Last time: scissors congruence.

Recall: If $P, Q \subseteq \mathbb{R}^3$ are two polyhedra such that

$D_f(P) \neq D_f(Q)$ for <u>some</u> $\mathbb{Q}$-linear function f with $f(\pi) = 0$

then P and Q are not scissors congruent.

What about the converse?

Theorem (Sydler)

Polyhedra P and Q are scissors congruent if $D_f(P) = D_f(Q)$ for every $\mathbb{Q}$-linear function f with $f(\pi) = 0$

Proof: is quite involved.

Example/Exercise

- Show that for the unit cube C and the tetrahedron T below

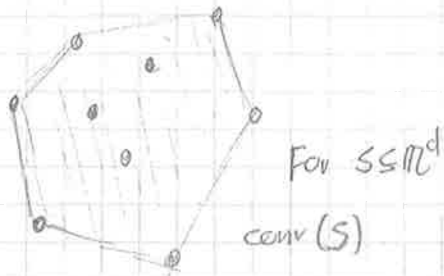
$D_f(C) = D_f(T) = 0$

for every f with $f(\pi) = 0$.

As a consequence, T is scissors congruent to a cube (of volume $1/6$)

- Try to find a dissection of T and reassemble the pieces to form a cube.

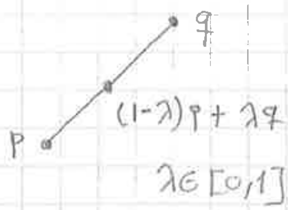
Today: Convex hulls



Before defining the convex hull, we start we some basic definitions.

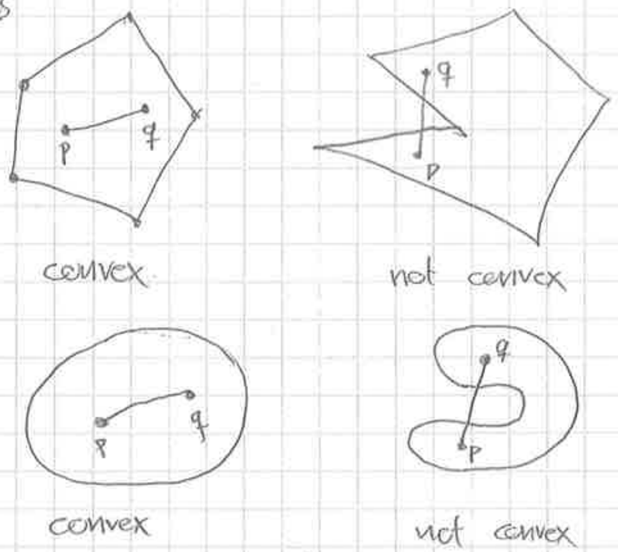
For $p, q \in \mathbb{R}^d$, the <u>line segement</u> between p and q is

$$\overline{pq} := \{ (1-\lambda)p + \lambda q \; : \; \lambda \in [0,1] \}$$



Def. A set $S \subseteq \mathbb{R}^d$ is <u>convex</u> if for every $p, q \in S$ the line segment $\overline{pq} \subseteq S$

<u>Examples</u>

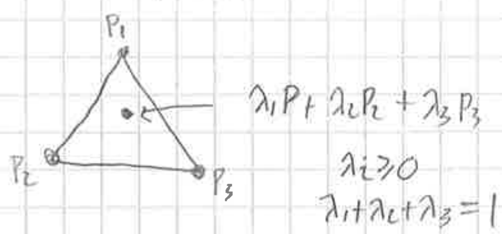


> **<u>Lemma</u>** If $\{S_i\}_{i \in I}$ are convex, then $\bigcap_{i \in I} S_i$ is also convex

<u>Proof</u> For $p, q \in \bigcap S_i$, the segment $\overline{pq} \subseteq S_i \Rightarrow \overline{pq} \subseteq \bigcap S_i$ ∎

For $p_1, p_2, p_3 \in \mathbb{R}^d$, the triangle with vertices p_1, p_2, p_3 is

$$\left\{ \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 \; : \; \begin{array}{l} \lambda_i \geq 0 \\ \lambda_1 + \lambda_2 + \lambda_3 = 1 \end{array} \right\}$$



In general:

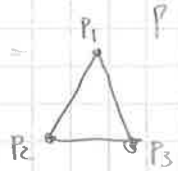
Def (Convex hull)

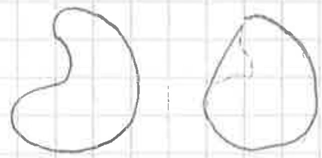
For $S \subseteq \mathbb{R}^d$, the <u>convex hull</u> of S is

$$\boxed{\operatorname{conv}(S) := \Big\{ \underbrace{\sum_{i=1}^{n} \lambda_i s_i}_{\text{convex linear combination}} \;:\; n \in \mathbb{N},\ s_i \in S,\ \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i = 1 \Big\}}$$

<u>Examples</u>:
- $S = \{p,q\} \Rightarrow \operatorname{conv}(S) = \overline{pq}$

- $S = \{p_1, p_2, p_3\} \Rightarrow \operatorname{conv}(S) =$



- 

S $\qquad$ $\operatorname{conv}(S)$

<u>Remarks</u>: The following hold:
- $S \subseteq \operatorname{conv}(S)$ ✓
- $S' \subseteq S \Rightarrow \operatorname{conv}(S') \subseteq \operatorname{conv}(S)$ ✓

$\boxed{\text{Lemma: The convex hull } \operatorname{conv}(S) \text{ is convex.}}$

<u>Proof</u> Let $p, q \in \operatorname{conv}(S)$.
Want to show that $\overline{pq} \subseteq \operatorname{conv}(S)$

Without loose of generality

$$p = \sum_{i=1}^{n} \lambda_i s_i, \quad q = \sum_{i=1}^{n} \mu_i s_i \qquad \lambda_i, \mu_i \geq 0 \qquad \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \mu_i = 1$$

for some set $\{s_1, \dots, s_n\} \subseteq S$

For $\alpha \in [0,1]$

$$(1-\alpha) p + \alpha q = \sum_{i=1}^{n} \underbrace{(\lambda_i (1-\alpha) + \mu_i \alpha)}_{\geq 0} s_i$$

and
$$\sum_{i=1}^{n} (\lambda_i (1-\alpha) + \mu_i \alpha) = (1-\alpha) \sum \lambda_i + \alpha \sum \mu_i$$
$$= (1-\alpha) + \alpha$$
$$= 1$$

∎

Lemma $S \subseteq \mathbb{R}^d$ is convex iff $\mathrm{conv}(S) = S$

Proof: ($\Leftarrow$) Assume $\mathrm{conv}(S) = S$

Let $p, q \in S$, then $(1-\lambda)p + \lambda q \in \mathrm{conv}(S) = S$

So, S is convex.

($\Rightarrow$) Assume S is convex. We already know that $S \subseteq \mathrm{conv}(S)$.

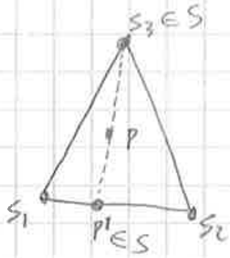
Let $p = \sum_{i=1}^{n} \lambda_i s_i \in \mathrm{conv}(S)$ for some $s_1, s_2, \dots, s_n \in S$
$\lambda_i \geq 0$
$\sum \lambda_i = 1$

We want to show that $p \in S$.

We proceed by induction on n.

$n = 1$ ✓

Assume $n \geq 2$.



Idea: • $p' \in S$ by induction
• $p \in S$ by convexity

Let $\lambda' = \sum_{i=1}^{n-1} \lambda_i$

and

$$p' := \sum_{i=1}^{n-1} \frac{\lambda_i}{\lambda'} s_i \in S \quad \text{by induction hypothesis.}$$

Moreover, since $\lambda_n = 1 - \lambda'$ we have

$$p = \lambda' \underbrace{p'}_{\in S} + (1-\lambda') \underbrace{s_n}_{\in S}$$

Therefore $p \in S$ by convexity of S. ∎

Theorem $\boxed{\mathrm{conv}(S) = \bigcap_{\substack{C \supseteq S \\ C \text{ convex}}} C}$

Proof

($\supseteq$) The set $\mathrm{conv}(S)$ is convex and contains S.

Thus $\bigcap_{\substack{C \supseteq S \\ C \text{ convex}}} C \subseteq \mathrm{conv}(S)$

($\subseteq$) If $S \subseteq C$ and C is convex, then

$$\mathrm{conv}(S) \subseteq \mathrm{conv}(C) = C.$$

Therefore

$$\mathrm{conv}(S) \subseteq \bigcap_{\substack{C \supseteq S \\ C \text{ convex}}} C$$

∎

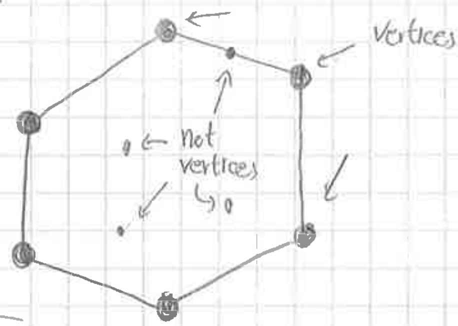
Corollary $\mathrm{conv}(S)$ is the smallest convex set containing S.

Vertices of convex hulls

Let $P \subseteq \mathbb{R}^d$ by a finite collection of points.

$p \in P$ is called a vertex of $\mathrm{conv}(P)$

if $p \notin \mathrm{conv}(P \setminus \{p\})$



Exercise

Let $V \subseteq P$ be the subset of vertices of P.
Then
$$\mathrm{conv}(V) = \mathrm{conv}(P)$$