Last time : Convex hulls

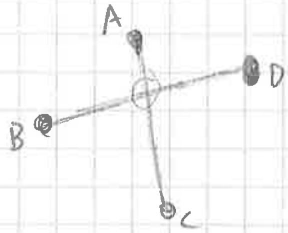
Today : Theorems about convex sets and point configurations

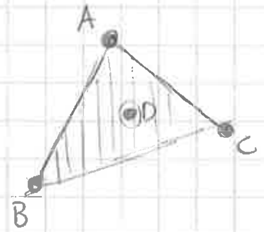
Radon's Theorem

Any set $P \in \mathbb{R}^d$ of $d+2$ points can be partitioned in disjoint sets P_1, P_2 such that

$$\mathrm{conv}(P_1) \cap \mathrm{conv}(P_2) \neq \emptyset$$

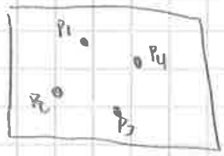
Example: $d=2$



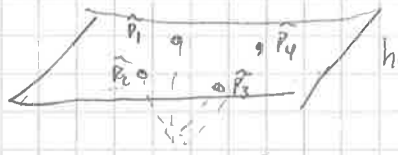


$P_1 = \{A, C\}$, $P_2 = \{B, D\}$ $\qquad$ $P_1 = \{A, B, C\}$, $P_2 = \{D\}$

Prop Let $P = \{p_1, \dots, p_{d+2}\} \subseteq \mathbb{R}^d$ and $\tilde{p}_i = (p_i, 1) \in \mathbb{R}^{d+1}$



$\mathbb{R}^2$



lift points at heigth 1, one dimension higher.

$\mathbb{R}^3$

Then $\tilde{p}_1, \dots, \tilde{p}_{d+2} \in \mathbb{R}^{d+1}$ are linearly dependent. So there is a linear combination

$$\sum_{i=1}^{d+2} \lambda_i \tilde{p}_i = 0$$

Let $Pos := \{ i : \lambda_i \geq 0 \}$, $Neg := \{ i : \lambda_i < 0 \}$

Then
$$\sum_{i \in Pos} \lambda_i \tilde{p}_i = \sum_{i \in Neg} (-\lambda_i) \tilde{p}_i \in \mathbb{R}^{d+1}$$

Since the last coordinate ($d+1$ coordinate) must be equal on both sides, then

$$\sum_{i \in Pos} \lambda_i = \sum_{i \in Neg} (-\lambda_i) = \lambda$$

Therefore

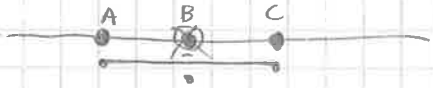
$$\underbrace{\sum_{i \in Pos} \frac{\lambda_i}{\lambda} p_i}_{\in \mathrm{conv}(Pos)} = \underbrace{\sum_{i \in Neg} \left(\frac{-\lambda_i}{\lambda}\right) p_i}_{\in \mathrm{conv}(Neg)} \quad \blacksquare$$

Tverbergs Theorem

Any set $P \subseteq \mathbb{R}^d$ of $(r-1)(d+1)+1$ points can be partitioned in r disjoint sets $P_1, \dots, P_r$ such that.

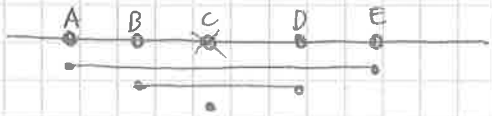
$$\bigcap_{i=1}^{r} \operatorname{conv}(P_i) \neq \emptyset$$

Example $d=1$, $r=2$ $\Rightarrow$ $1 \cdot 2+1 = 3$ points in $\mathbb{R}^1$



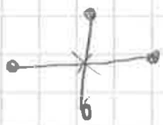
$P_1 = \{A, C\}$ $P_2 = \{B\}$

$d=1$, $r=3$ $\Rightarrow$ $2 \cdot 2+1 = 5$ points.



$P_1 = \{A, E\}$ $P_2 = \{B, D\}$ $P_3 = \{C\}$

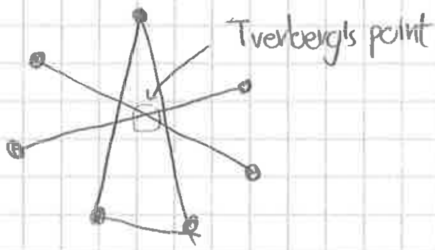
$d=2$, $r=2$ $\Rightarrow$ $1 \cdot 3+1 = 4$ points in $\mathbb{R}^2$

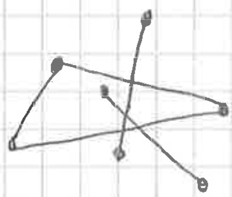


Radon's Theorem.

Remark For $r=2$, we recover Radon's Theorem.

$d=2$, $r=3$ $\Rightarrow$ $2 \cdot 3+1 = 7$ points in $\mathbb{R}^2$





Tverberg's partition

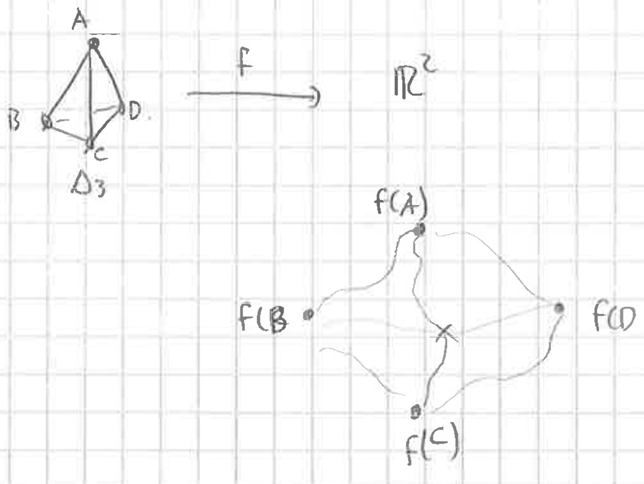
Without proof.

<u>Problems</u> - Efficient algorithm to find Tverberg partitions. Polynomial?
- Topological generalization.

<u>Conjecture</u> (Topological version of Tverberg's theorem)
Let $f : \Delta_m \to \mathbb{R}^d$ be a continuous function from an m-dimensional simplex Δ_m to $\mathbb{R}^d$. If $m \geq (r-1)(d+1)$ then there are r pairwise disjoint faces of Δ_m whose images have a point in common.

Proven when r is prime: Bárány, Shlosman and Szucs
prime power: Özaydin. Volovikov. Sarkaria.
General case is still open.

<u>Example</u> $r=2$, $d=2$ $\Rightarrow$ $m = 1 \cdot 3 = 3$.



$f(AC) \cap f(BD) \neq \emptyset$

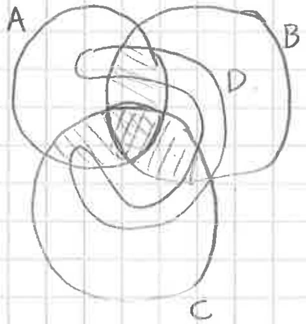
Helly's Theorem

Let $n \geq d+1$ and $C_1, \dots, C_n \subseteq \mathbb{R}^d$ be convex subsets such that any $(d+1)$-subset of them has a non-empty intersection. The

$$\bigcap_{i=1}^{n} C_i \neq \phi$$

Examples: $d=1$, $n=3$

$d=2$, $n=4$



Proof: Induction on n.

$n = d+1$ ✓

Assume $n \geq d+2$. Consider $D_i = \bigcap_{\substack{j \neq i \\ i=1,\dots,n}} C_j$ for $i = 1, \dots, n$

By induction hypothesis: $D_i \neq \phi$.

Let $p_i \in D_i$. By Radon's Theorem, $\{p_1, \dots, p_n\}$ can be partitioned into two sets P_1, P_2 such that $conv(P_1) \cap conv(P_2) \neq \phi$

Let $p \in conv(P_1) \cap conv(P_2)$. We want to show that

$$p \in \bigcap_{j=1}^{n} C_j$$

Equivalently, we want to show $p \in C_j$ for all j.

For $\boxed{j \text{ fixed}}$, $\boxed{p_i \in C_j \text{ for all } j \neq i}$.

Now we consider two cases: $p_j \in P_1$ or $p_j \in P_2$

Case 1: $p_j \in P_1 \Rightarrow p \in conv(P_2) = conv\{\underbrace{p_{i_1}}_{\in C_j}, \dots, \underbrace{p_{i_k}}_{\in C_j}\}$ with $i_1, \dots, i_k \neq j$

$\Rightarrow p \in C_j$

Case 2: $p_j \in P_2 \Rightarrow p \in conv(P_1) = conv\{\underbrace{p_{i_1'}}_{\in C_j}, \dots, \underbrace{p_{i_\ell'}}_{\in C_j}\}$ with $i_1', \dots, i_\ell' \neq j$

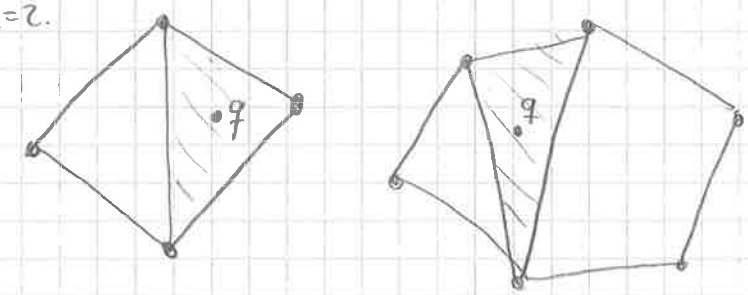
$\Rightarrow p \in C_j$ ∎

Caratheodory's Theorem

For $S \subseteq \mathbb{R}^d$ and $q \in \mathrm{conv}(S)$, there exist $k \leq d+1$ points $p_1, p_2, \dots, p_k \in S$ such that

$$q \in \mathrm{conv}(\{p_1, p_2, \dots, p_k\})$$

Example $d=2$.



Proof If $q \in \mathrm{conv}(S)$, there is a finite convex combination

$$q = \sum_{i=1}^{n} \lambda_i p_i \quad \text{for some } p_i \in S, \ \lambda_i \geq 0, \ \sum_{i=1}^{n} \lambda_i = 1$$

Assume that n is the minimal number s.t. this is possible. (so $\lambda_i > 0$)
We want to show that $n \leq d+1$.

- Assume $n \geq d+2$.

Let $A = \begin{bmatrix} 1 & & 1 \\ p_1 & \cdots & p_n \end{bmatrix}$ with columns $\binom{1}{p_i} \in \mathbb{R}^{d+1}$

and $\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$

We know that $\boxed{A\lambda = \binom{1}{q}}$

But, since $n \geq d+2$, the columns of A are linearly dependent. Then $\exists \mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}$ such that $\boxed{A\mu = 0}$

In particular $\sum_{i=1}^{n} \mu_i = 0$ and at least one $\mu_i < 0$

Let $\overline{\lambda(t)} = \begin{pmatrix} \bar\lambda_1(t) \\ \vdots \\ \bar\lambda_n(t) \end{pmatrix} = \lambda + t\mu$ for $t \in \mathbb{R}_{\geq 0}$ $\Rightarrow \boxed{A\bar\lambda(t) = \binom{1}{q}}$

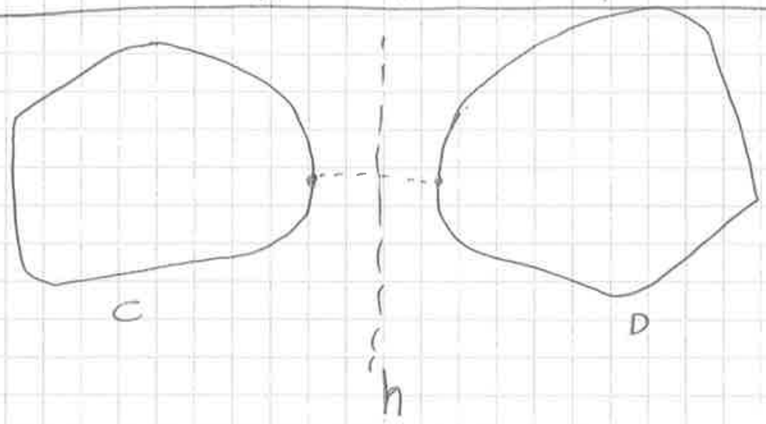
So $\boxed{\sum_{i=1}^{n} \bar\lambda_i(t) = \overbrace{\sum \lambda_i}^{1} + t \overbrace{\sum \mu_i}^{0} = 1}$ and $\boxed{q = \sum_{i=1}^{n} \bar\lambda_i(t) p_i}$

Take the smallest $t \geq 0$ s.t. one of the $\bar\lambda_i(t) = 0$. Then $\bar\lambda_j(t) \geq 0 \ \forall j$, $\bar\lambda_i(t) = 0$

This contradicts the minimality on n ∎

Separation Theorem

Any two compact convex sets $C, D \subseteq \mathbb{R}^d$ with $C \cap D = \emptyset$ can be strictly separated by a hyperplane h, i.e. C and D are on opposite open half-spaces with respect to h



Excercise:

- Prove this theorem
- Find an exemple of two disjoint, non-empty, closed convex sets that can not be strictly separated