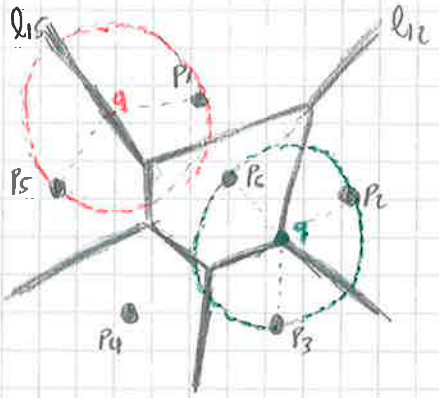
Last time : Voronoi diagrams.

Today : Delaunay triangulations



Recall:

Let $P = \{p_1, ..., p_n\}$ be a set of n points in the plane, called sites. Let $Vor(P)$ be its Voronoi diagram.

For $q \in \mathbb{R}^2$, let $C_P(q)$ be the largest empty circle centered at q with respect to P (no p_i in its interior)

<u>Theorem</u> (characterization of vertices and edges of $Vor(P)$)

q is a vertex of $Vor(P)$ $\iff$ $C_P(q)$ contains three or more sites on its boundary

l_{ij} defines an edge of $Vor(P)$ $\iff$ $\exists q \in l_{ij}$ such that $C_P(q)$ contains p_i and p_j on its boundary and no other site

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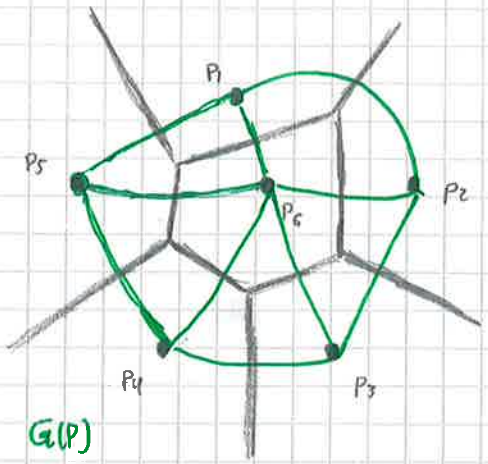
↳ Relevant result for Delaunay triangulations.

Delaunay triangulations

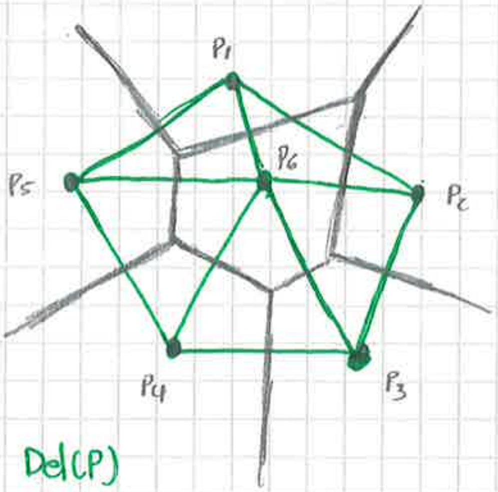


Now, we are interested in the dual graph of the Voronoi diagram $Vor(P)$: $G(P)$

Vertices of $G(P)$: regions of $Vor(P)$ (or sites $p_1, ..., p_n$)

Edges of $G(P)$: two vertices are connected by an arc if the corresponding regions share an edge

This means that $G(P)$ has an arc for every edge of $Vor(P)$



$P = \{p_1, \dots, p_n\}$

The Delaunay graph Del(P) is the straight-line embedding of G(P)

Vertices : $p_1, \dots, p_n$

$\overline{p_i p_j}$ is an edge $\Leftrightarrow$ bisector l_{ij} defines an edge of Vor(P)

Observe : in this example, Del(P) is a plane graph, i.e. no two edges in the embedding cross.

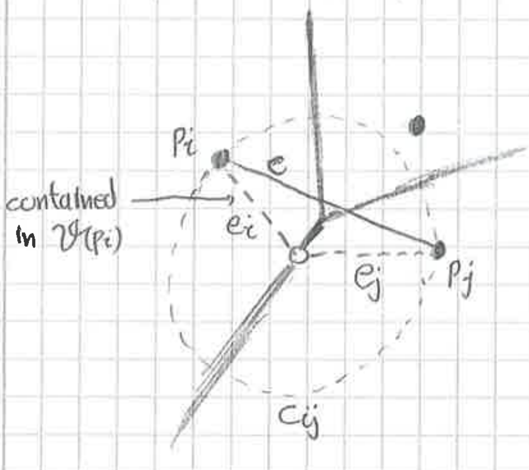
This is always the case!

Theorem

The Delaunay graph Del(P) of a planar point set P is a plane graph

Proof : By the characterization of edges in Vor(P)

$e = \overline{p_i p_j}$ is an edge of Del(P) $\Leftrightarrow$ $\exists$ a closed disc C_{ij} with p_i and p_j on its boundary and no other site of P contained in it (its center lies on the common edge of $\mathcal{V}(p_i)$ and $\mathcal{V}(p_j)$)



Let t_{ij} be the triangle with vertices p_i, p_j and the center of C_{ij}

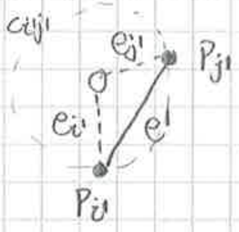
Its edges are $e = \overline{p_i p_j}$ and e_i, e_j

where e_i connects p_i with the center of C_{ij}
e_j " p_j " " "

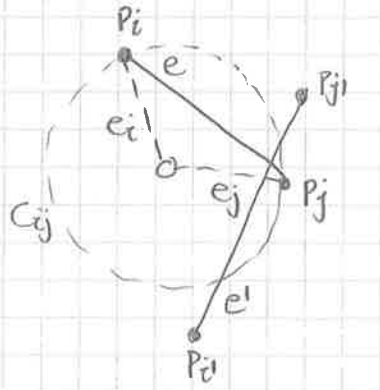
Observe that e_i is contained in the Voronoi region $\mathcal{V}(p_i)$
e_j " " " " " $\mathcal{V}(p_j)$

Now, suppose for a contradiction, that there is another edge $e' = \overline{p_{i'} p_{j'}}$ of Del(P) such that e and e' cross in their interior

Let $t_{i'j'}$, $e_{i'}$, $e_{j'}$ as before



Again
$e_{i'}$ is contained in $\mathcal{V}(p_{i'})$
$e_{j'}$ " " " $\mathcal{V}(p_{j'})$



Now, e' crosses e, but its end points $p_{i'}, p_{j'}$ have to be outside of C_{ij}.

Thus e' intersects one of e_i, e_j

Viceversa

e intersects one of $e_{i'}, e_{j'}$

Thus (by a simple case study), one of e_i, e_j must intersect one of $e_{i'}, e_{j'}$

This intersection point would be in the interior of two Voronoi regions (because the edges $e_i, e_j, e_{i'}, e_{j'}$ are contained in different regions)

This is a contradiction ∎

- Duality

We have the following duality between the Voronoi diagram $Vor(P)$ and the Delaunay graph $Del(P)$:

$$\begin{pmatrix} \text{Vertices of } Del(P) \\ p_1, \dots, p_n \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \text{Regions of } Vor(P) \\ \mathcal{V}(p_1), \dots, \mathcal{V}(p_n) \end{pmatrix}$$

$$\begin{pmatrix} \text{edges} \\ \overline{p_i p_j} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \text{edges} \\ \text{defined by bisector } l_{ij} \end{pmatrix}$$

$$\begin{pmatrix} \text{faces} \\ \text{K-gon } conv(p_{i_1}, \dots, p_{i_k}) \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \text{vertices} \\ q \text{ shared in regions } \mathcal{V}(p_{i_1}), \dots, \mathcal{V}(p_{i_k}) \end{pmatrix}$$

In particular, if $p_{i_1}, \dots, p_{i_k}$ are the vertices of a face on $Del(P)$ then they are on the boundary of the largest empty circle $C_P(q)$ where q is the corresponding vertex in $Vor(P)$, and no other site is inside nor on the boundary of $C_P(q)$.

Rephrasing the characterization theorem for vertices and edges of the Voronoi diagram, we obtain:

Theorem Let $P = \{p_1, \dots, p_n\}$ be a set of points in the plane

(i) Three points $p_i, p_j, p_k \in P$ are vertices of the same face of $Del(P)$ iff the circle through p_i, p_j, p_k contains no point of P in its interior.

(ii) Two points $p_i, p_j \in P$ form an edge in $Del(P)$ iff there is a closed disk C that contains p_i, p_j on its boundary and does not contain any other point of P