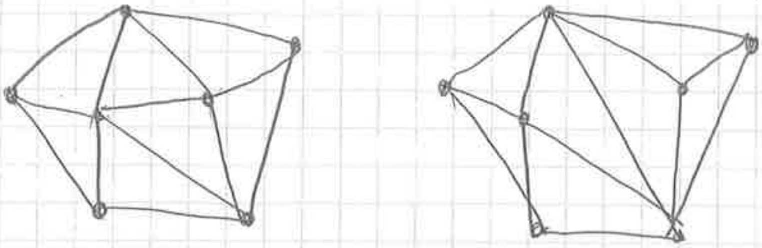
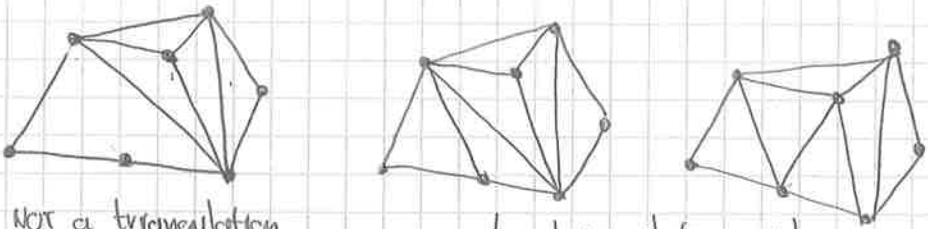
Today : Triangulations of point sets in the plane



Let S be a finite collection of points in the plane.

Definition A triangulation of a planar point set S is a subdivision determined by a maximal set of noncrossing edges whose vertex set is S.

↓ any other edge intersects the interior of at least one edge in the triangulation

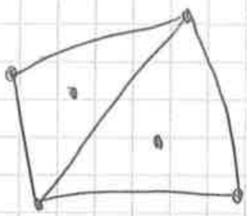


NOT a triangulation ↳ triangulations ↲

Question 1: Is it always possible to triangulate? YES

Existence of triangulations

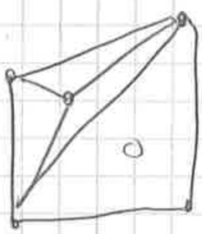
Algorithm 1 : Triangle-Splitting.



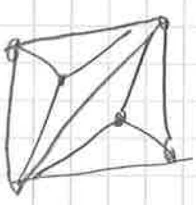
(1) Find the convex hull of S.

(2) triangulate this hull

(3) choose interior point and draw edges to the three vertices of triangle containing it.



(4) Continue process until all interior vertices are exhausted ✓



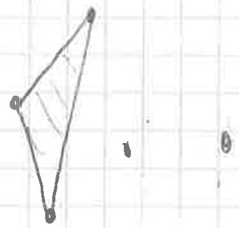
Algorithm 2: Incremental Algorithm

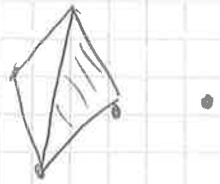
(1) Sort points of S according to their x-coordinates

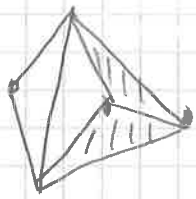
(2) First three points determine a triangle.

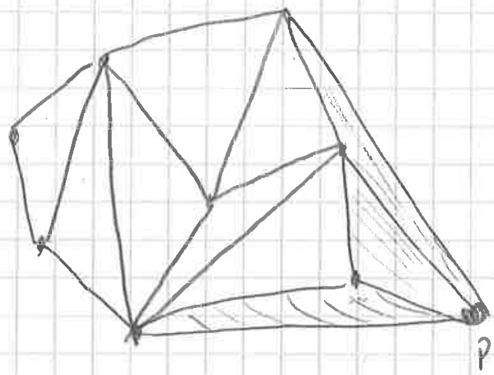
(3) Next point p is connected with previous points which are visible from p.

(4) Continue adding on point at a time until all of S has been processed







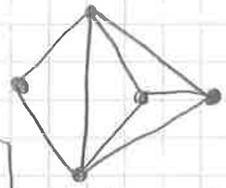


Question 2: Is the number of triangles / edges fixed for all triangulations? YES

Theorem Let $S \subseteq \mathbb{R}^2$ be a point set with h points on the hull and k in the interior, and so $n = k+h$ in total.
If not all points are collinear, then any triangulation of S has exactly

$$2k + h - 2 \quad \text{triangles} \quad \text{, and}$$
$$3k + 2h - 3 \quad \text{edges}$$

Proof based on Euler's Formula.



$h=4$ $\quad 2k+h-2 = 4$ triangles
$k=1$ $\quad 3k+2h-3 = 8$ edges.

Euler's Formula

Let G be a connected planar graph with V vertices, E edges, and F faces on the plane (where the outer face is unbounded). Then

$$V - E + F = 2$$

Proof of Euler's Formula

Induction on the number of edges.

- If $E = 0 \implies G = \bullet \qquad V=1, E=0, F=1 \implies V-E+F=2$ ✓
- If $E \geq 1$, choose an edge e of G.

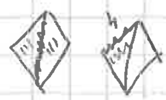
  If  connects two vertices then contract it reducing V and E by one.

  If  is a loop, delete it reducind E and F by one.

  In either case, the new graph remains connecter and planar, and the formula follows by induction ∎

Proof of Theorem

Let t be the number of triangles of a triangulation T of S.
Then, $F = t+1$ faces and $3t+h = 2E$ because each edge is counted twice. So.



$$F = t+1 \qquad E = \frac{3t+h}{2} \qquad V = n = h+k$$

By Euler's Formula:

$$n - \frac{3t+h}{2} + (t+1) = 2$$

$$\implies t = 2n - h - 2 = 2k + h - 2 \quad \text{and} \quad E = \frac{3t+h}{2} = 3k+2h-3$$ ∎

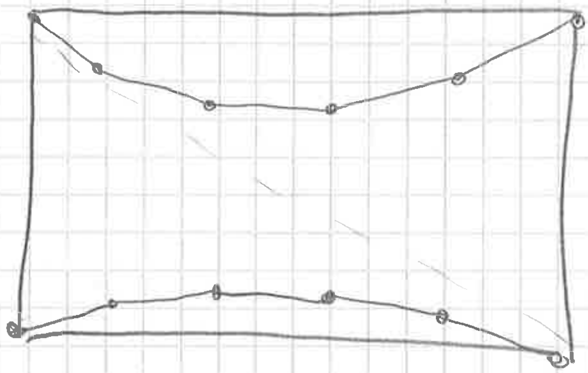
Question 3 : What is the maximal number of triangulations possible if $|S| = n$?

convex position
↳ C_{n-2} triangulations
Catalan many

Answer : Unknown.
Finding a good upper bound seems difficult.

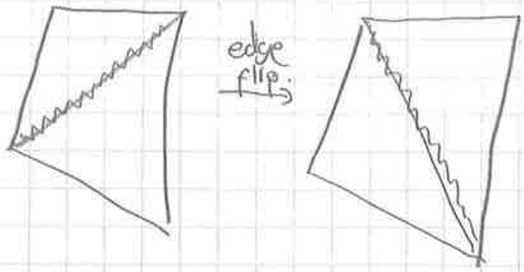
Sharir-Sheffer-Welzl 2009 :
If S is a planar point set of n points, then S has no more than 30^n distinct triangulations.

Exercise : How many distinct triangulations has a "double chain" with n points on each chain ?

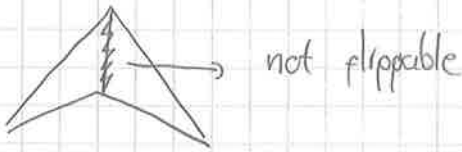


The flip graph

An *edge flip* or just *flip* on a triangulation removes one diagonal and replaces it another (unique) diagonal



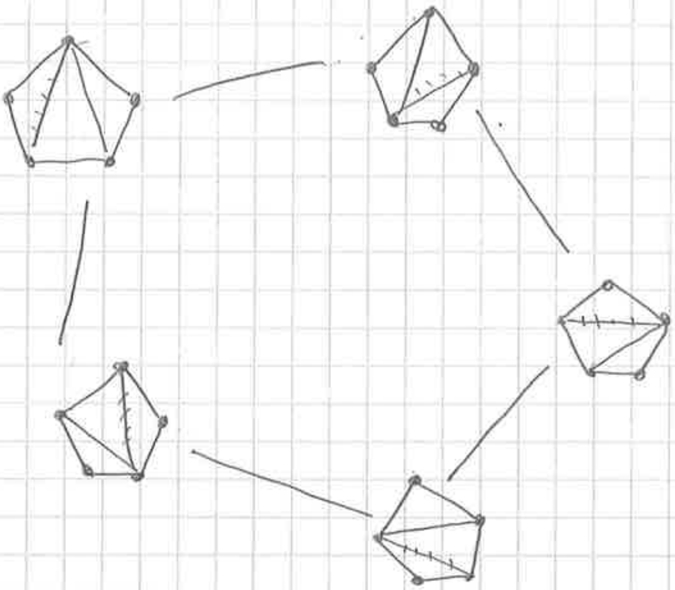
Def. For a point set S, the *flip graph* of S is the graph whose nodes are the triangulations of S. Two nodes T_1 and T_2 are connected by an arc iff they are connected by a flip.



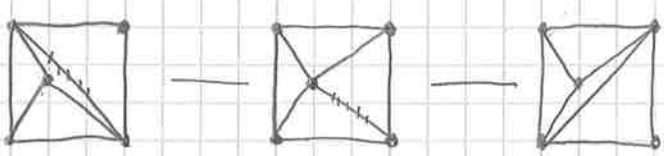
Examples:

(1)

(2)



(3)



Question 4: Is the flip graph connected? YES

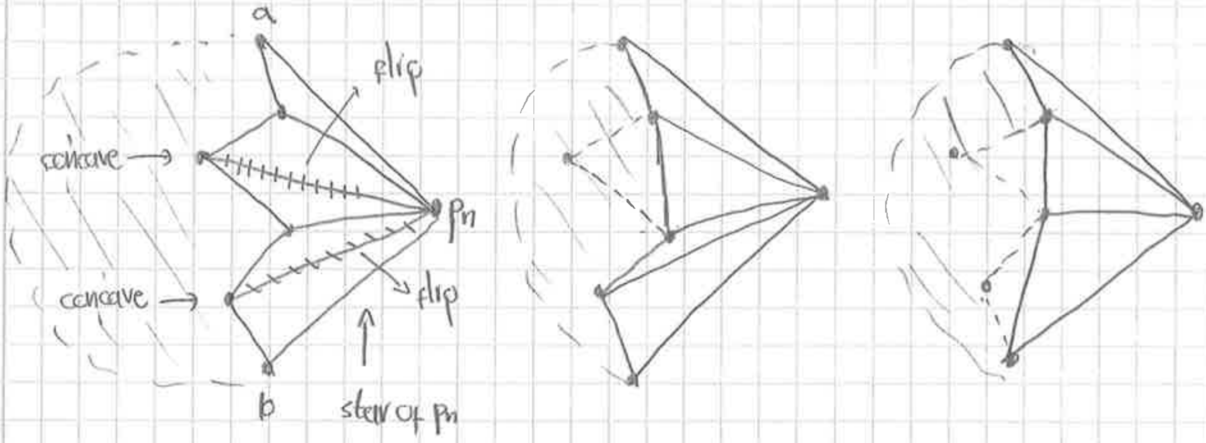
Theorem (Lawson '71)
The flip graph of any point set in the plane is connected

Proof Order the points of S according to their x-coordinate (rotate slightly if necessary)

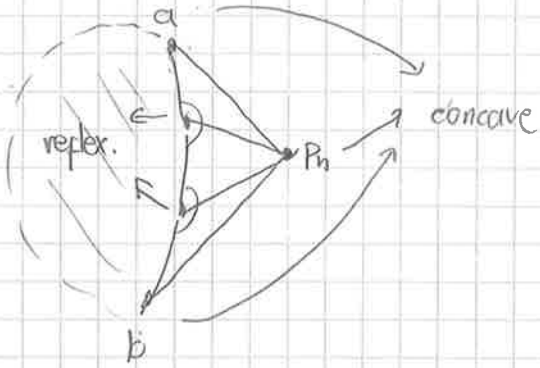
Label the resulting ordering of S $\{p_1, \dots, p_n\}$

Let T_* be the triangulation obtained from the incremental algorithm.

We show that every triangulation T can be connected to T_* via a sequence of flips.



In T_*, the star of p_n looks like



We flip diagonals/edges adjacent to p_n corresponding to concave vertices from top to bottom in order until obtaining a reflex chain connected to p_n.

Once p_n is fixed we continue with $p_{n-1}, p_{n-2}, \dots$
At the end, we get the triangulation T_*. ∎

Corollary For a planar point set S of n points, the diameter of its flip graph is at most $(n-2)(n-3)$

Proof We show that $d(T, T_*) \leq \frac{(n-2)(n-3)}{2}$ for any triang. T.

This follows from the previous proof:
in order to "fix" p_n we need at most $n-3$ flips.
p_{n-1} $n-4$
$\vdots$ $\vdots$

in total: $1 + 2 + \dots + (n-3) = \frac{(n-2)(n-3)}{2}$ ∎

A beautiful nice bound:

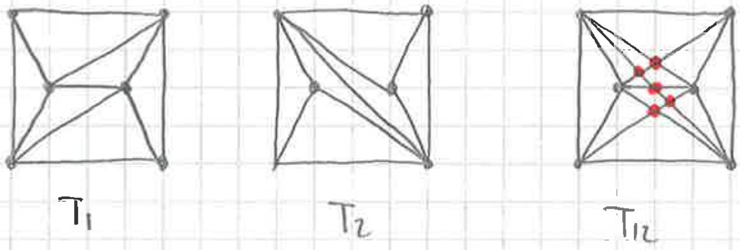
Theorem (Hanke, Ottmann, Schuierer 1996)

Let S be a point set in general position in the plane, and let T_1 and T_2 be two triangulations of S.

Let T_{12} be the result of overlapping T_1 and T_2.

The distance between T_1 and T_2 in the flip graph is at most the number of crossings between edges in T_{12}.

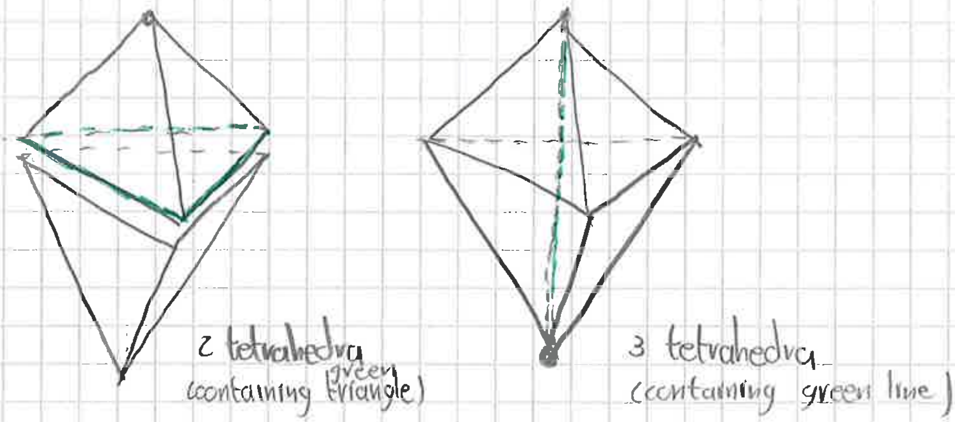
(Without proof)



$d(T_1, T_2) \leq 5$

Remarks (higher dimensions)

(1) The notion of flips can be extended to higher dimensions



(2) Santos 2000: The flip graph is disconnected for point sets in dimension $d \geq 5$

Unknown in dimensions 3 and 4.