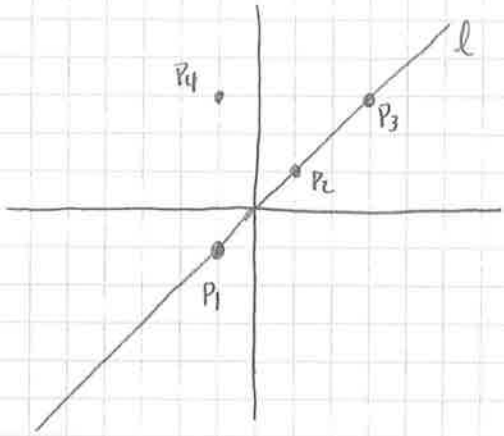
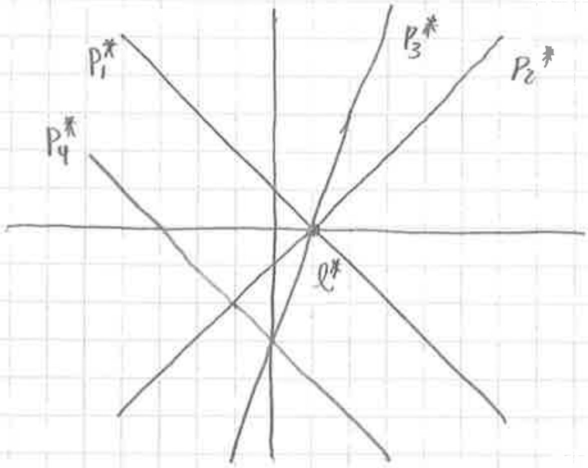
Today : line arrangements and duality

duality.

Primal plane. $\longleftrightarrow$ Dual plane.





lines ℓ $\longleftrightarrow$ points ℓ^*
points p $\longleftrightarrow$ lines p^*

point set $\longleftrightarrow$ arrangement of lines

Information/properties about the point set can be translated to information/properties about the arrangement of lines (and viceversa)

This duality is often very useful, ie: - Computer graphics (computing discrepancies of random point samples)

Duality :

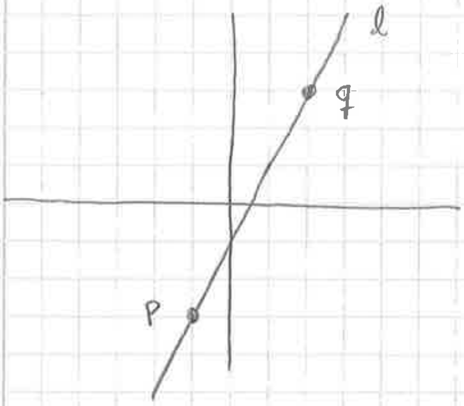
$\boxed{\begin{array}{l}\text{Points in } \mathbb{R}^2 \\ p=(a,b)\in\mathbb{R}^2\end{array}}$ $\longrightarrow$ $\boxed{\begin{array}{l}\text{non-vertical lines} \\ \ell: \{(x,y): \; y = mx+n\} \text{ with } (m,n)\in\mathbb{R}^2\end{array}}$

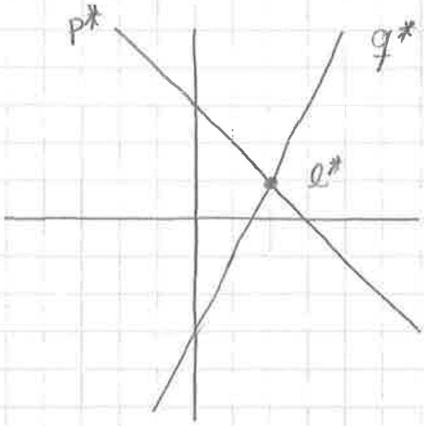
infinitely many bijections.

Standard duality :

$$\boxed{\begin{array}{lcl} p=(a,b) & \xrightarrow{\quad D \quad} & D(p)=p^* : \; y = ax-b \\ \ell: \; y=mx+n & & D(\ell)=\ell^* = (m,-n) \end{array}}$$

Example:





$l: y = 2x - 1$
$p = (-1, -3)$
$q = (2, 3)$

$D(l) = l^* = (2, 1)$
$D(p) = p^*: \; y = -1 \cdot x + 3$
$D(q) = q^*: \; y = 2 \cdot x - 3$

This duality has nice properties:

(1) $D^2 = Id$.

$p = (a, b) \longrightarrow D(p) = p^*: y = ax - b \longrightarrow D^2(p) = D(p^*) = (a, b)$

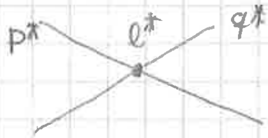
$l: y = mx - n \longrightarrow D(l) = l^* = (m, n) \longrightarrow D^2(l) = D(l^*): y = mx - n$

(2) D is incidence preserving: $\underset{\text{point } p \text{ is on line } l}{p \in l} \iff \underset{\text{point } l^* \text{ is on line } p^*}{l^* \in p^*}$

$p \in l \iff b = ma - n \iff n = am - b \iff (m; n) \in \text{line } y = ax - b$

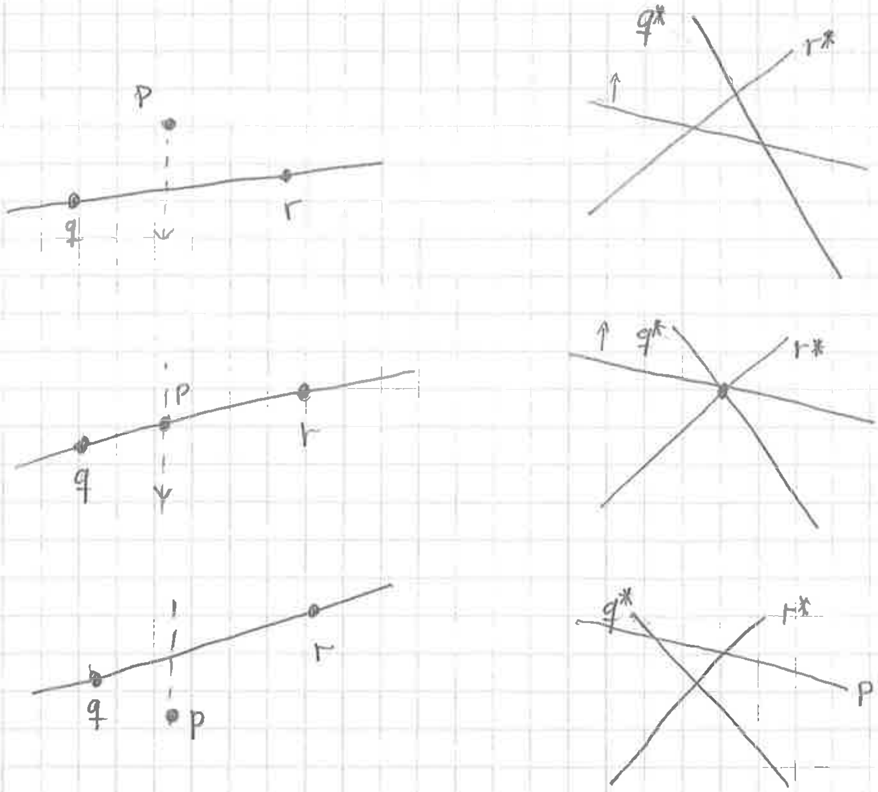
$\iff l^* \in p^*$

(3) The line l through two points p, q corresponds to the intersection point l^* between the two lines p^*, q^*



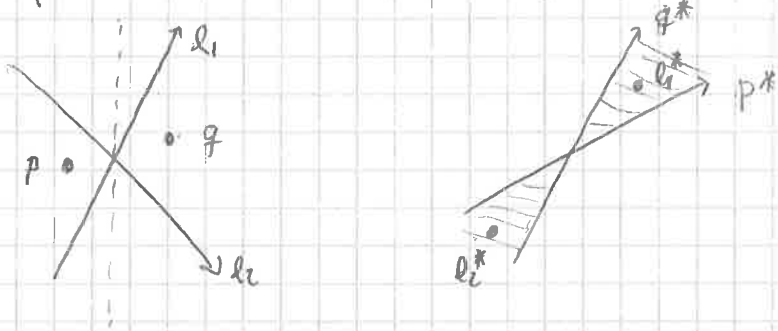
(4) D is order preserving: (Exercise)
A point p is above the line l iff the point l^* is above the line p^*

move p down preserving x-coordinate

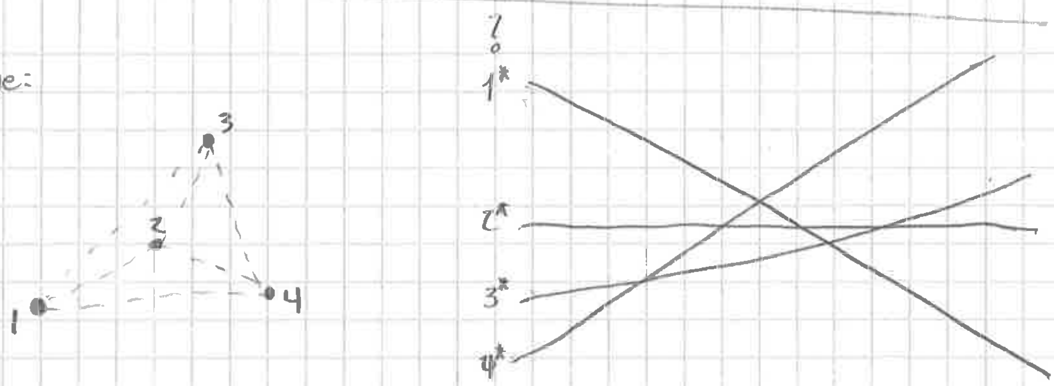


move p* up preserving slope

(5) A line that separates two points p and q corresponds to a point that has one of p and q above and the other below



Challange:

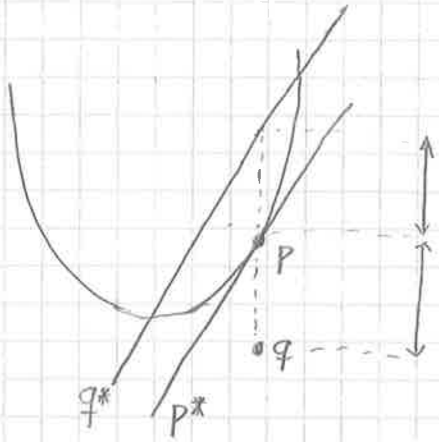


who is who?

Parabola interpretation of duality

$$p = (a,b) \xrightarrow{D} D(p) = p^* : \ y = ax - b$$
$$\ell : \ y = mx + n \qquad D(\ell) = \ell^* = (m, -n)$$

Consider the parabola $y = x^2/2$



(1) If p is a point on the parabola, then p^* is the tangent line to the parabola at p.

(2) For a point q, let p its vertical projection to the parabola.

Then, q^* is the line parallel to p^* at a vertical distance opposite to the distance from p to q

Proof: Exercise.