Curvature theory based on parallel meshes

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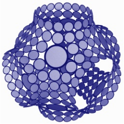
DFG Research Unit “Polyhedral Surfaces”

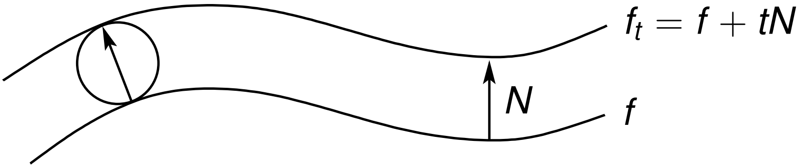


This talk. Curvature of parallel meshes

based on Pottmann, Liu, Wallner, Bobenko, Wang [SIGGRAPH '07]

- Quadrilateral Surfaces (discrete parametrized surfaces) with line congruences = Geometric support structures
- Curvatures
- Discrete Minimal Surfaces
- Discrete Constant Mean Curvature Surfaces
- Generalizations (projective geometry, relative geometry)

Curvature via parallel surfaces



Steiner's formula

$$A(f_t) = \int (1 - 2Ht + Kt^2) dA(f)$$

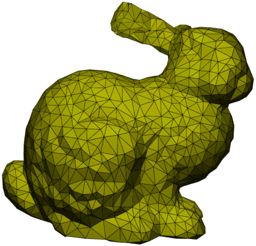
H mean curvature, K Gaussian curvature of f, t small.

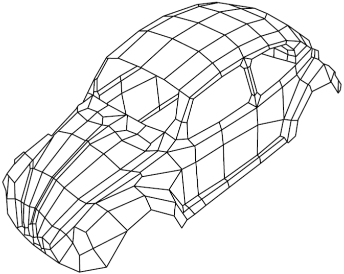
Previous work

Curvature for discrete surfaces via Steiner's formula

- Steiner's formula for simplicial surfaces [Nishikawa, Jinnai, Koga, Hashimoto, Hyde '98,'01]
- Steiner's formula for circular surfaces [Schief '06]

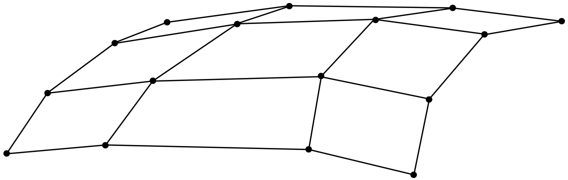
Quadrilateral Surfaces





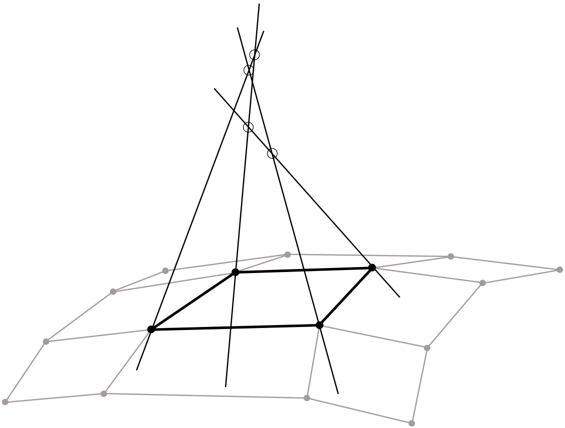
Quadrilateral surfaces as discrete parametrized surfaces

Quadrilateral Surfaces



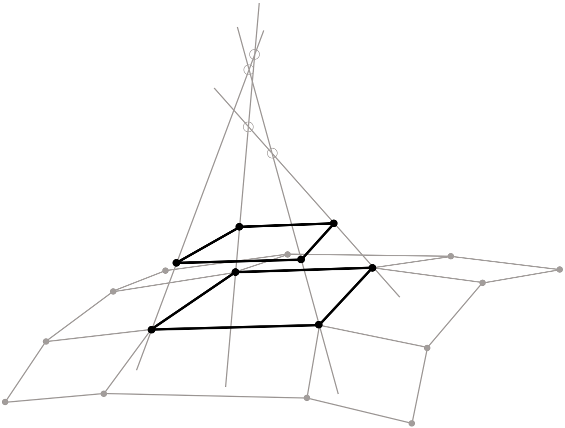
Quadrilateral surface

Quadrilateral Surfaces



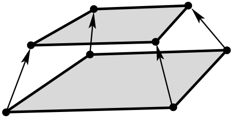
Quadrilateral surface with line congruence = Geometric support structure

Quadrilateral Surfaces



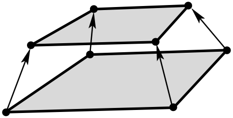
Line congruence net with parallel surfaces

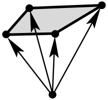
Quadrilateral Surfaces



A quad of a line congruence net

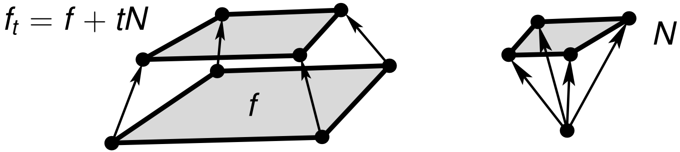
Quadrilateral Surfaces





A quad of a line congruence net and its "Gauss" map

Curvature via offsets



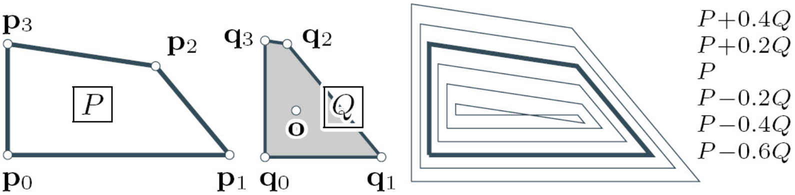
$$A(f_t) = A(f + tN, f + tN) = (1 - 2tH + t^2 K)A(f)$$

- Gaussian curvature $K = \dfrac{A(N)}{A(f)}$
- mean curvature $H = -\dfrac{A(f, N)}{A(f)}$

Vector space of polygons with parallel edges.
Mixed area $A(P, Q)$.

Mixed area

Vector space of polygons with parallel edges



Mixed area $A(P, Q)$ is the symmetric bilinear form corresponding to the quadratic form area $A(P) = A(P, P)$.

$$A(P,Q) = \frac{1}{2}(A(P+Q) - A(P) - A(Q)),$$

$$A(P,Q) = \frac{1}{4}\sum_{i=0}^{k-1}([p_i, q_{i+1}] + [p_i, q_{i+1}]), \quad [,] \text{ area form.}$$

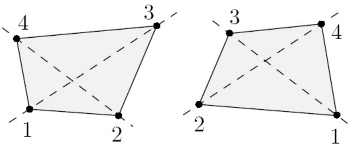
Principal curvatures

$$A(f_t) = (1 - 2tH + t^2 K)A(f) = (1 - tk_1)(1 - tk_2)A(f),$$

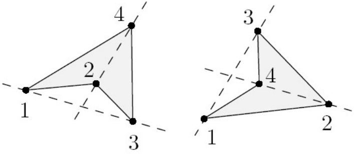
$$K = k_1 k_2, \quad H = \frac{1}{2}(k_1 + k_2)$$

Equivalent:

- principal curvatures k_1, k_2 real,
- the area form $A : \{\text{quads with parallel edges}\} \to \mathbb{R}$ indefinite,
- empty convex hull



indefinite A, real k_1, k_2



definite A, complex k_1, k_2

Discrete surfaces with constant curvature

Definition.

- $H = 0$ - minimal
- $H = const$ - constant mean curvature (CMC)
- $K = const$ - constant Gaussian curvature

Theorem (f, N) CMC with $H = H_0 \neq 0$, then

- parallel surfaces $f + tN$ are linear Weingarten $aH + bK = 1$
- $(f + \frac{1}{2H_0}N, N)$ has constant Gaussian curvature $4H_0^2$
- $(f + \frac{1}{H_0}N, N)$ has constant mean curvature $-H_0$

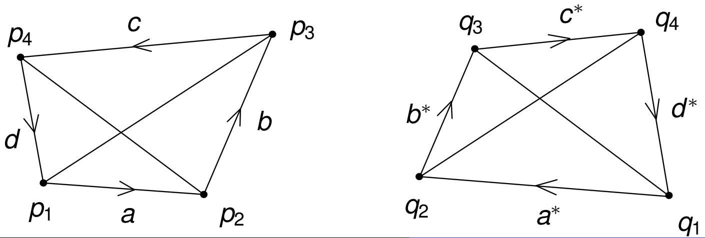
Discrete minimal surfaces

- **Definition.**
  Discrete minimal surface is a line congruence net (f, N) with $H = 0$ for all faces.
- **Mixed area characterization**
  Minimal $\Leftrightarrow$ mixed area $A(f, N) = 0$ for all corresponding quads of f and N.

$\Rightarrow$Dual quadrilaterals and dual quad-nets.

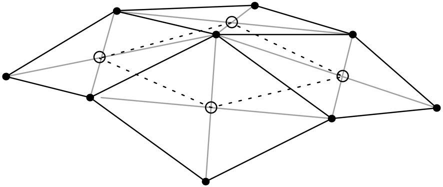
Dual quadrilaterals

- **Definition.** Two quadrilaterals P, Q with parallel edges are called **dual** to each other if their mixed area vanishes, $A(P, Q) = 0$.
- **Existence and uniqueness** For every planar quadrilateral a dual one exists and is unique up to scaling and translation. (Two dimensional vector space with a bilinear symmetric form A.)
- Two quadrilaterals with parallel edges are dual if and only if their diagonals are **antiparallel**:
  $(p_1, p_3) \parallel (q_2, q_4), \quad (p_2, p_4) \parallel (q_1, q_3)$.



Discrete Koenigs nets

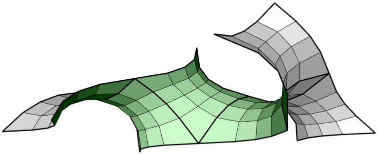
- **Definition.** A quad-surface f with planar faces is called a discrete Koenigs net if it admits a dual net f^*.
- **Projective characterization** A discrete surface $f : \mathbb{Z}^2 \to \mathbb{R}^3$ with planar faces and non-planar vertices is a discrete Koenigs net if and only if the intersection points of diagonals of any four quadrilaterals sharing a vertex are co-planar. [B., Suris '07]

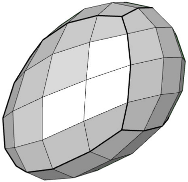


Discrete minimal and CMC surfaces "for free"

Let f be a discrete Koenigs net. Then:

- (f, N) with $N = f^*$ is discrete minimal,
- (f, N) with $N = f - f^*$ is discrete CMC,



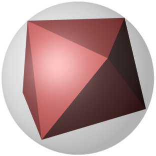


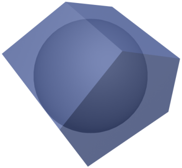
Discrete minimal surface $f = N^*$ and its Gauss image N (Koenigs net). [Schröder]

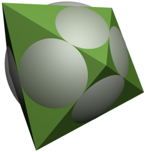
Discrete spheres

Three natural types of spherical polyhedra

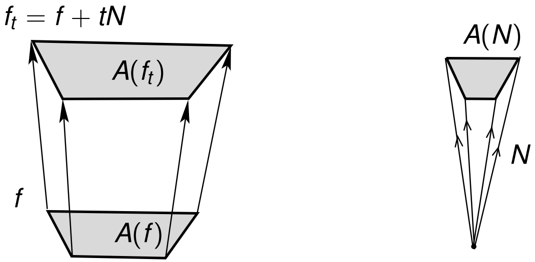
- vertices on S^2 (circular nets)
- faces tangent to S^2 (conical nets)
- edges tangent to S^2 (Koebe polyhedra)







Circular nets. Curvatures

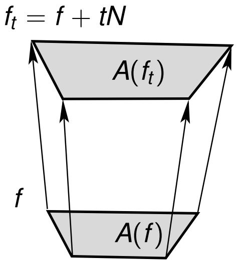


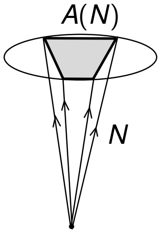
- parallel circular surface $f_t = f + tN$:

$$\begin{aligned} A(f_t) &= (1 - 2tH + t^2 K)A(f) \\ &= (1 - tk_1)(1 - tk_2)A(f) \end{aligned}$$

- mean curvature $H = -\frac{A(f,N)}{A(f)}$, Gaussian curvature $K = \frac{A(N)}{A(f)}$, principal curvatures k_1, k_2 (real)

Circular nets. Curvatures





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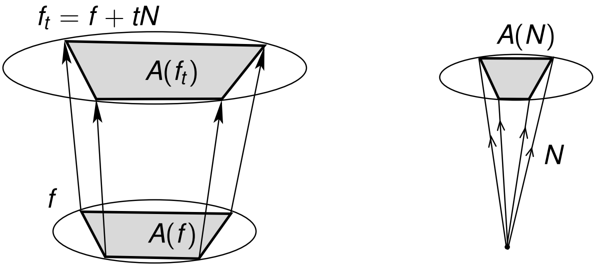
Circular nets. Curvature



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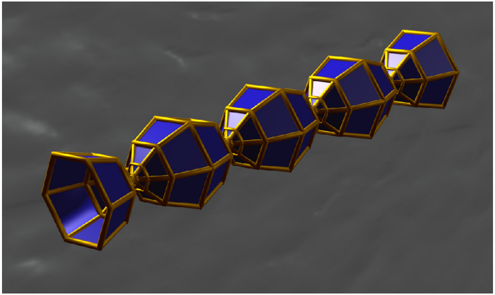
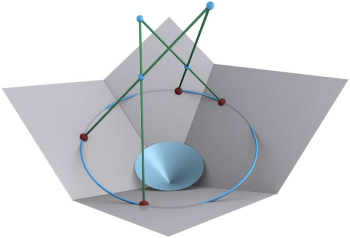
Circular nets. Isothermic, minimal, CMC

- Circular Koenigs nets = Discrete isothermic surfaces [B., Pinkall '96]
- $H = 0 \Rightarrow$ Discrete (circular) minimal surfaces of [B., Pinkall '96] = dual to discrete isothermic in S^2
- $H = H_0 \neq 0 \Rightarrow$ Discrete (circular) CMC surfaces of [B., Hertrich-Jeromin, Hoffmann, Pinkall '99] = isothermic f and its dual at constant distance $|f - f^*| = const$.



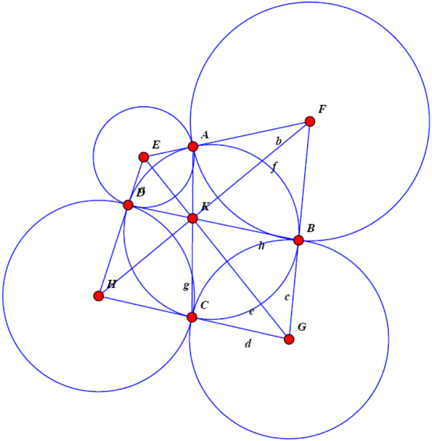
Principal contact element nets as discrete curvature parametrization

- Principal contact element nets = neighboring contact elements share a common (principal curvature) sphere (Lie geometry)
- Circular and conical nets merged [Pottmann,Walner '06], [B.,Suris '06].



Points ⇒ Circular (Möbius); Planes ⇒ Conical (Laguerre)

S-isothermic surfaces



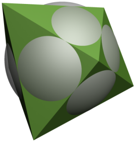
- Touching spheres with an orthogonal circle
- Dual to circumscribed quads are circumscribed
- General S-isothermic surface = T-net of spheres

- **Theorem.** Centers of spheres of an S-isothermic surface build a Koenigs net.

$$R_E/R_G = |EK|/|KG|$$

Koebe polyhedra

Edges of a Gauss polyhedron N
touch a sphere $\Rightarrow$ Koebe polyhedra



- Koebe polyhedra are Koenigs (S-isothermic $\Rightarrow$ dualizable)
- Most developed (based on the theory of circle packings)
- **Theorem.** Every polytopal cell decomposition of the sphere can be realized by a polyhedron with edges tangent to the sphere. This realization is unique up to projective transformations which fix the sphere.

Construction method for discrete minimal surfaces of Koebe type

[B., Hoffmann, Springborn '06]

continuous minimal surface

$\Downarrow$

image of curvature lines under Gauss-map

$\Downarrow$

cell decomposition of (a branched cover of) the sphere

$\Downarrow$

Koebe polyhedron (variational principle)

$\Downarrow$

discrete minimal surface

- Geometry from combinatorics of curvature lines
- Existence and uniqueness
- Boundary conditions and symmetries can be implemented

Variational principle for orthogonal circle patterns

[B., Springborn '04]

- Orthogonal circle pattern in a plane (stereographic projection)
- Minimize the convex function

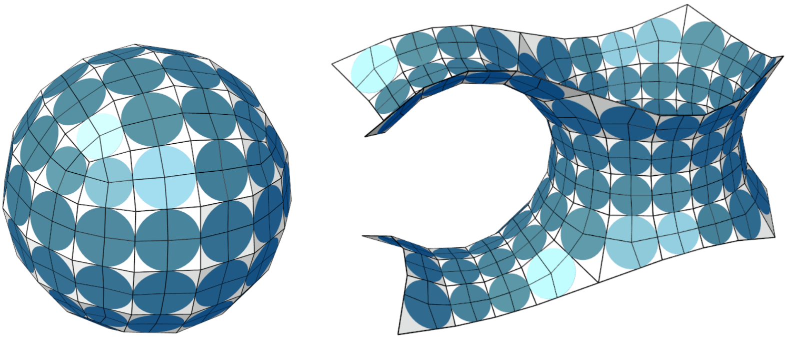
$$S(\rho) = \sum_{j\circ\!-\!\circ k} \left(\operatorname{Im} \operatorname{Li}_2\left(ie^{\rho_k - \rho_j}\right) + \operatorname{Im} \operatorname{Li}_2\left(ie^{\rho_j - \rho_k}\right) - \frac{\pi}{2}(\rho_j + \rho_k) \right) + 2\pi \sum_{\circ j} \rho_j$$

  logarithmic radii: $r = e^{\rho}$

  dilogarithm function: $\operatorname{Li}_2(z) = \frac{z}{1^2} + \frac{z^2}{2^2} + \frac{z^3}{3^2} + \dots$

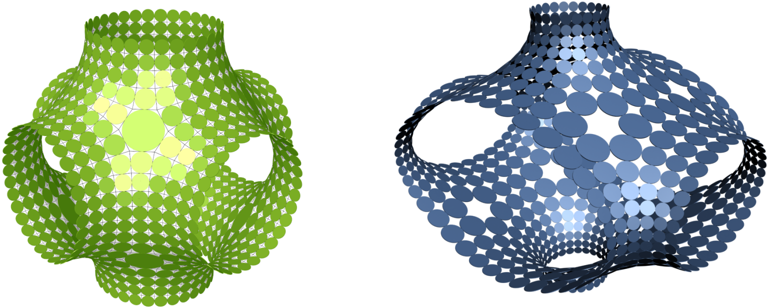
- Explicit formula, no constraints, easy to compute
- Convexity $\Rightarrow$ uniqueness. Existence more delicate
- Generalization for circle patterns on a sphere

Construction method



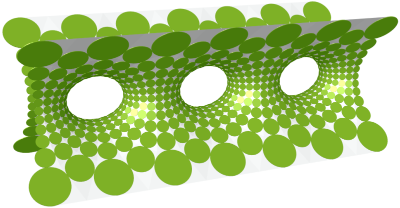
Koebe polyhedron and the dualized minimal surface

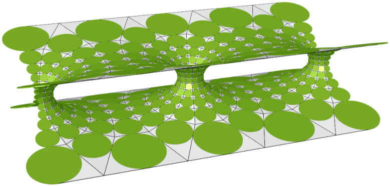
Examples. [Sechelmann, Bücking]



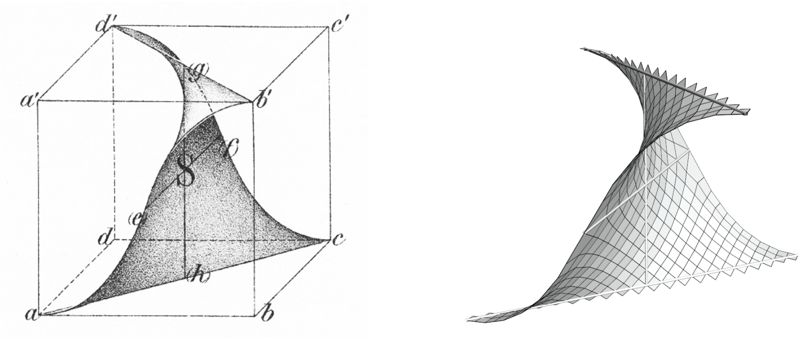
Symmetric and unsymmetric Schwarz P-surfaces

Examples



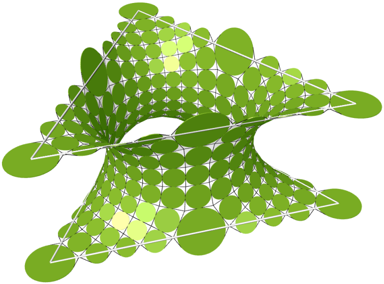


Symmetric and unsymmetric Scherk's towers



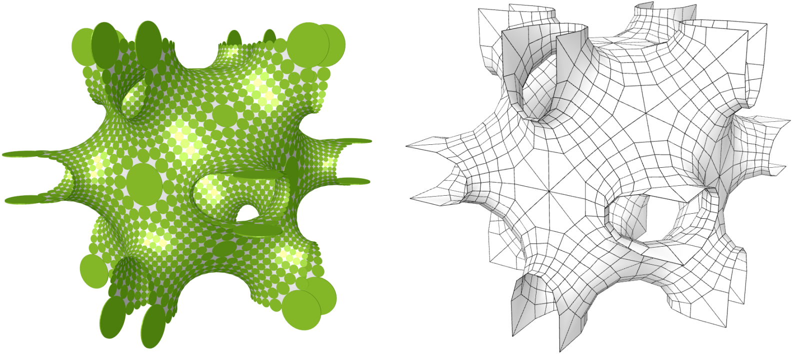
Gergone’s surface by Schwarz and discrete analog

Examples



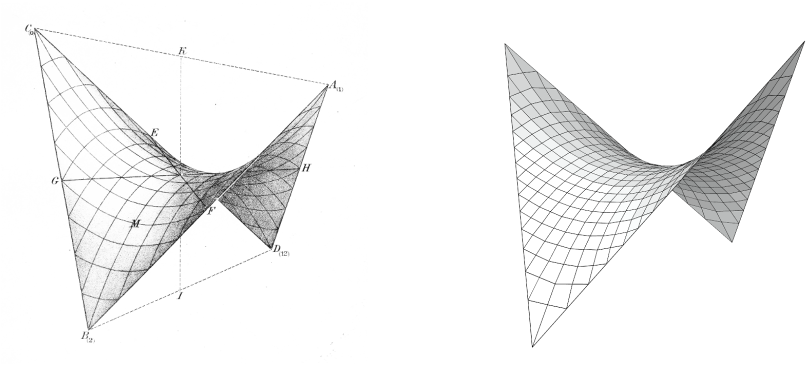
Schwarz' H-surfaces

Examples



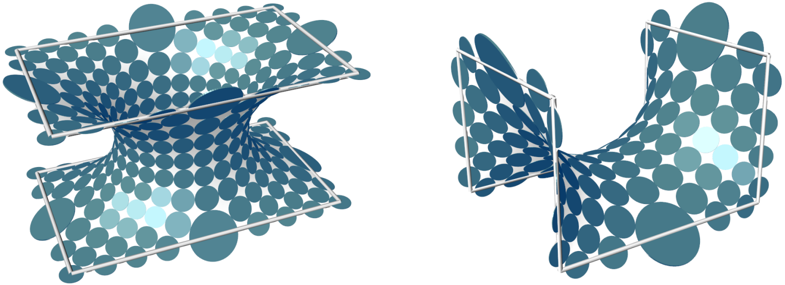
Neovius' surface

Examples



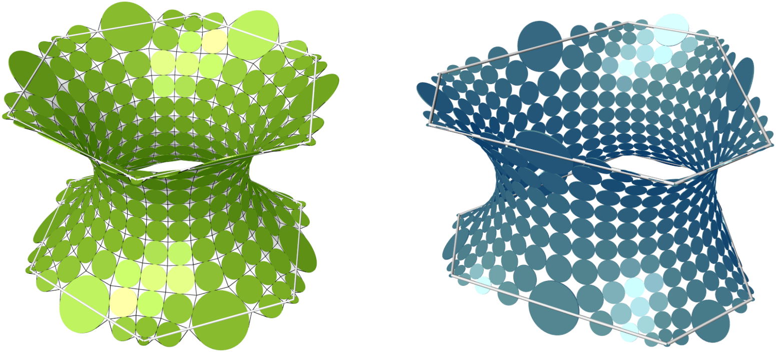
Quadrilateral minimal surface by Schwarz and discrete analog

Examples



Schoen's I-6 surface and a cuboid boundary frame

Examples



Symmetric and unsymmetric catenoid approximations