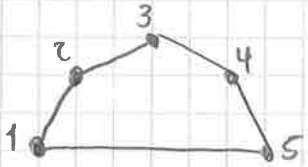
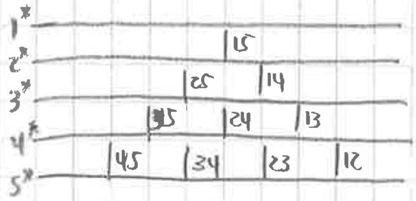
Today: Pseudotriangulations

Recall from last time:

For a set P of points in convex position, we have an <u>interpretation of triangulations of P</u> <u>in terms of pseudoline arrangements</u> supported at $\mathcal{N}_P^{*1}$



point set P



Network $\mathcal{N}_P$

Network $\mathcal{N}_P^{*1}$
(removing first and last levels)

edge $\overline{ij}$ $\longleftrightarrow$ commutator ij

(more precisely $\overline{ij}^*$)
simplify notation: ij

internal diagonal $\overline{ij}$ of conv(P) $\longleftrightarrow$ commutator ij in $\mathcal{N}_P^{*1}$

Theorem (Pilaud–Pocchiola 2012, based on Pocchiola–Vegter 1996)
Let P be a set of points in convex position.

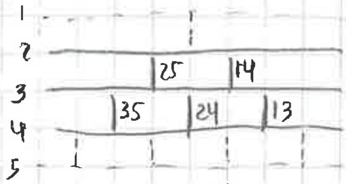
(1) A set T of <u>edges</u> of P is a triangulation iff (including boundary edges)
removing commutators T^* (dual to T) from $\mathcal{N}_P$ yields a pseudoline arrangement supported at $\mathcal{N}_P^{*1}$

Equivalently,

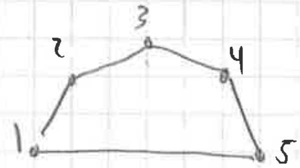
(2) A set T of <u>diagonals</u> of P forms a triangulation iff
removing commutators T^* from $\mathcal{N}_P^{*1}$ yields a pseudoline arrangement.

(i.e. each pair of pseudolines crosses exactly once, hence the exit permutation is reversed: $n-1, \dots, 3, 2$.)

Example:



N_P^{*1}

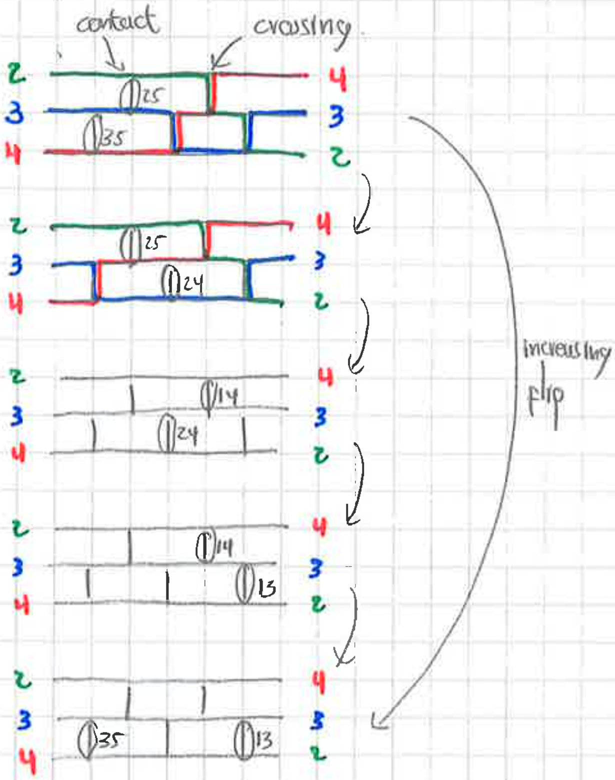


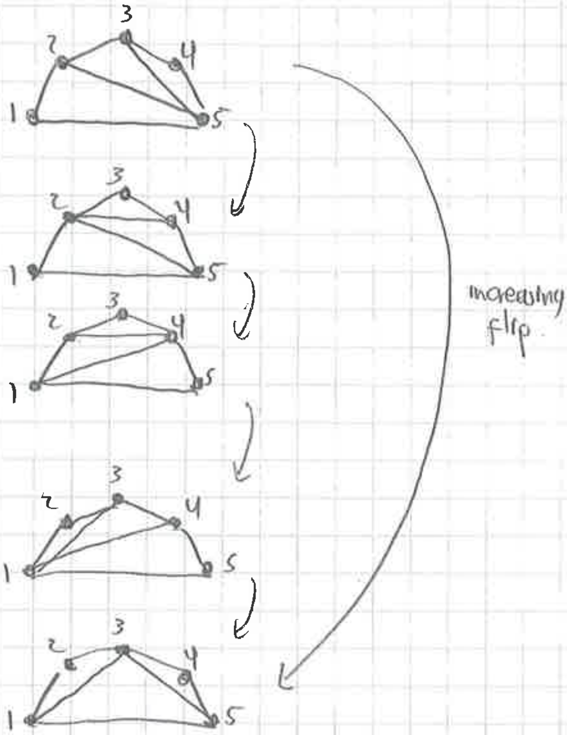
P

5 pseudoline arrangements supported at N_P^{*1}

5 triangulations of P

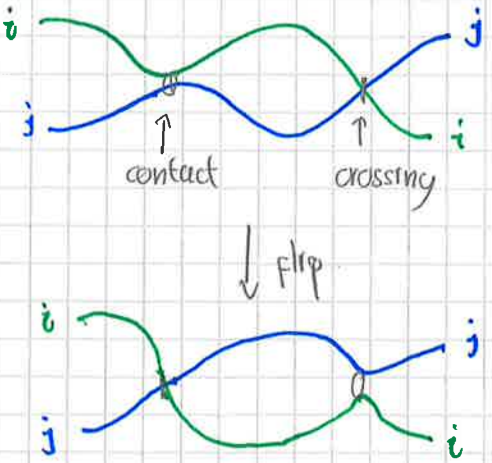
remove circled "bridges" cross all others while you walk





Question: What is the dual interpretation of flips on triangulations?

We call a commutator ⓘ that is "removed" a contact
the other commutators | that are kept are crossings



Every contact between two pseudolines i & j can be replaced by the corresponding crossing between ij (this crossing is unique!)

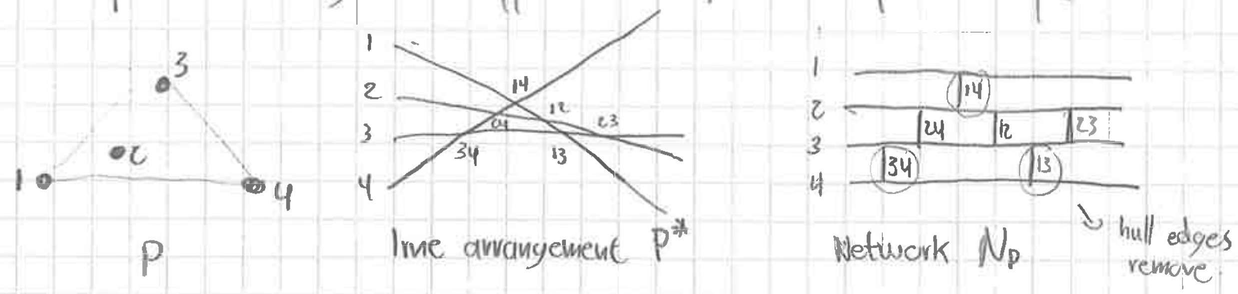
The result is a new pseudoline arrangement. This operation is called a <u>flip</u>.

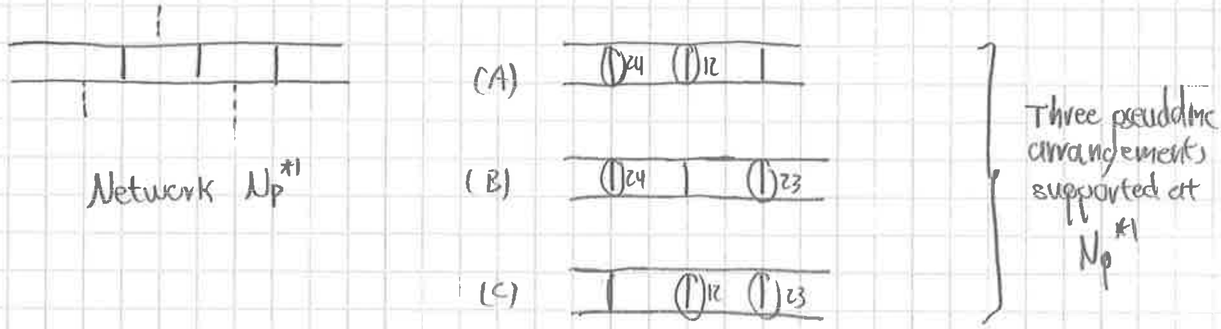
The flip is <u>increasing</u> if the contact is on the left of the crossing and decreasing otherwise.

Observe: Increasing flips on pseudoline arrangements supported at N_P^{*1} $\Longleftrightarrow$ increasing diagonal flips on triangulations of P.

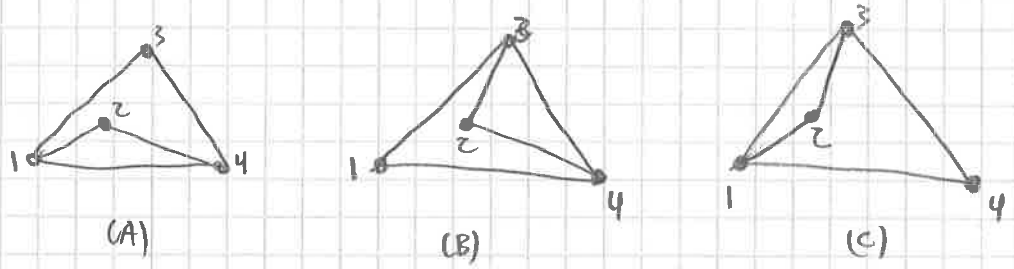
Questions: • What happens if P is not in convex position?

• Do pseudoline arrangements supported at N_P^{*1} have a primal interpretation?





Corresponding primal collections of edges

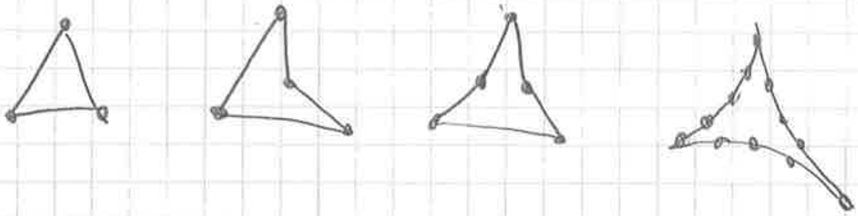


What are they? NOT triangulations!

- Pseudotriangulations

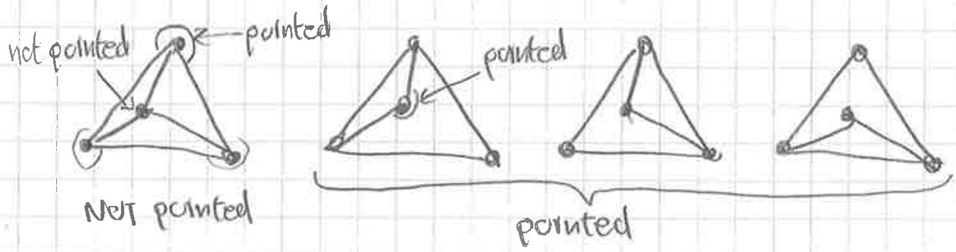
Let $P \subseteq \mathbb{R}^2$ be a finite set of points in the plane.

A pseudotriangle is a polygon with exactly 3 convex vertices:



A pseudotriangulation of P is a subdivision into pseudotriangles.

all hull vertices are pointed



A vertex is pointed if one of its angles is bigger than 180°

A pseudotriangulation is pointed if all its vertices are pointed

Note: Not all pseudotriangulations have the same number of triangles!

Theorem: A pseudotriangulation of a point set P with p pointed vertices and q nonpointed vertices has $p+2q-2$ pseudotriangles and $2p+3q-3$ edges

Corollary: Every pointed pseudotriangulation of a point set P with n points has $n-2$ pseudotriangles and $2n-3$ edges.

Proofs: Exercise

↳ Pointed pseudotriangulations behave well!

What about flips?

Theorem (Flips)

(1) For a pointed pseudotriangulation T of P, there is exactly one flip for each interior edge of T, i.e. $\exists e'$ such that $T \setminus e \cup e'$ is again a pointed pseudotriangulation.

(2) The flip graph of pointed pseudotriangulations is connected.

Proof: Exercise.