Gluing a convex polytope: a constructive proof of Alexandrov's theorem

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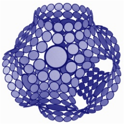

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An Observation

Every convex polytope can be unfolded (self-overlaps allowed).

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More exactly...

- A convex polytope can be cut into flat pieces:
  - for example, into faces (cut along all edges);
  - or just in one piece (unfolding);
  - in general, the cut lines need not be along edges.
- Given a set of flat pieces and a gluing rule, when is it possible to glue a convex polytope?
  - The pieces may be bent.

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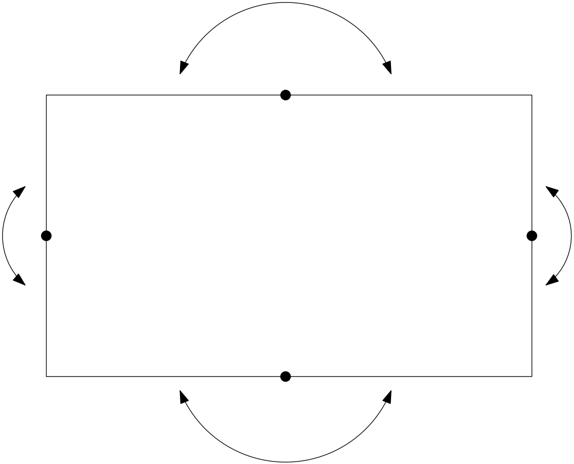
An Example

Does this fold to a convex polytope?



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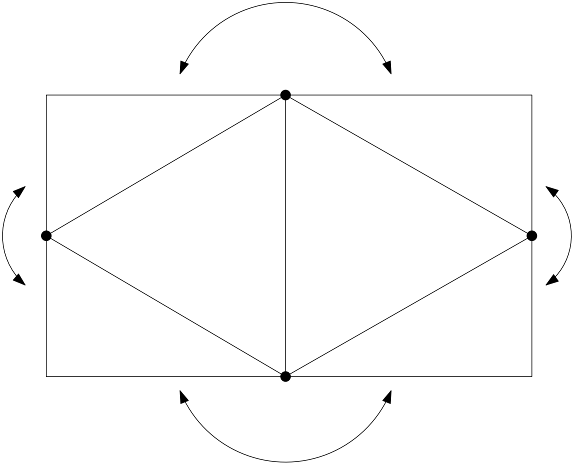

Yes, it does.

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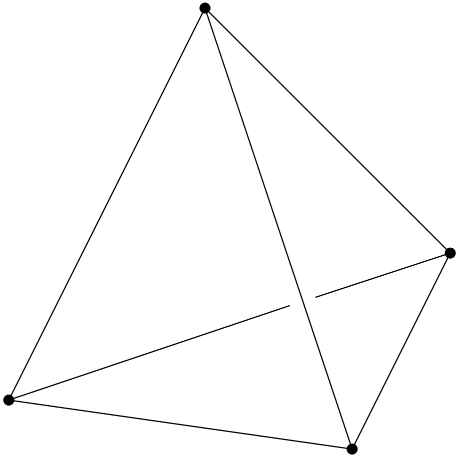
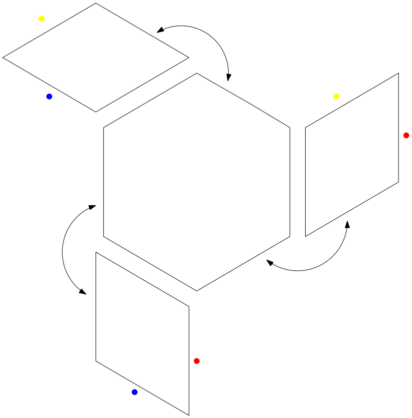


Here it is.

Abstract polyhedral surface

Definition

An *abstract polyhedral surface* is a collection of polygons glued side-to-side.



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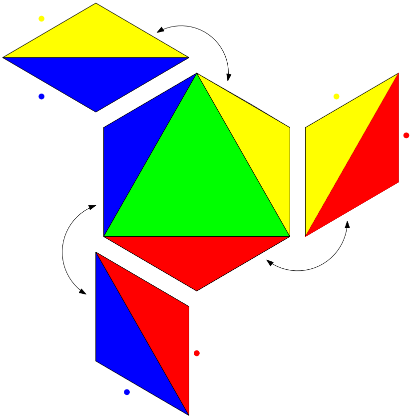

The Question reformulated

The Question

S an abstract polyhedral surface

- Does *S* fold to a convex polytope?
- If yes, then how to construct this polytope?

Difficulty

The polygons of *S* need not be the faces of the polytope.

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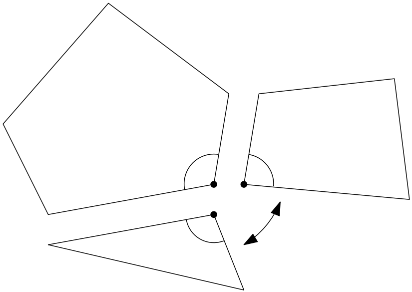
Positive curvature

Necessary condition

If S folds, then every vertex has positive (or zero) curvature.



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A vertex has *positive curvature*, if the angle sum around it $< 2\pi$.

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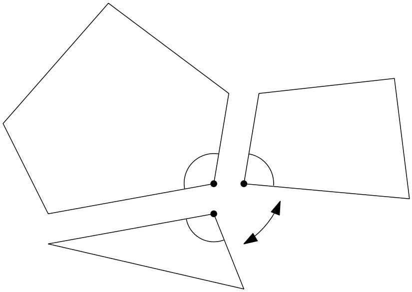

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Alexandrov's theorem

Theorem (A. D. Alexandrov, 1942)

Every abstract polyhedral surface with vertices of non-negative curvature folds to a convex polytope.
Furthermore, this polytope is unique.

The uniqueness part is the Cauchy-Alexandrov theorem on rigidity of convex polytopes.

How to CONSTRUCT the polytope?

- Alexandrov: The polytope exists, but how to construct it?
- Volkov gave a constructive proof in 1955.
- We give a new proof and implement it in a computer program.
- This is a joint work with Alexander Bobenko; algorithm developed with participation of Boris Springborn; the program written by Stefan Sechelmann.

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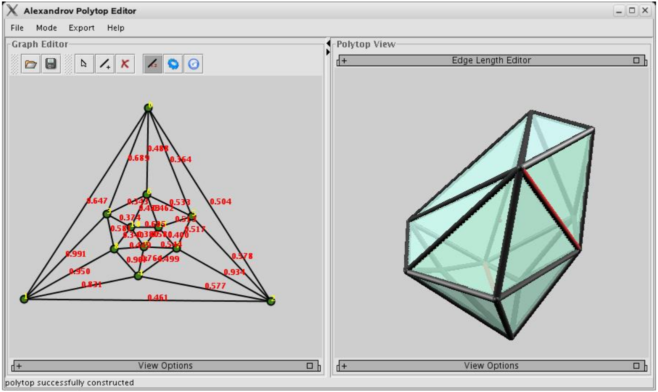
The input data.

- The input is a triangulation with specified edge lengths.
- The program checks the positive curvature condition.
- It cannot work with vertices of zero curvature.



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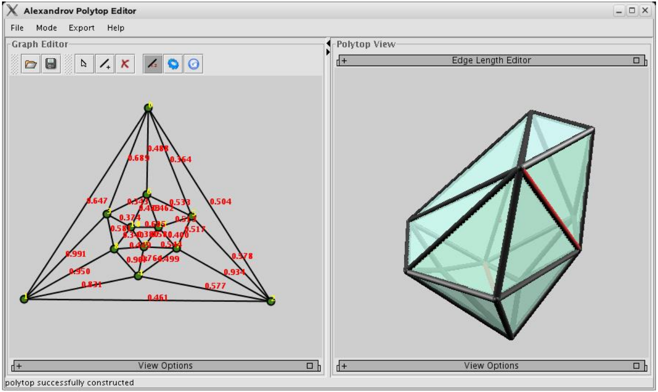

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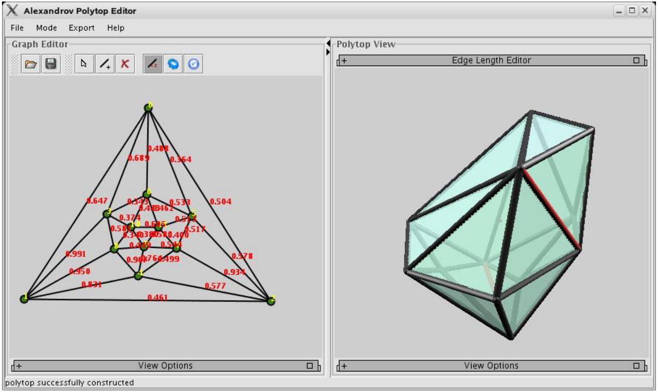
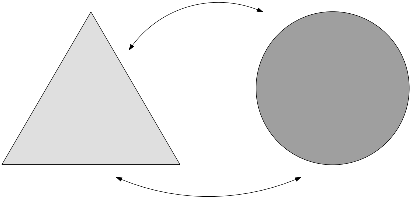
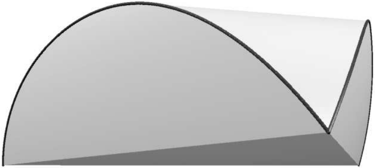


Any two convex figures of equal perimeter can be glued along the boundaries.



This follows from Alexandrov's theorem by approximation.



dForms

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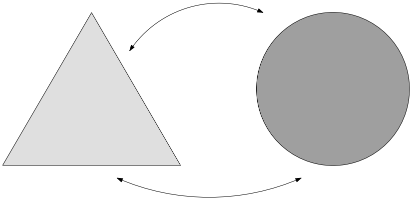

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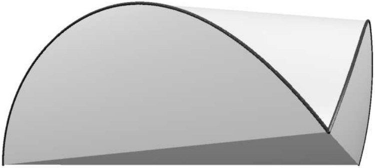
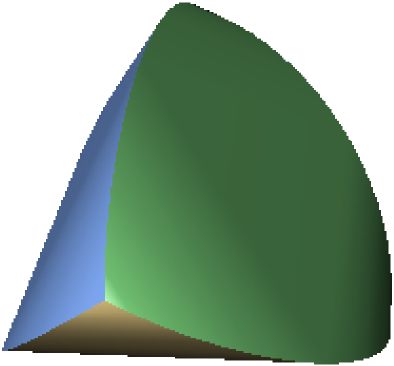


dForms in the real life

Releaux triangle tetrahedron

One can also glue several pieces together, as long as the positive curvature condition is satisfied.



Tetrahedron glued from four Releaux triangles.

Soccer ball

A soccer ball is a convex body glued from flat pieces.

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Springer Monographs in Mathematics. Springer-Verlag, 2005.

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J. O'Rourke. *Computational Geometry Column 49.*

Software available at:
`http://www.math.tu-berlin.de/geometrie/ps/software.shtml`
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dForms: a concept by Tony Wills
`http://local.wasp.uwa.edu.au/%7Epbourke/surfaces_curves/dform/`

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To fold a surface S to a convex polytope, we need to know

- where to bend: a triangulation of S;
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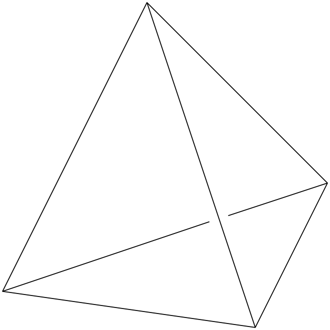

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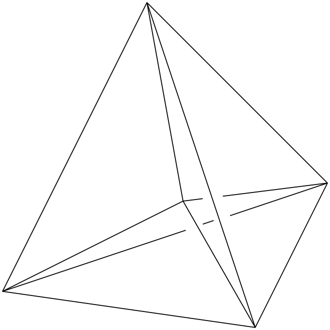
Assume we have a polytope P.
Choose a point inside and cut P into pyramids.



The polytope is determined by

- a triangulation T
- and edge lengths (r_i) of the pyramids over triangles of T.

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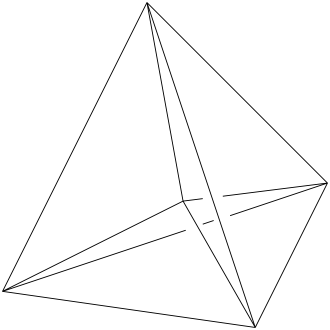

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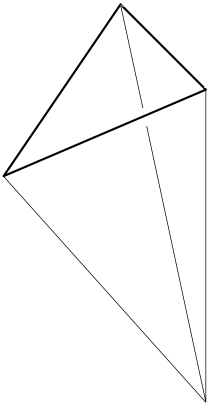

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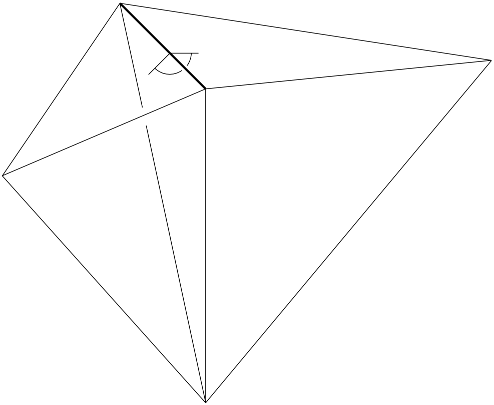

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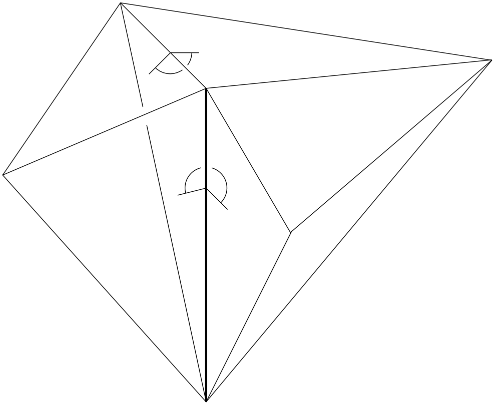
So we look for a pair (T, r) such that

- pyramids with side lengths r_i over the triangles of T exist,
- the dihedral angle θ_{ij} at every edge ij is $\leq \pi$,
- and the pyramids around every radial edge fit together.



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The convexity condition:

- the dihedral angle θ_{ij} at every edge ij is $\leq \pi$

is equivalent to:

- T is the weighted Delaunay triangulation of S with weights (r_i^2).

Lemma

The weighted Delaunay triangulation is unique, if exists.

Corollary

T is determined by r.

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The fitting condition:

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The map $r \mapsto \kappa$ is a local diffeomorphism, under certain restrictions.

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Take a pair $(T(1), r(1))$, where

- $T(1)$ is the Delaunay triangulation of S;
- $r_i(1) = R$ for all i, with R large.

Then construct a family $(T(t), r(t)), t \in [0, 1]$ such that $\kappa(t)$ goes to 0 proportionally to t:

$$\kappa_i(t) = t \cdot \kappa_i(1).$$

As t moves from 1 to 0, the triangulation $T(t)$ gets transformed by flips. At $t = 0$ we have $\kappa = 0$, and we are done.

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