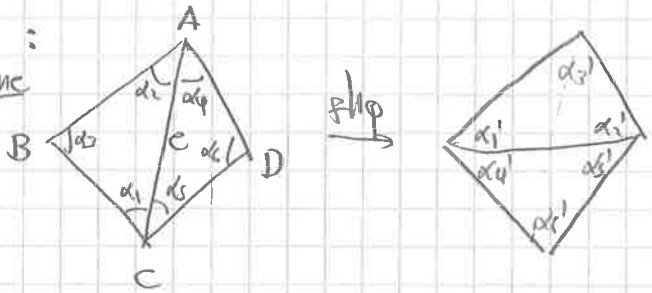
Last time : Angle optimal triangulations (Terrain reconstruction)
Legal triangulations

Today : Voronoi diagrams.

Recall :
from last time



Edge e is <u>illegal</u> if $\min_{i=1,\dots,6} \alpha_i$ increases when flipping e :

$$\min_{i=1,\dots,6} \alpha_i < \min_{i=1,\dots,6} \alpha_i'$$

> **Theorem**
> e is illegal $\iff$ circumcircle(ABC) contains D in its interior

A triangulation is <u>legal</u> if it does not contain any illegal edge.

We can obtain a legal triangulation as follows :

(1) start with some triangulation T
(2) As long as there are illegal edges, flip one of them.

This process terminates because the sequence A(T) of angles (ordered in non-decreasing order) increases at each step, and the number of triangulations is finite.

Questions:

(a) Is there a unique legal triangulation? No
(b) Is a legal triangulation angle optimal? No
} (Exercise)

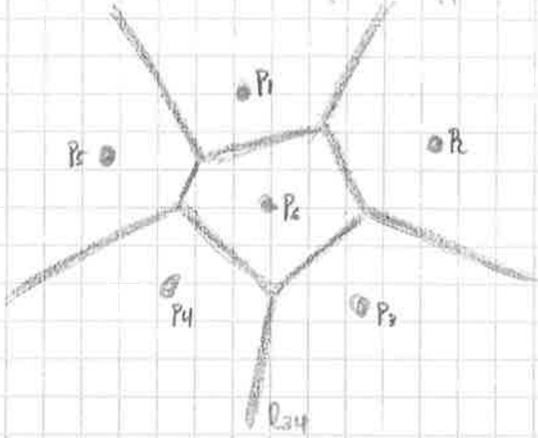
But if no four point lie on a circle then the answer to both questions is yes.

Voronoi diagrams

Motivation: Post office Problem

We are given a finite set of points $P = \{p_1, \dots, p_n\} \subseteq \mathbb{R}^2$ called <u>sites</u> (post offices)

For each point q, we want to know what is the closest site (post office) to it.



That is, find $p \in P$ such that the distance

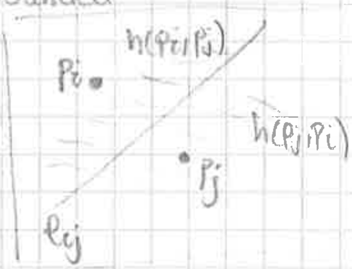
$$\|p - q\| = \sqrt{(p_x - q_x)^2 + (p_y - q_y)^2}$$

is the smallest among P.

The line $\ell_{ij} := \{x \in \mathbb{R}^2 : \|x - p_i\| = \|x - p_j\|\}$ is called the <u>bisector</u> of p_i and p_j.

We denote be $h(p_i, p_j)$ the open half-space bounded by ℓ_{ij} containing p_i.

$h(p_j, p_i)$ is the other open half-space containing p_j

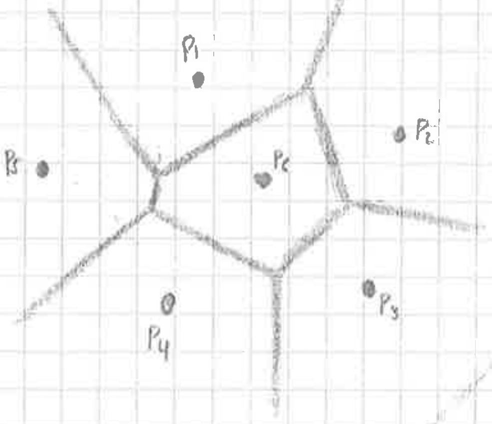


The <u>Voronoi region</u> of p_i is

$$\boxed{\mathcal{V}_P(p_i) := \bigcap_{j \neq i} h(p_i, p_j)}$$

This is the set of points with p_i as the only nearest neighbour.

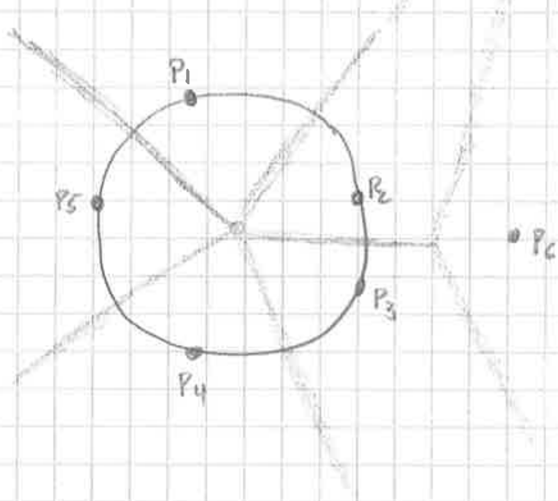
Each $\mathcal{V}_P(p_i)$ is convex (intersection of convex sets) and bounded by line segments / half-lines / lines (portions of the bisectors)



The <u>Voronoi diagram</u> Vor(P)

is the subdivision of the plane whose <u>faces</u> are the Voronoi regions.
The <u>edges</u> are parts of bisectors l_{ij} such that along the edge p_i and p_j are the closest sites (<u>Voronoi edges</u>)

The <u>vertices</u> are points with 3 or more closest sites (<u>Voronoi vertices</u>)



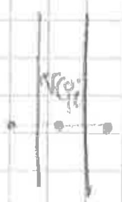
<u>Proposition</u> Let P be a set on n point sites in the plane. If all the sites are colinear then Vor(P) consits of n-1 parallel lines.
Otherwise, the Voronoi edges are either segments or half lines, and the union of all Voronoi edges and vertices is connected

<u>Proof</u> The first part is clear. ✓

For the second part, assume that not all sites of P are collinear.

The Voronoi edges are parts of the bisector lines l_{ij}.
suppose for a contradiction that there is an edge e of Vor(P) that is a full line l_{ij}, bounding the Voronoi regions $\mathcal{V}(p_i)$ and $\mathcal{V}(p_j)$

Let $p_k \in P$ be a point not collinear with p_i and p_j.
Then l_{jk} is not parallel to e, so they intersect.
But the part of e that lies in the interior of $h(p_k, p_j)$ can not be in the boundary of $\mathcal{V}(p_j)$, because it is closer to p_k than p_j.
This is a contradiction.



For connectedness, assume that this was not the case.
Then, there would be a Voronoi region $\mathcal{V}(p_i)$ splitting the plane into two. Since $\mathcal{V}(p_i)$ is convex, it must be a strip bounded by two parallel lines.
This is a contradiction to the previous statement ∎

Let $P = \{p_1, \dots, p_n\}$.

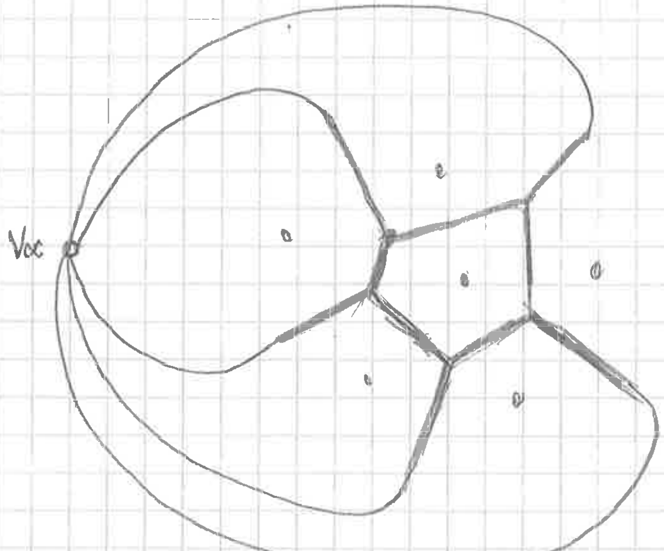
Proposition For $n \geq 3$, the Voronoi diagram $Vor(P)$ has

- n regions
- at most $2n-5$ vertices
- at most $3n-6$ edges

Proof If all sites p_i are collinear ✓ (n regions, 0 vertices, $n-1$ edges)

If not, we prove the result using Euler's formula.

For this, we transform the edge graph of $Vor(P)$ into a planar graph by connecting all half lines to an extra vertex "at infinity" v_∞.



If n_v = # vertices in $Vor(P)$
n_e = # edges in $Vor(P)$

For the resulting planar graph we have

faces = # regions = n
edges = n_e
vertices = $n_v + 1$

By Euler's formula:

$$(n_v+1) - (n_e) + n = 2 \qquad (*)$$

Now, since every vertex has degree ≥ 3, then

$$2n_e \geq 3(n_v+1)$$

By $(*)$ we have

$$\cancel{3}n_e - 3n + 6 = 3(n_v+1) \leq \cancel{2}n_e$$

$$\Rightarrow \quad n_e \leq 3n-6 \qquad (**) \quad ✓$$

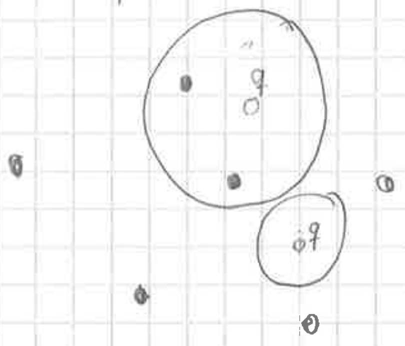
By $(*)$ we also have

$$n_v = n_e - n + 1 \overset{(**)}{\leq} (3n-6) - n + 1 = 2n-5 \quad ✓$$

∎

Characterization of edges and vertices of Vor(P)

Now, we want a characterization of the edges and vertices of the Voronoi diagram.



Let $P = \{p_1, \dots, p_n\}$
We say that a circle is empty if no site lies in its interior.

For $q \in \mathbb{R}^2$, let $C_P(q)$ be the largest empty circle centered at q with respect to P.

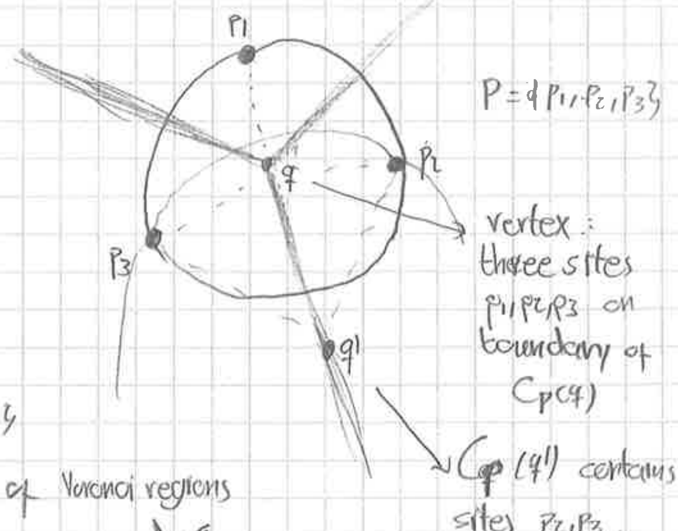
Theorem For the Voronoi diagram Vor(P) of a set of points P the following hold:

(i) A point q is a vertex of Vor(P) if and only if its largest empty circle $C_P(q)$ contains three or more sites on its boundary

(ii) The bisector l_{ij} contains an edge of Vor(P) if and only if there is a point q on the bisector such that $C_P(q)$ contains p_i and p_j on the boundary and no other site.

Proof

Observe that site s is on the boundary of $C_P(q)$ iff s is closest site to q.



(i) $(\Leftarrow)$ For the backward direction consider $q \in \mathbb{R}^2$ such that $C_P(q)$ has three or more sites on its boundary $\{p_i, p_j, p_k, \dots\}$

Then q is on the boundary of Voronoi regions $\mathcal{V}(p_i), \mathcal{V}(p_j), \mathcal{V}(p_k), \dots$

So q must be a vertex of Vor(P)

$(\Rightarrow)$ On the other hand, if q is a vertex of Vor(P), it belongs to at least three edges, hence to at least three Voronoi regions $\mathcal{V}(p_i), \mathcal{V}(p_j), \mathcal{V}(p_k), \dots$ and no other site is closer to q.

Hence the interior of the circle centered at q with p_i, p_j, p_k on its boundary does not contain any site.
This circle is $C_P(q)$

($\Leftarrow$)

(ii) Let q be a point such that $C_P(q)$ contains p_i, p_j on its boundary but no other site.

Then $q \in l_{ij}$ and.

$$\| q - p_i \| = \| q - p_j \| < \| q - p_k \|$$

for any other site $p_k \neq p_i, p_j$. So q lies on an edge of $Vor(P)$ defined by the bisector l_{ij}.
By part (i) q can not be a vertex.
Hence, q lies on the edge of $Vor(P)$ defined by l_{ij}.
This shows the backward direction of statement (ii)

($\Rightarrow$) Conversely, assume that the bisector l_{ij} defines a Voronoi edge. The largest empty circle of any point q in the interior of this edge must contain p_i and p_j on its boundary and no other sites