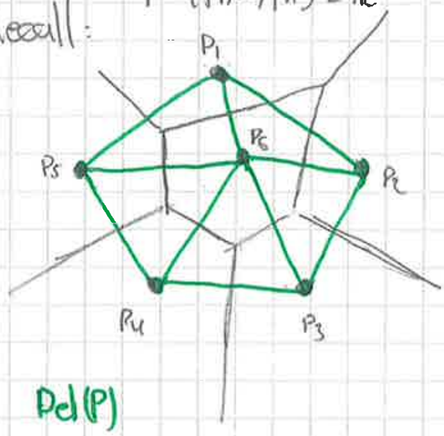
Last time : Delaunay graph

Today : Delaunay triangulations
The parabolic lift

Recall: $P = \{p_1, \dots, p_n\} \subseteq \mathbb{R}^2$

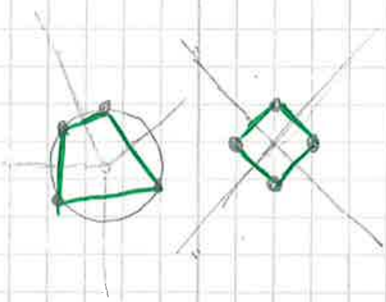


Del(P)

The Delaunay graph Del(P) is the straight-line embedding of the dual graph of the Voronoi diagram Vor(P)

We have the following duality dictionary:

Duality:



Del(P)	Vor(P)
Vertices $p_1, \dots, p_n$	Regions $\mathcal{V}(p_1), \dots, \mathcal{V}(p_n)$
Edges $\overline{p_i p_j}$	Edges defined by bisector ℓ_{ij}
Faces k-gon $\mathrm{conv}\{p_{i_1}, \dots, p_{i_k}\}$	Vertices point q shared by regions $\mathcal{V}(p_{i_1}), \dots, \mathcal{V}(p_{i_k})$

Rephrasing the characterization theorem for vertices and edges of Vor(P):

Theorem Let $P = \{p_1, \dots, p_n\}$ be a set of points in the plane. Then

(i) Three points $p_i, p_j, p_k \in P$ are vertices of the same face of Del(P) iff the circle through p_i, p_j, p_k has no point of P in its interior.

(ii) Two points $p_i, p_j \in P$ form an edge in Del(P) iff there is a closed disk that contains p_i, p_j on its boundary and does not contain any other point of P.

- No four points on a circle

Observe that if no four points of P lie on a circle (general position) then all the vertices of Vor(P) have degree 3

So, all the faces of Del(P) are triangles

In this case

Del(P) is called the Delaunay triangulation of P

- General case

In the general case, we have to be a bit more careful.

We define a Delaunay triangulation to be any triangulation obtained by adding edges to the Delaunay graph Del(P).

(Since all faces of Del(P) are convex polygons, this is easy)

with vertices on a circle.

Theorem Let $P = \{p_1, \ldots, p_n\}$ be a set of points in the plane.

A triangulation $\mathcal{T}$ of P is a Delaunay triangulation iff the circumcircle of any triangle does not contain any point of P in its interior

Theorem A triangulation $\mathcal{T}$ of P is legal iff $\mathcal{T}$ is a Delaunay triangulation of P

Proof (Exercise).

Corollary If no four points of P lie on a circle (general position)

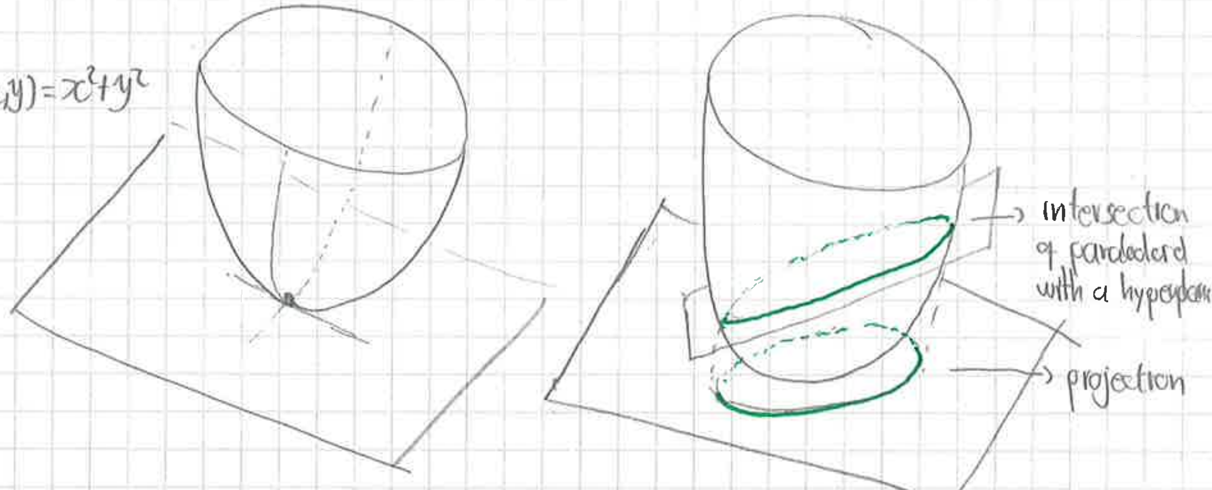
Then the Delaunay triangulation Del(P) is the unique legal triangulation of P. Hence, it is the unique angle-optimal triangulation.

The parabolic lift

Goal: Obtain the Delaunay triangulation (or subdivision determined by the Delaunay graph for points no in general position) as the projection as the lower hull of a polytope in one higher dimension.

Let $P = \{p_1, \dots, p_n\}$ be a set of points in the plane.
Consider the paraboloid $z = x^2 + y^2$

$f(x,y) = x^2 + y^2$



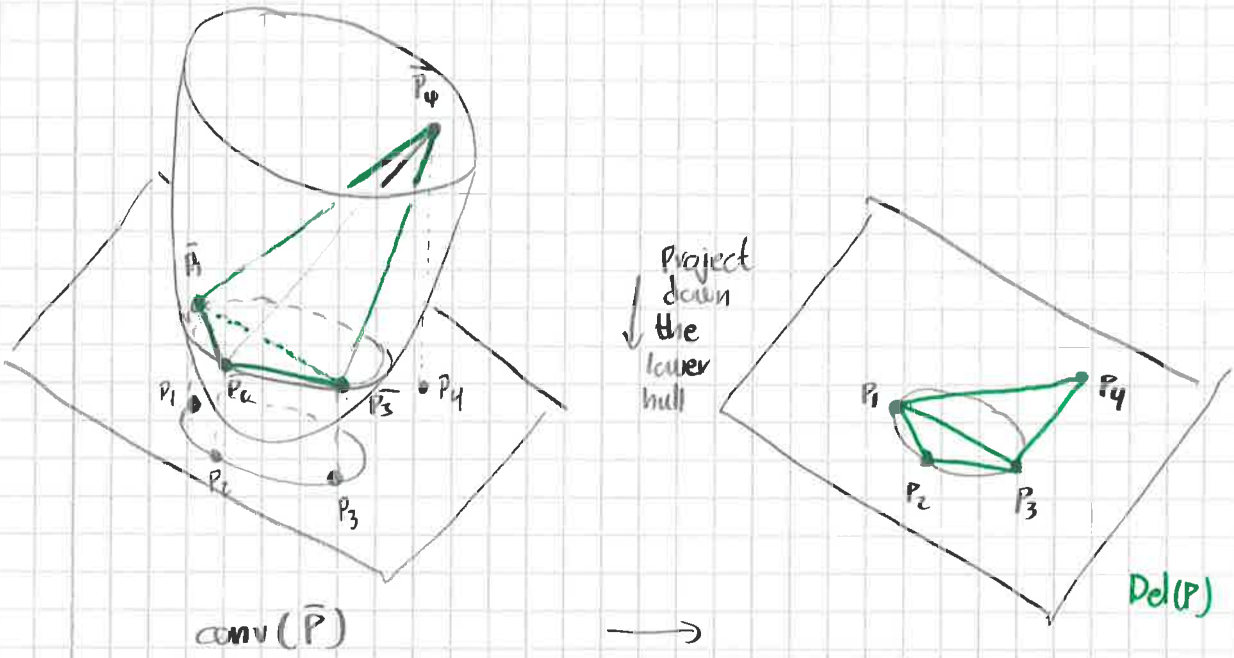
Interesting property of the paraboloid: If we intersect it with a plane a project the resulting curve to the xy-plane then we obtain a circle!

Moreover, the points in the paraboloid above the plane project outside the circle, and the points below the hyperplane project inside the circle. (see proof of theorem below)

For $p = (x,y) \in P$, let $\bar{p} = (x, y, x^2+y^2)$ be the corresponding lifted point.
Let $\bar{P} = \{\bar{p}_1, \bar{p}_2, \dots, \bar{p}_n\}$ and $conv(\bar{P})$ be the convex hull of the lifted points in $\mathbb{R}^3$ (a 3-dimensional polytope)
The lower hull of $conv(\bar{P})$ is the set of faces of $conv(\bar{P})$ that "you can see from below", i.e such that $conv(\bar{P})$ is above the face

Theorem

The subdivision determined by the Delaunay graph Del(P) (The Delaunay triangulation if no four points are in circle) is the projection of the lower hull of conv($\bar{P}$)



Proof

Idea: projection of the intersection of the paraboloid with a plane is a circle in the xy-plane

In order to show this, consider the tangent plane of the paraboloid at the point $(x_0, y_0, x_0^2+y_0^2)$

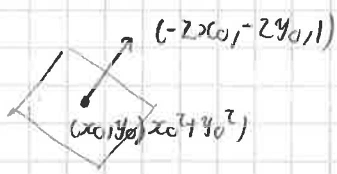
Two vectors spanning this plane are

$$\text{"}\frac{\partial}{\partial x_0}\text{"} \longrightarrow (1, 0, 2x_0)$$

$$\text{"}\frac{\partial}{\partial y_0}\text{"} \longrightarrow (0, 1, 2y_0)$$

The normal vector is

$$(-2x_0, -2y_0, 1)$$



The equation of the tangent plane is

$$-2x_0(x-x_0) - 2y_0(y-y_0) + (z - x_0^2 - y_0^2) = 0$$

$$\Updownarrow$$

$$\boxed{z = 2x_0 x + 2y_0 y - x_0^2 - y_0^2}$$

Now, a plane E intersecting the paraboloid is parallel to some tangent plane.

Thus the equation of E is of the form

$$z = 2x_0 x + 2y_0 y - x_0^2 - y_0^2 + r^2$$

The intersection of E with the paraboloid $z = x^2 + y^2$ is

$$x^2 + y^2 = 2x_0 x + 2y_0 y - x_0^2 - y_0^2 + r^2$$

$$\Updownarrow$$

$$\boxed{(x-x_0)^2 + (y-y_0)^2 = r^2}$$ the equation of a circle

Hence, the projection of the faces of the lower hull of $conv(\bar{P})$ satisfy the property that

p_i, p_j, p_k are a (projected) face $\iff$ the circle throug p_i, p_j, p_k has no point of P in its interior

Thus the projection of the lower hull is the subdivision determined by the Delaunay graph $Del(P)$.

In the case of general position (no four points of P in a circle) this is the unique Delaunay triangulation.

$\blacksquare$

Algorithmic Consequence

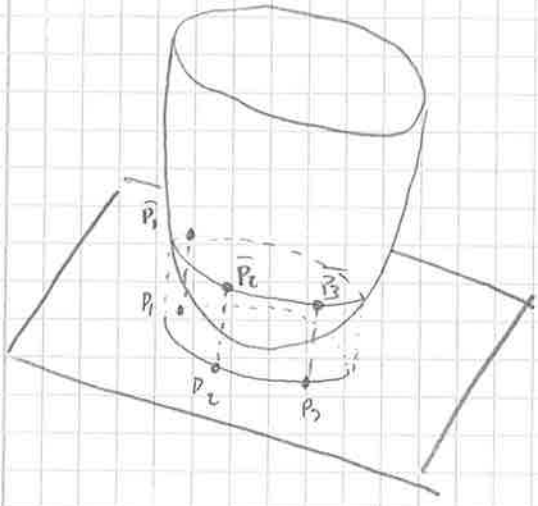
Since convex hulls in 3D can be computed in $O(n \log n)$ time (for example using a Divide-and-Conquer algorithm ◦ Exercise).

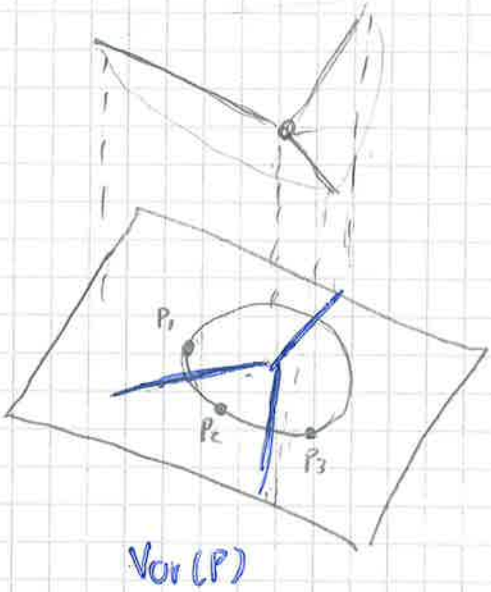
Then, we can compute Delaunay triangulations in $O(n \log n)$ time. (as well as Voronoi diagrams)

Remark: Duality between Delaunay triangulations and convex hulls holds in higher dimensions.

Application: Solid meshing of 3D objects using Delaunay tetrahedralizations.

Questions
How can we obtain the Voronoi diagram using the parabolic lift?





Answer

Consider the tangent planes of the paraboloid at the lifted points $\overline{p_1}, \dots, \overline{p_n}$

Look at these hyperplanes and their intersections from above. $(z = +\infty)$

The projection of the upper envelope (pieces not obscured by any other plane) yield the Voronoi diagram.