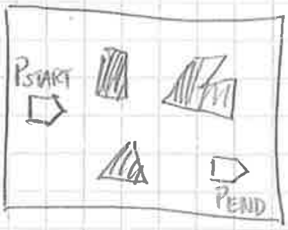
Last time : Robot Motion Planning
Today : Complexity of Minkowski sums

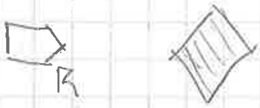
Last time:



We observed:

(1) Complexity of motion planning is related to the complexity of the free space C_{FREE} (or equivalent the forbidden space)

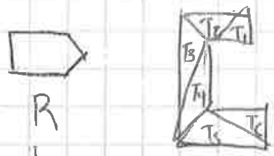
(2) Determined the complexity of the forbidden space $CP = P \oplus (-R)$ for one convex robot R and one convex obstacle P.



Theorem

R, P convex polygons with m, n edges resp. $\Rightarrow$ $P \oplus R$ is a convex polygon with at most $m+n$ edges

Next step: R convex, P not convex



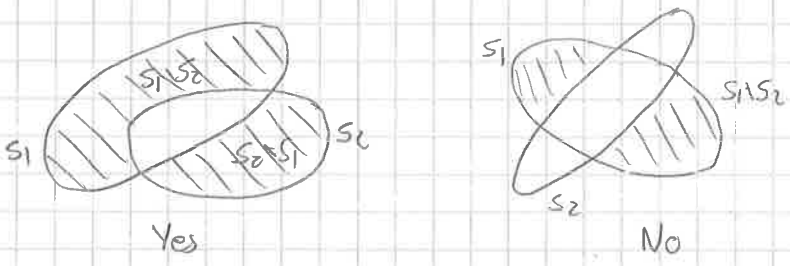
R → m edges

P → n edges

Question: What is the complexity of $P \oplus R$?

Idea: - triangulate $P \Rightarrow T_1, \dots, T_t$
- compute $T_1 \oplus R \cup T_2 \oplus R \cup \dots \cup T_t \oplus R$

We say that a pair (S_1, S_2) of convex sets has the <u>pseudodisc property</u> if $S_1 \setminus S_2$ and $S_2 \setminus S_1$ are connected



A collection of convex sets has the <u>pseudodisc property</u> if every pair has.

<u>Observation</u>: For any pair of convex sets with the pseudodisc property, there are at most two <u>proper</u> intersection points of their boundaries

Lemma: If $P_1, \dots, P_n$ are interior disjoint convex polygons and R is a convex polygon, then

$$P_1 \oplus R,\ P_2 \oplus R,\ \dots,\ P_n \oplus R$$

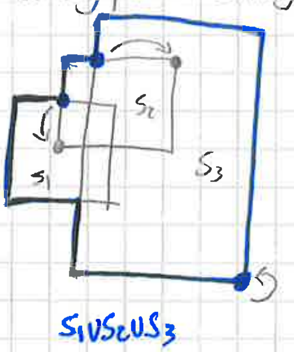
has the pseudodisc property

(without proof).

Proposition: Let S be a collection of convex polygons with the pseudodisc property having n edges in total, then the complexity of their union is $O(n)$.
# of vertices (or # of edges)

Proof: We want to count # vertices of the union.

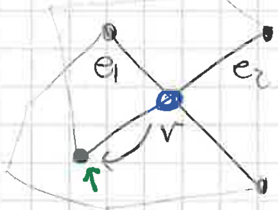
Strategy: Charge every vertex of the union to a vertex of some polygon in S, such that every vertex is charged at most twice.



Two types of vertices in the union:

(1) Vertices from a polygon in S: charge to itself

(2) Intersections of two edges of polygons in S.



e_1 edge of polygon P_1
e_2 edge of polygon P_2

claim at least one end point of e_1 or e_2 lies in the interior of $P_1 \cup P_2$.

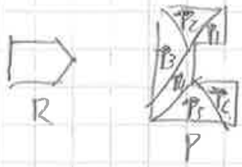
Proof: If both end points of e_1 are outside of P_2 then e_1 crosses the boundary of P_2 twice.
But then, because of the pseudodisc property, e_2 can not go through another such intersection point.

Charge v to end point in the interior.
Each point is charged at most twice, once per incident edge ∎

10/9

Theorem Let R be a convex polygon with m vertices, and P be an arbitrary polygon with n vertices. Then $P \oplus R$ has $O(m \cdot n)$ vertices

Proof: Let $T_1, \dots, T_{n-2}$ be a triangulation of P



Then

$$P \oplus R = \bigcup_{i=1}^{n-2} T_i \oplus R.$$

Since $T_1, \dots, T_{n-2}$ are convex interior disjoint then

$T_1 \oplus R, \dots, T_{n-2} \oplus R$ have the pseudodisc property (by Lemma)

Furthermore, $T_i \oplus R$ has at most $m+3$ vertices

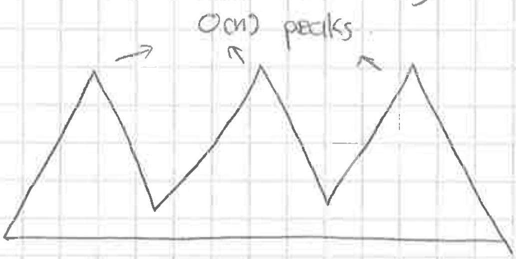
So, the total number of edges of the collection is at most $(n-2)\cdot(m+3)$

Thus, by proposition, the complexity of the union is

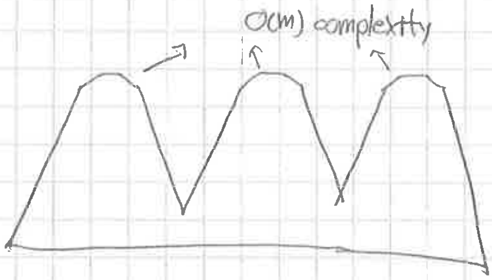
$$O((n-2)\cdot(m+3)) = O(n \cdot m)$$

Observe: The bound is tight:

$P =$



$R =$  m segments



Total complexity $O(m \cdot n)$