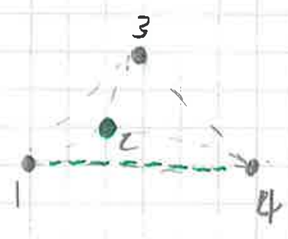
Last time: Line arrangement and duality

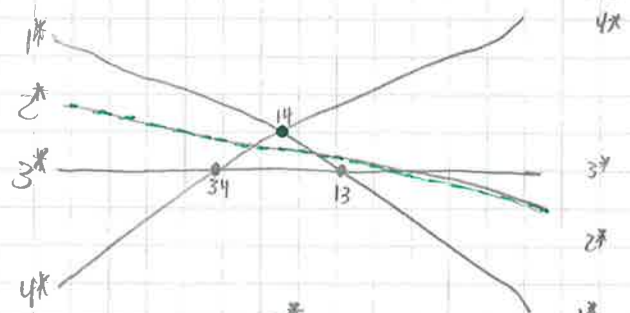
Today : Duality : Convex hulls and envelopes
Pseudoline arrangements & Wiring diagrams
Triangulations for points in convex position.

Recall: duality

point $p=(a,b)\in\mathbb{R}^2$	$\overset{D}{\longleftrightarrow}$	line $D(p)=p^*: y=ax-b$
non-vertical line $\ell: y=mx+n$	$\longleftrightarrow$	point $D(\ell)=\ell^*=(m,-n)$



P



P^*

Properties :	$p\in\ell$	$\Leftrightarrow$	$\ell^*\in p^*$
	p below ℓ / above	$\Leftrightarrow$	ℓ^* below p^* / above

An arrangement is called simple if no three lines meet at a point
$\Updownarrow$ dual
point sets in general position (i.e. no three points on a line)

For instance:

2 is above line $\overline{14}$ $\Leftrightarrow$ $\overline{14}^*$ is above line 2^*

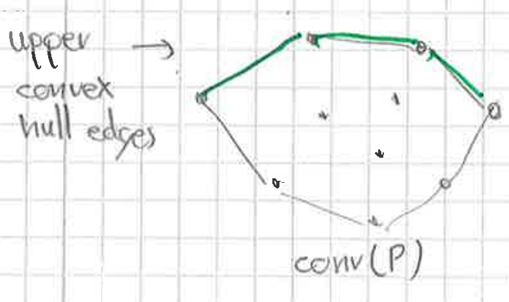
These properties allow us to make a nice interpretation of the convex hull edges.

Convex hulls and envelopes

Let $P\subseteq\mathbb{R}^2$ be a finite set of points and P^* be its dual arrangement of lines.

Let $p,q\in P$

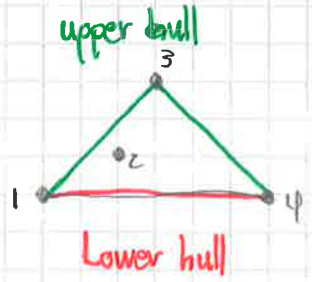
Note: The edge pq is an upper convex hull edge

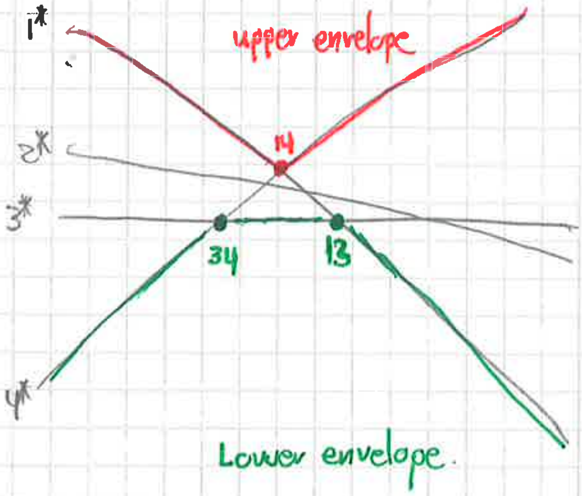


$\Leftrightarrow$ Every other point $r\in P$ is below line $\overline{pq}$

$\Leftrightarrow$ The intersection point of the lines p^* and q^* ($\overline{pq}^*$) is below any other line r^*

$\Leftrightarrow$ $\overline{pq}^*$ is in the "lower envelope" of the line arrangement





$\overline{pq}$ is an upper hull edge	$\Longleftrightarrow$	$\overline{pq}^*$ is a point in the lower envelope

similarly

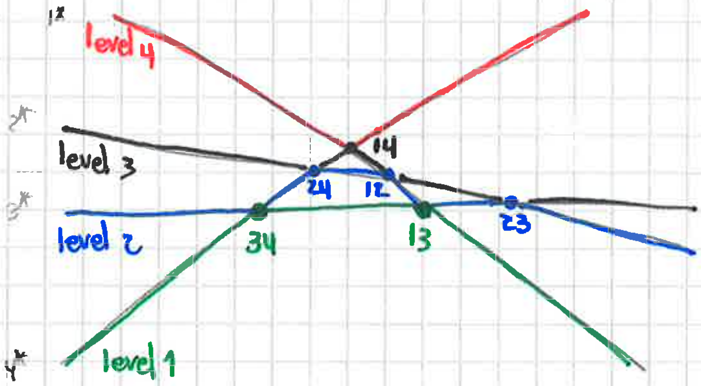
$\overline{pq}$ is a lower hull edge	$\Longleftrightarrow$	$\overline{pq}^*$ is a point in the upper envelope

Computing conv(P)

$\Updownarrow$

Computing Upper and lower envelope of the line arrangement.

In general, we can assign levels to the arrangement:

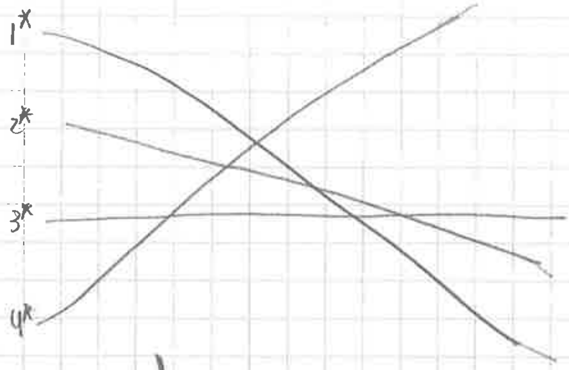


level 1 :	$\overline{34}^*$, $\overline{13}^*$	dual lines	$\overline{34}$, $\overline{13}$:	0 points above
level 2 :	$\overline{24}^*$, $\overline{12}^*$, $\overline{23}^*$		$\overline{24}$, $\overline{12}$, $\overline{23}$:	1 " "
level 3 :	$\overline{14}^*$		$\overline{14}$:	2 points above

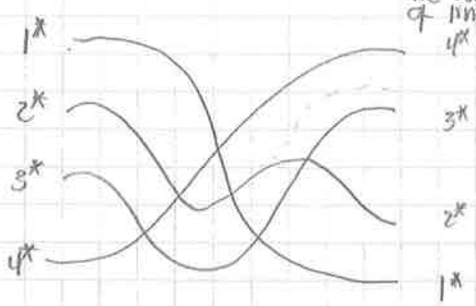
Property: points in level K that are not in level K-1 correspon to lines in the primal having K-1 points of P above.

Line arrangements vs pseudoline arrangements

- Two pseudolines intersect once.
- important condition the "out" order of lines is reversed



line arrangement.
(straight lines)



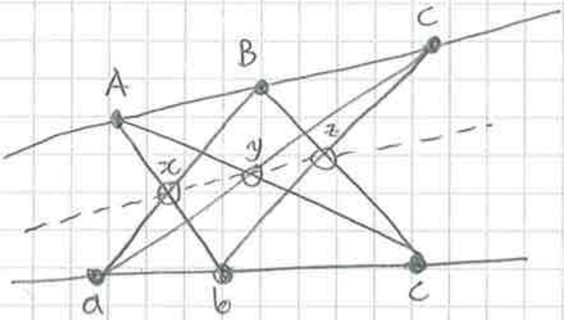
pseudoline arrangement.
(pseudolines are not necessarily straight lines)

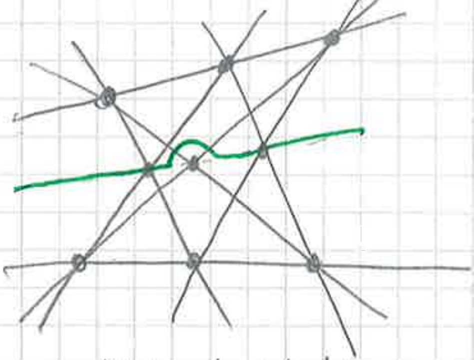
line arrangements usually need a lot of space to be drawn.
while the pseudoline arrangement saves space while keeping the combinatorial properties of the arrangement.

Question: Can every pseudoline arrangement be stretched to a line arrangement?

Goodman & Pollak '80 : Every arrangement of at most 8 pseudolines is stretchable

Richter '88 : Every simple arrangement of at most 9 pseudolines is stretchable except the simple non-Pappus arrangement.





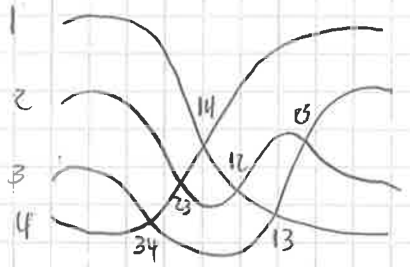
Not stretchable

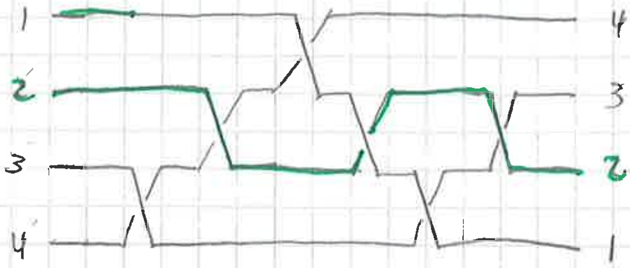
Pappus's Theorem:
A, B, C collinear
a, b, c collinear
then
x, y, z collinear
where
$x = Ab \cap aB$
$y = Ac \cap aC$
$z = Bc \cap bC$

$\rightsquigarrow$ Pseudoline arrangements are more general.

Pseudoline arrangements vs wiring diagrams

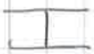
A simple way to represent pseudoline arrangements is via their wiring diagram

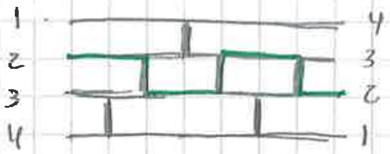




order is reversed

Another simple way to represent is via its sorting network:

replace  by 



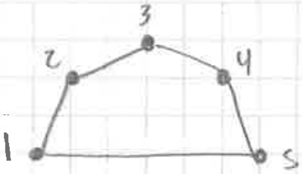
We can think of vertical segments, called "commutators", as bridges. When we walk along line i and encounter a bridge we cross it.

The commutators are naturally labeled by the pair of lines crossing at it:

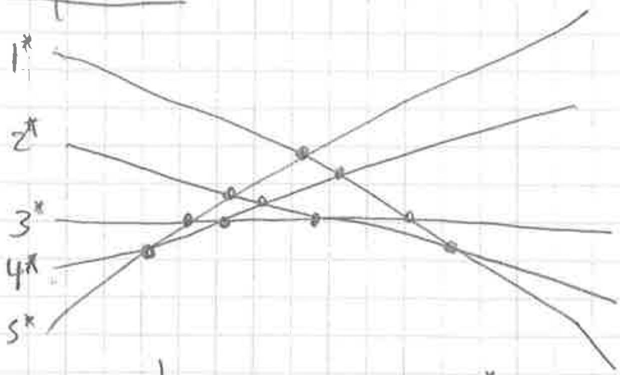
point 12 is above pseudoline 3

Advantage: Figures of sorting networks are much simpler than those of line arrangements and they keep their combinatorial information. (i.e. position of intersection with respect to pseudolines)

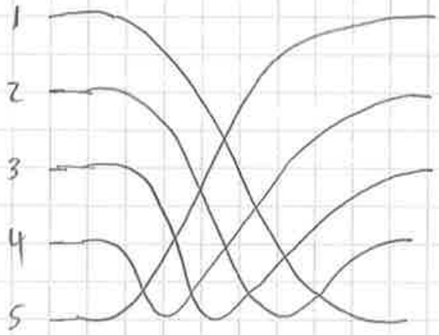
Revisiting points in convex position



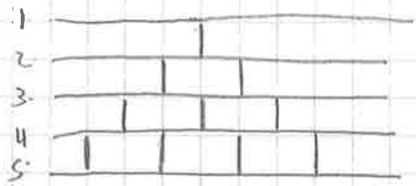
point set P
in convex position



line arrangement P^*

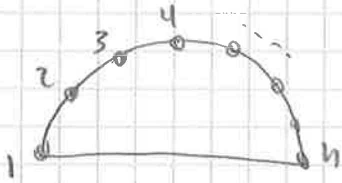


pseudoline arrangement

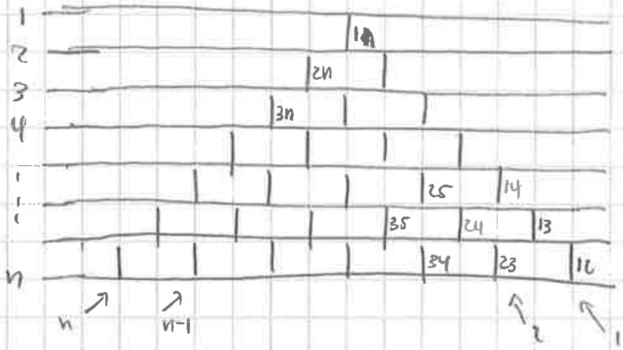


Sorting network

In general, sorting network of → triangular shape.



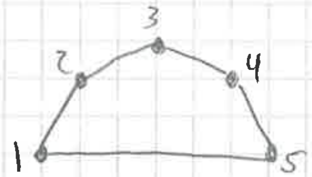
P



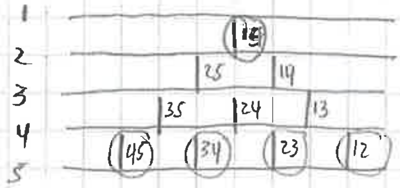
Question: What is the interpretation of the dual of a triangulation of a convex polygon?

diagonals
(including boundary edges) → points of pseudoline arrangement
⇕
commutators of the sorting Network

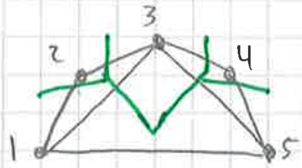
point set P in convex position



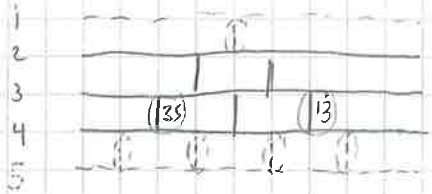
Networ N_P

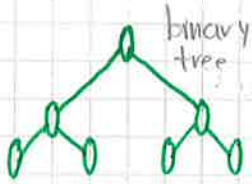


boundary edges: remove first and last level.



triangulation T





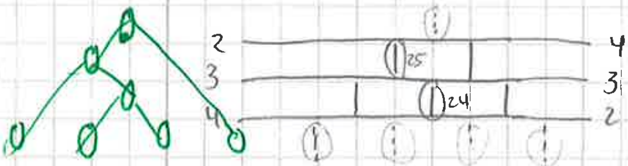
binary tree

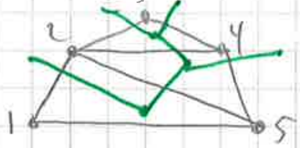
remove commutators / bridges corresponding to edges / diagonals of the triangulation

Walking along remaining commutators / bridges reverses the permutation $2, 3, --, n-1$

reversed

Viceversa, any choice of commutators to be removed, such that the result is a pseudoline arrangement reversing the permutation $2, 3, --, n-1$, supported at N_P after removing first and last level corresponds to a triangulation (N_P^{*1})





Binary tree rotated 180°

pseudoline arrangements supported at N_P^{*1} — Bijection — Triangulations of convex polygon conv(P)

Let N_P^{*1} be the network obtained from N_P by removing the first and last levels

Theorem (Pilaud-Pocchiola 2012, following Pocchiola-Vegter '96)

Let P be a set of points in convex position in the plane.
T is a triangulation of P $\Leftrightarrow$ T^* (the dual of the edges of T) removing from N_P is a pseudoline arrangement supported at N_P^{*1}

(i.e. The permutation $2, 3, --, n-1$ is reversed
Any two pseudolines intersect exactly once)