Last time : Triangulations of point sets

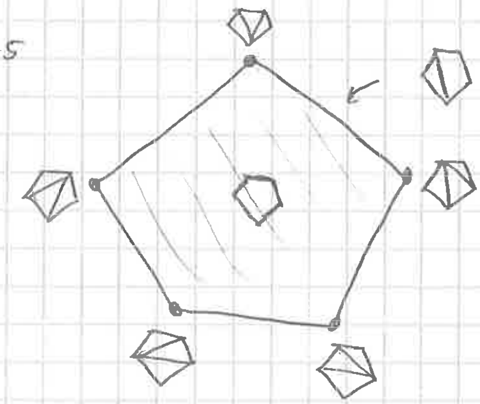
Today : - Triangulations for points in convex position
- Associahedron

- The associahedron

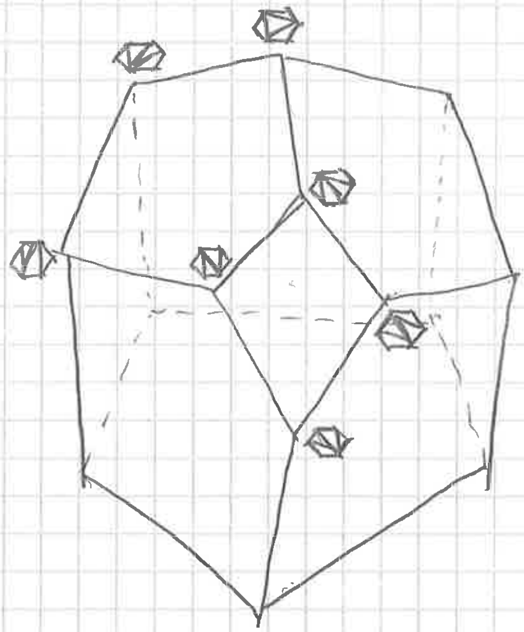
We are interested in the structure of the flip graph of triangulations of a set S of n points in convex position in the plane.

Examples $n = 4$

$n=5$



$n=6$



Theorem

convex hull of finitely many points.

There exists an $(n-3)$-dimensional convex polytope, called the associahedron, whose

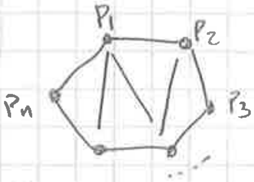
vertices	$\longleftrightarrow$	triangulations of a convex n-gon
edges	$\longleftrightarrow$	flips.
k-dimensional faces	$\longleftrightarrow$	subdivisions using $n-3-k$ diagonals

(without proof)

History of this polytope goes back to Tamari 1950's (Homotopy of H-spaces)

First geometric constructions: Haiman 1984, Lee 1989, ...

There is a nice construction via secondary polytopes by Gelfand-Kapranov-Zelevinsky, which assigns precise coordinates to each triangulation:



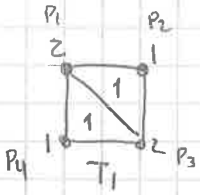
triangulation T

$$\Rightarrow \quad \phi_T(p_i) = \sum_{p_i \in \Delta \in T} \text{area}(\Delta)$$

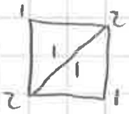
↳ triangles containing p_i

$$\phi(T) = (\phi_T(p_1), \dots, \phi_T(p_n)) \in \mathbb{R}^n$$

Example:



$$\to \quad \phi(T_1) = (2,1,2,1) \in \mathbb{R}^4$$



$$\to \quad \phi(T_2) = (1,2,1,2) \in \mathbb{R}^4$$

And the convex hull of $\{\phi(T_1), \phi(T_2)\}$ is a segment

= flip graph.

Note the difference on dimension!

For $n=5$, although we have 5 points (corresponding to the 5 triangulations) in $\mathbb{R}^5$, their convex hull is a 2-dimensional pentagon in $\mathbb{R}^5$! (Exercise)

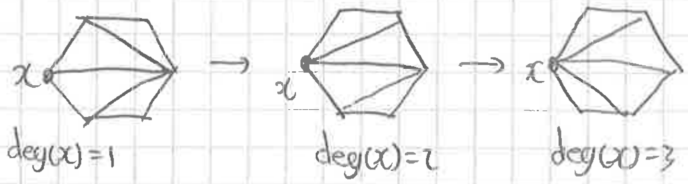
The diameter

We denote by $d(n)$ the diameter of the flip graph of triangulations of a set $S \subseteq \mathbb{R}^2$ of n points in convex position.
So, $d(n)$ = diameter of the $(n-3)$-dimensional associahedron.

Lemma (Sleator-Tarjan-Thurston 1988)

$d(n) \leq 2n - 10$ for all $n > 12$

Proof: Any triangulation of a convex n-gon has $n-3$ diagonals.
Given a vertex x of degree $\deg(x) < n-3$, we can increase $\deg(x)$ by one using a suitable flip.



So we can connect any two triangulations T_1 and T_2 in $n-3-\deg_1(x) + n-3-\deg_2(x) = 2n-6-\deg_1(x)-\deg_2(x)$

$$d(T_1,T_2) \leq 2n-6-\deg_1(x)-\deg_2(x).$$

Now $\sum_x \deg_i(x) = 2(n-3) \Rightarrow$ avarage of $\deg_i(x)$ in T_i is $2 - 6/n$

$\Rightarrow$ avarage of $\deg_1(x) + \deg_2(x)$ is $4 - 12/n$.

If $n > 12$, then there exist a vertex x such that

$$\deg_1(x) + \deg_2(x) \geq 4$$

Thus

$$d(T_1,T_2) \leq 2n-6-4 = 2n-10 \quad \blacksquare$$

Using hyperbolic geometry
Sleator-Tarjan-Thurston: $d(n) = 2n-10$ when n is large enough!

Combinatorial proof: by Pournin

Theorem (Pournin 2014)

$d(n) = 2n-10$ when $n > 12$

Complexity

What is the complexity of finding a shortest flip sequence between two triangulations of a convex polygon?

↳ equivalent to rotations on rooted binary trees
  ↳ central data structure for algorithms

This is a big open problem!

NP-hard? } both?
polynomial time algorithm? }