two polyhedra that are not scissors congruent

Theorem The unit cube and the regular tetrahedron of volume 1 are not scissors congruent.

Proof The dihedral angles of a regular tetrahedron T are all equal to

$$\alpha = \arccos \frac{1}{3} \qquad \text{(Exercise)}.$$

This is not a rational multiple of π (see Proofs from THE BOOK chapter 6)

Let $M = \{\pi, \underset{\alpha}{\arccos \frac{1}{3}}\}$ and $f : V(M) \to \mathbb{Q}$ s.t. $f(\pi)=0$ and $f(\alpha) = 1$.

Then

$$\left.\begin{aligned} D_f(T) &= 6 \cdot l(e) \\ D_f(C) &= 0 \end{aligned}\right\} \Rightarrow$$

while

since $D_f(T) \neq D_f(C)$ then T and C are not scissors congruent!

∎