Hexagonal meshes as minimal surfaces

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Notation

Mesh $\mathcal{M}$

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- planar faces F_i

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- corresponding edges e_i' and e_i are parallel

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vertex offset mesh $\mathcal{M}'$ to $\mathcal{M}$ at constant distance d

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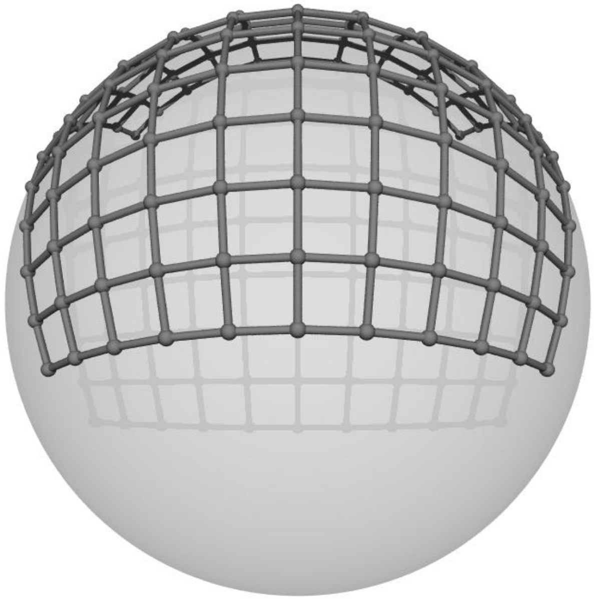
- $\mathcal{M}'$ is a parallel mesh to $\mathcal{M}$
- $\|m_i' - m_i\| = d$ for all vertices

Notation

discrete Gauss image

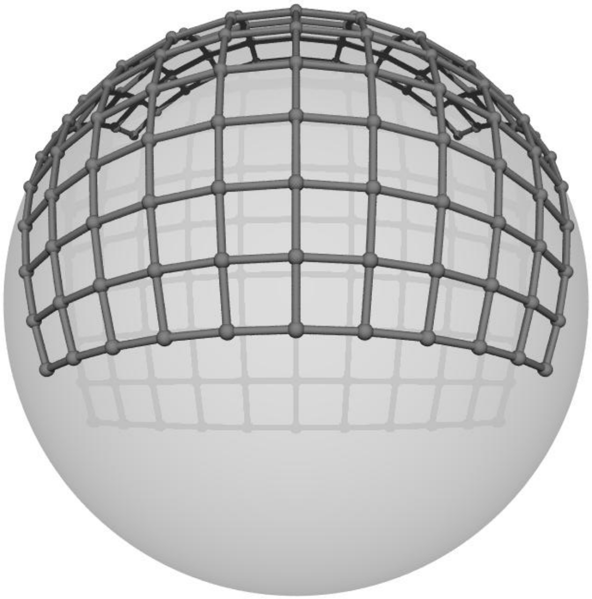
- $\sigma(\mathcal{M}) := (\mathcal{M}' - \mathcal{M})/d$



Notation

discrete Gauss image

- $\sigma(\mathcal{M}) := (\mathcal{M}' - \mathcal{M})/d$
- $\Longrightarrow$ The vertices of $\sigma(\mathcal{M})$ are contained in the unit sphere.

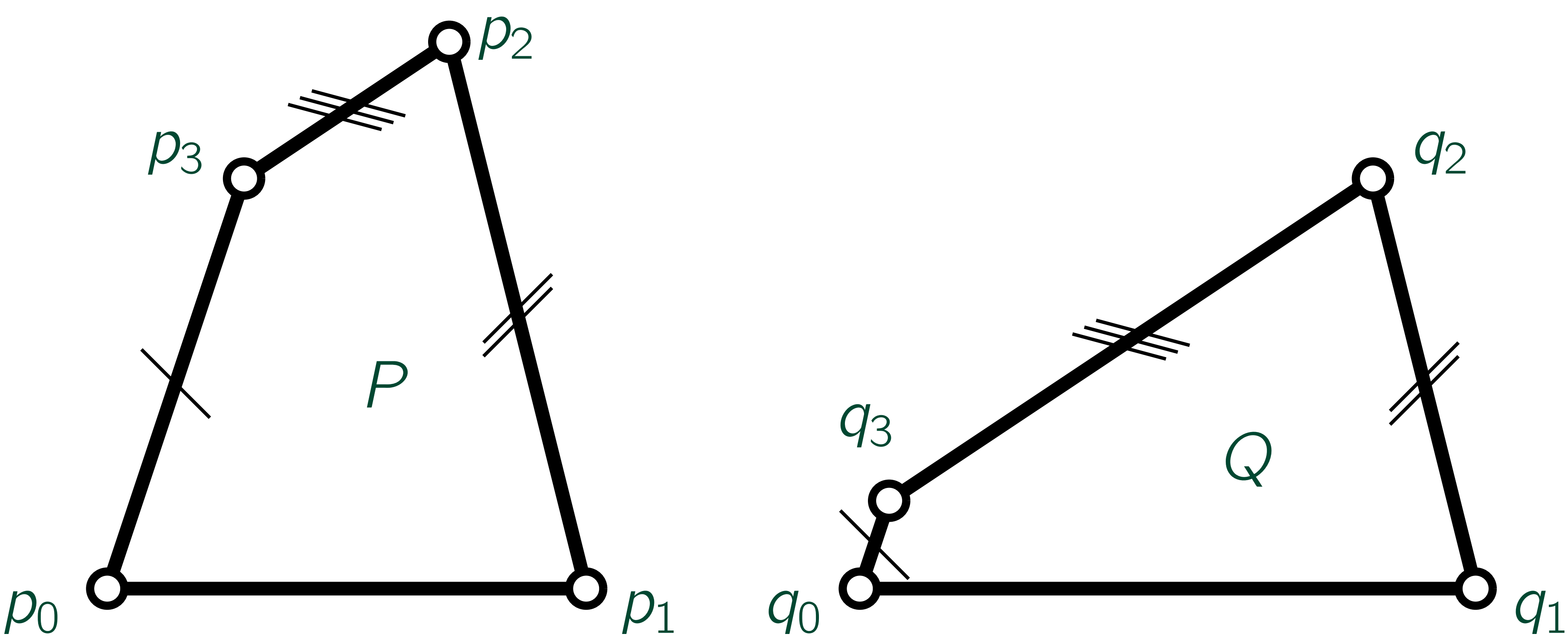


Notation

Two planar polygons $P = (p_0, \ldots, p_{k-1})$ and

$Q = (q_0, \ldots, q_{k-1})$ are called *parallel*, if

$p_i - p_{i+1} \parallel q_i - q_{i+1}$.



Notation

Two convex sets $P, Q \subseteq \mathbb{R}^2$ and $d \geq 0$:

- $\text{area}(P + dQ) = \text{area}(P) + 2d\,\text{area}(P, Q) + d^2\,\text{area}(Q)$
- $\text{area}(P, Q)$ mixed area

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Two parallel polygons P, Q and $d \in \mathbb{R}$:

- oriented area defined by
  $$\text{area}(P + dQ) = \frac{1}{2} \sum_{i=0}^{k-1} \det(p_i + d\,q_i, p_{i+1} + d\,q_{i+1}, n)$$

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Notation

The *mixed area* of two parallel polygons P and Q is defined as

$$\text{area}(P, Q) := \frac{1}{4} \sum_{i=0}^{k-1} (\det(p_i, q_{i+1}) + \det(q_i, p_{i+1}))$$

(see e.g. [Pottmann et al. SIGGRAPH 07])

Definitions

smooth

surface $f(U)$

discrete

mesh $\mathcal{M}$

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offset surface $f^d(U)$ where $f^d = f + d \cdot n$

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discrete

mesh $\mathcal{M}$

offset mesh $\mathcal{M}^d$ where $\mathcal{M}^d = \mathcal{M} + d \cdot \sigma(\mathcal{M})$

Definitions

smooth

Steiner's formula:

$$\text{area}(f^d(U)) = \int_{f(U)} (1 - 2dH(\mathbf{x}) + d^2K(\mathbf{x}))d\mathbf{x}$$

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$$\text{area}(\mathcal{M}^d) = \sum_{F_i:\ \text{Face of}\,\mathcal{M}} (1 - 2dH_{F_i} + d^2K_{F_i})\,\text{area}(F_i)$$

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where H_{F_i} and K_{F_i} are discrete analogues of mean and Gaussian curvature

Definitions

$\Longrightarrow$ *discrete mean curvature*

$$H_{F_i} = -\frac{\text{area}(F_i, \sigma(F_i))}{\text{area}(F_i)}$$

[Pottmann et al. SIGGRAPH 07]

Definitions

$\Longrightarrow$ *discrete mean curvature*

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$\Longrightarrow$ *discrete Gaussian curvature*

$$K_{F_i} = \frac{\mathrm{area}(\sigma(F_i))}{\mathrm{area}(F_i)}$$

[Pottmann et al. SIGGRAPH 07]

Definitions

smooth

$f(U)$ is a minimal surface $\iff$ $\boxed{H(\mathbf{x}) = 0}$ for all $\mathbf{x} \in U$

discrete

Definitions

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$f(U)$ is a minimal surface $\Longleftrightarrow$ $\boxed{H(\mathbf{x}) = 0}$ for all $\mathbf{x} \in U$

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$\mathcal{M}$ is a discrete minimal surface $\Longleftrightarrow$ $\boxed{H_{F_i} = 0}$ for all faces F_i of the mesh $\mathcal{M}$

Definitions

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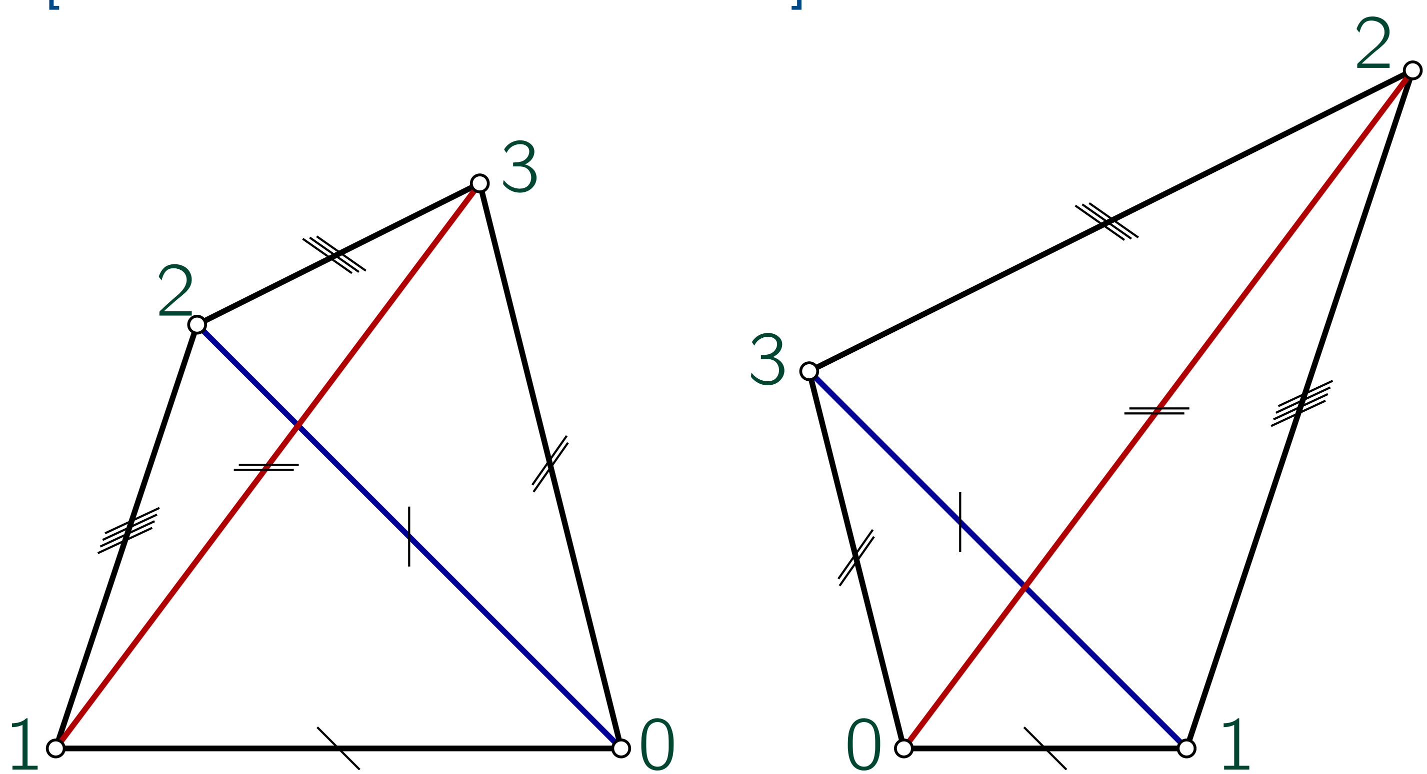
$\mathcal{M}$ is a discrete minimal surface if and only if

$\text{area}(F_i, \sigma(F_i)) = 0$ for all faces F_i of the mesh $\mathcal{M}$.

Vanishing mixed area

Two parallel quads have vanishing mixed area if and only if they have antiparallel diagonals.

[Pottmann et al. SIGGRAPH 07]



Vanishing mixed area

In the following:

Assume that k is even, and that $P = (p_0, \ldots, p_{k-1})$ and $Q = (q_0, \ldots, q_{k-1})$ are planar polygons.

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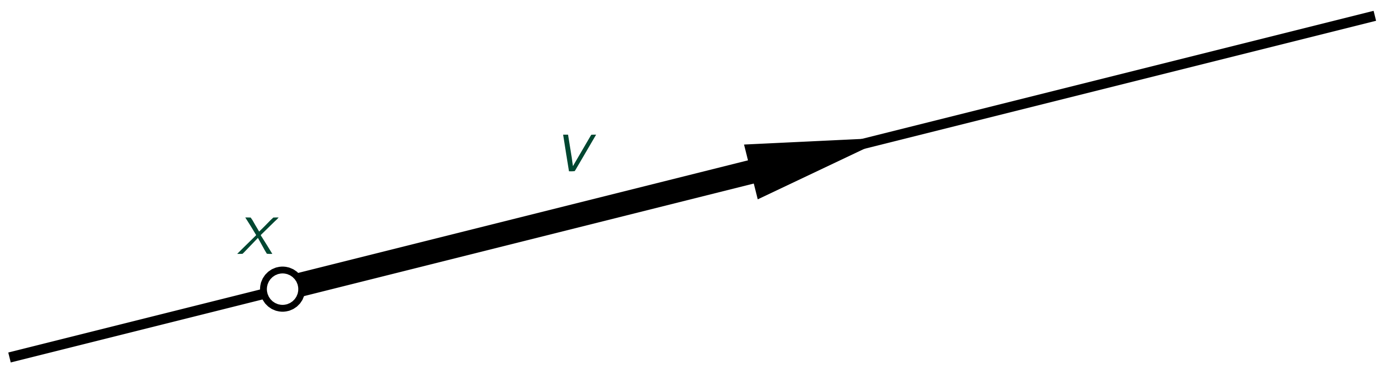
Assume that k is even, and that $P = (p_0, \ldots, p_{k-1})$ and $Q = (q_0, \ldots, q_{k-1})$ are planar polygons.

So P and Q have an even number of vertices.

Vanishing mixed area

Notation:

$L(x, v) := \{x + \lambda v \mid \lambda \in \mathbb{R}\}$ denotes a straight line through the point x with the direction v.

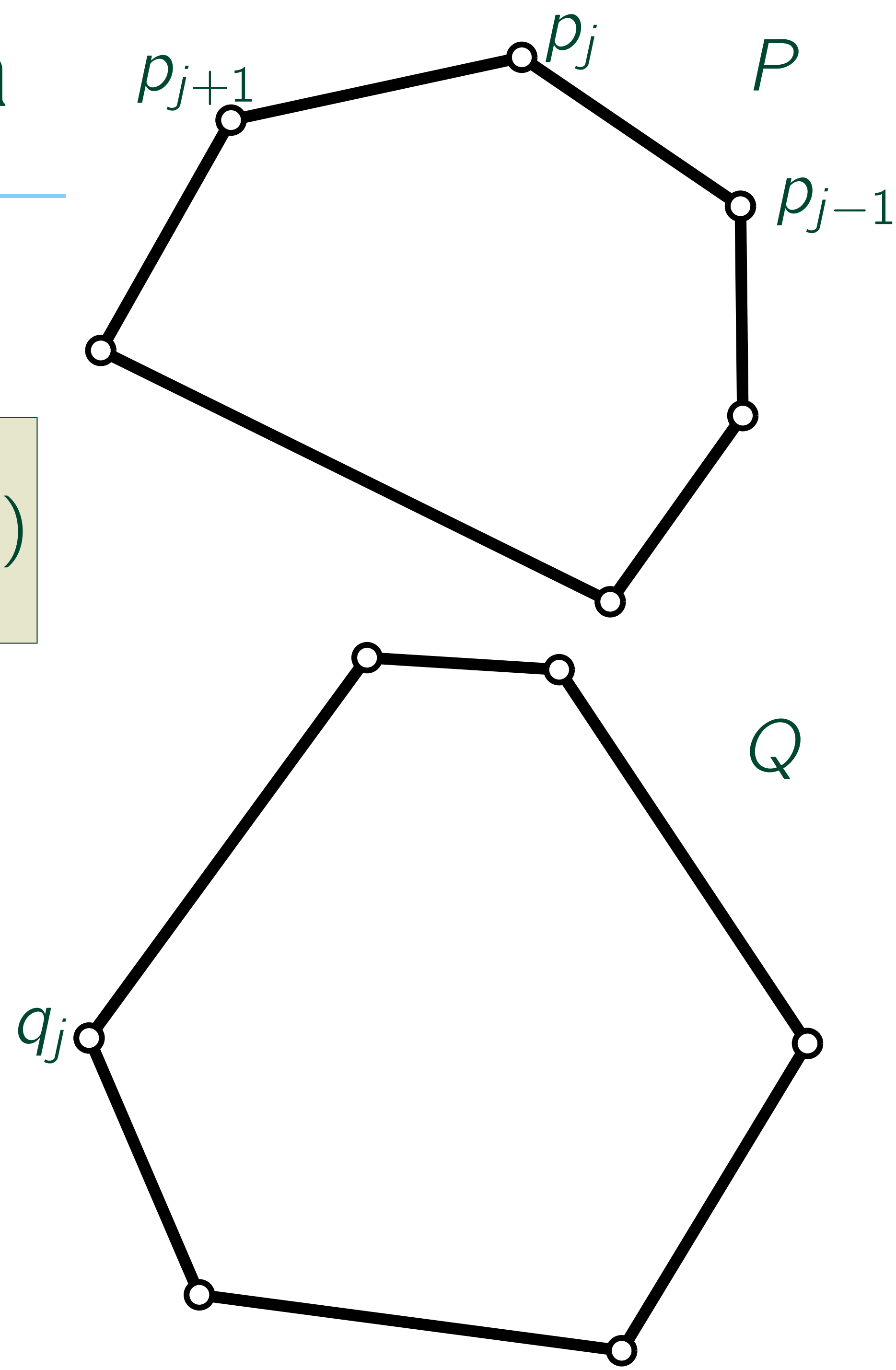


Vanishing mixed area

For the two polygons P and Q,

$$\sum_{i=0}^{k-1}(\det(p_i, q_{i+1}) + \det(q_i, p_{i+1}))$$

is zero, if both



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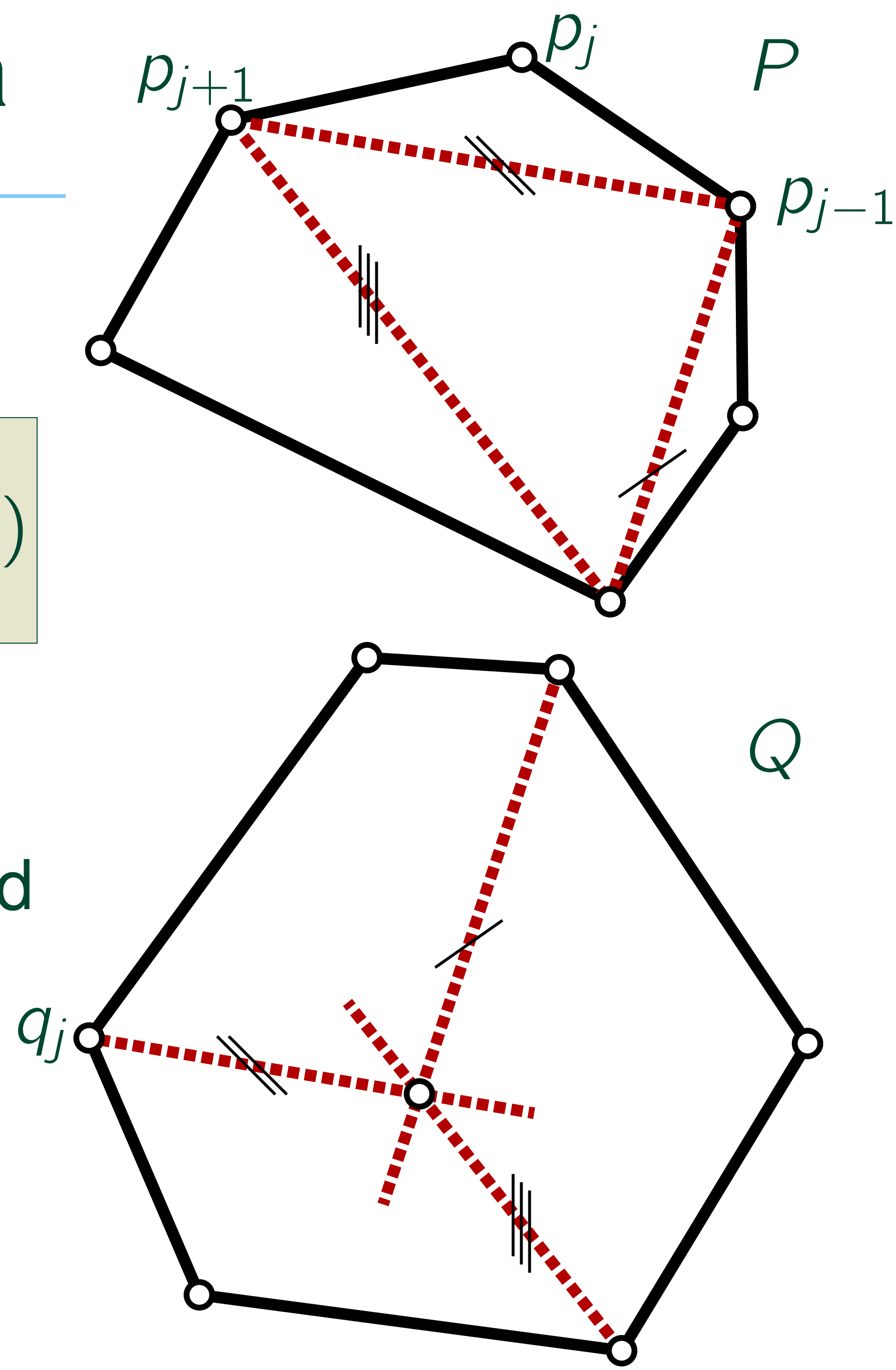
$\{L(q_j, p_{j-1} - p_{j+1}) \mid j \text{ odd}\}$ and

$\{L(q_j, p_{j-1} - p_{j+1}) \mid j \text{ even}\}$

are concurrent.



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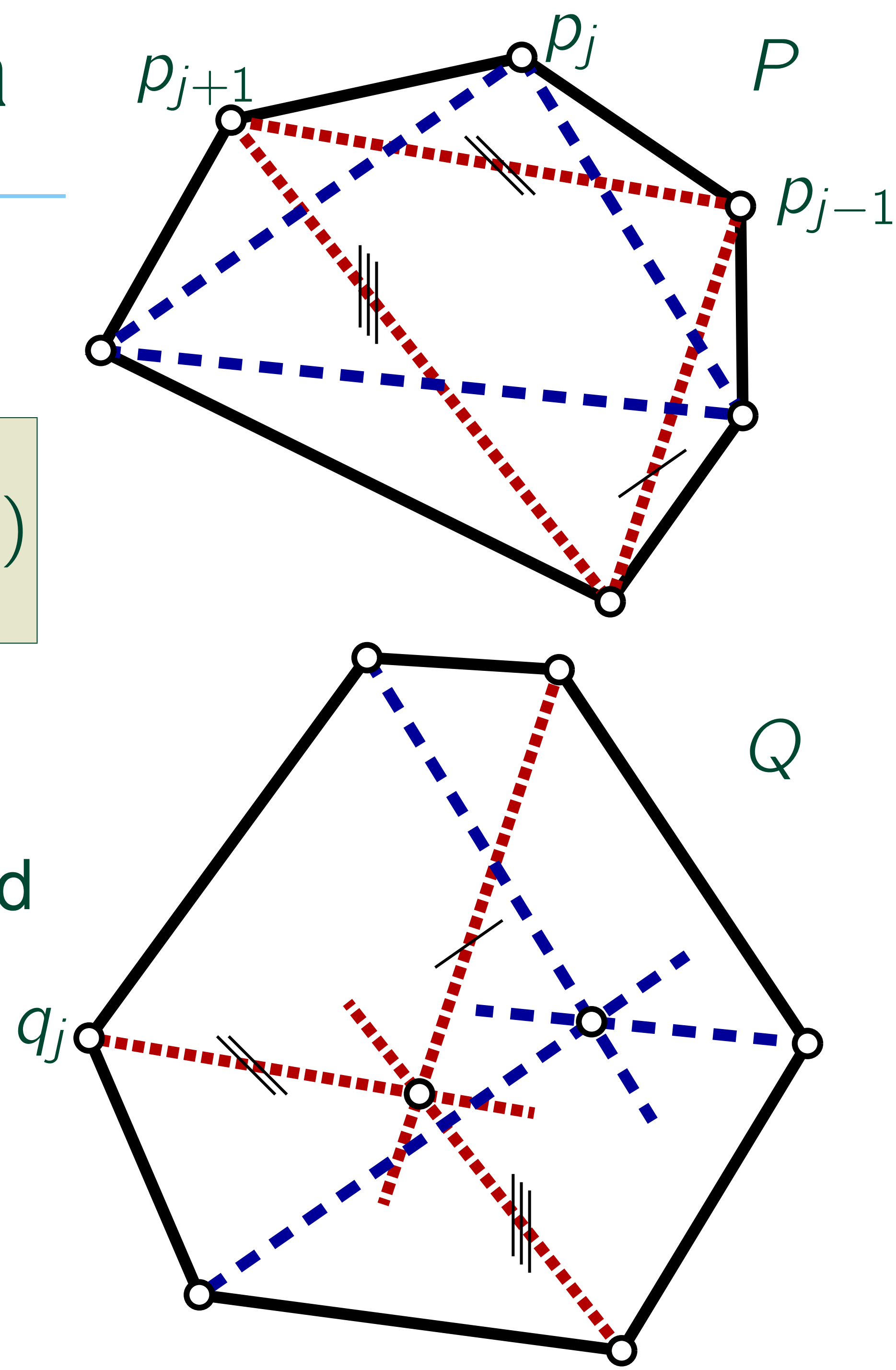
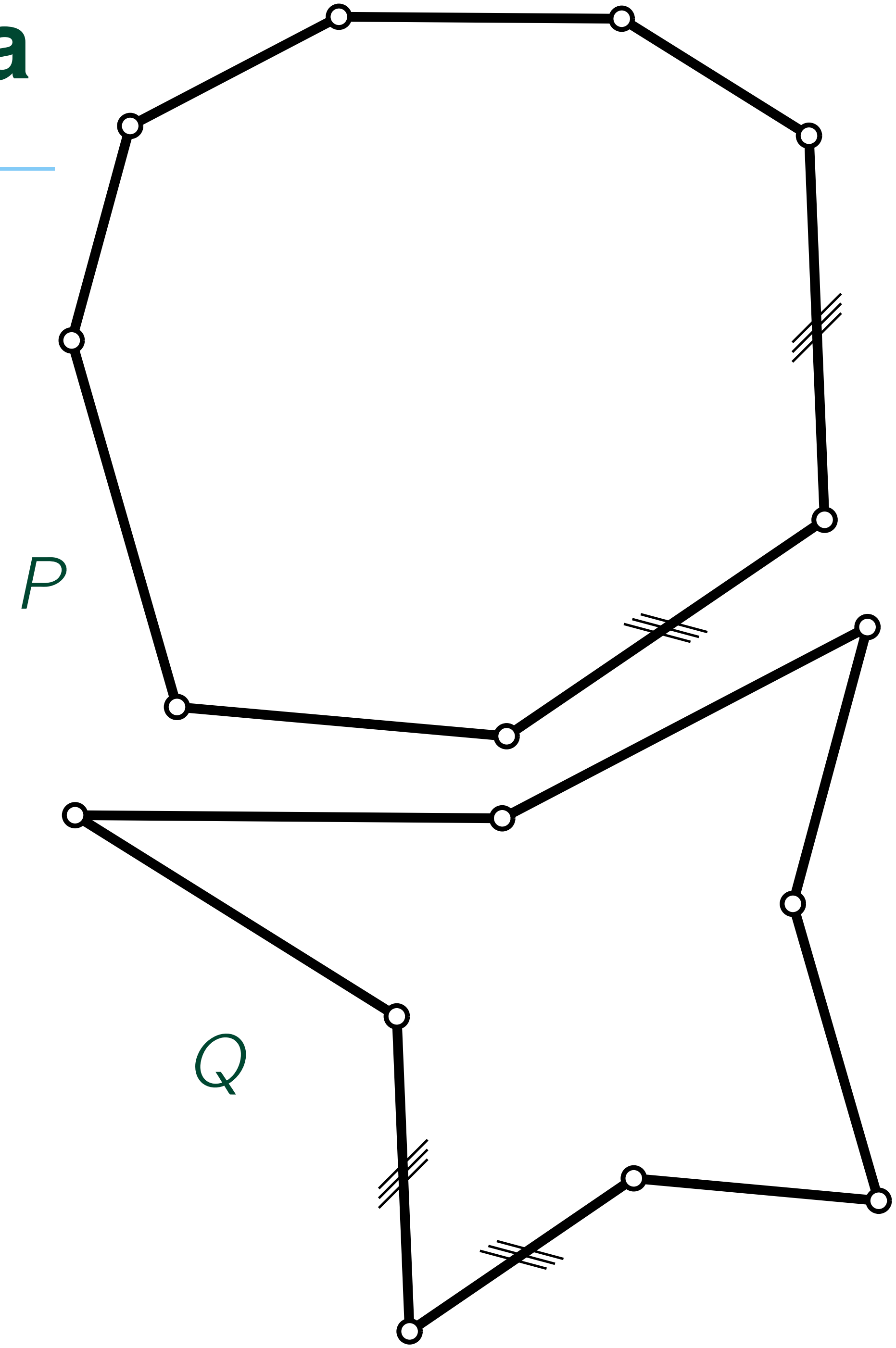


Vanishing mixed area

For two parallel polygons

P and Q, the mixed area

$\text{area}(P, Q)$ is zero, if either



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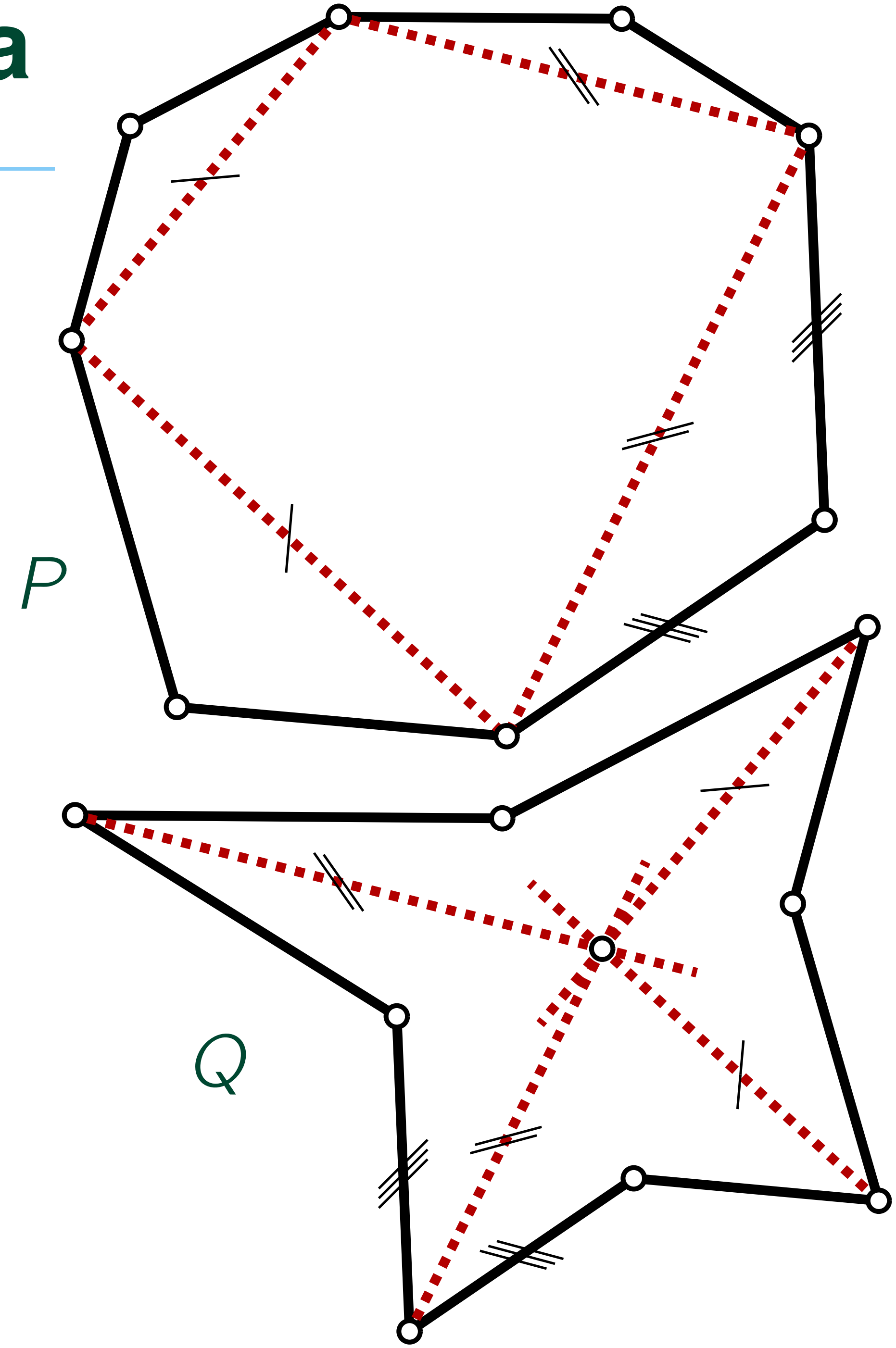
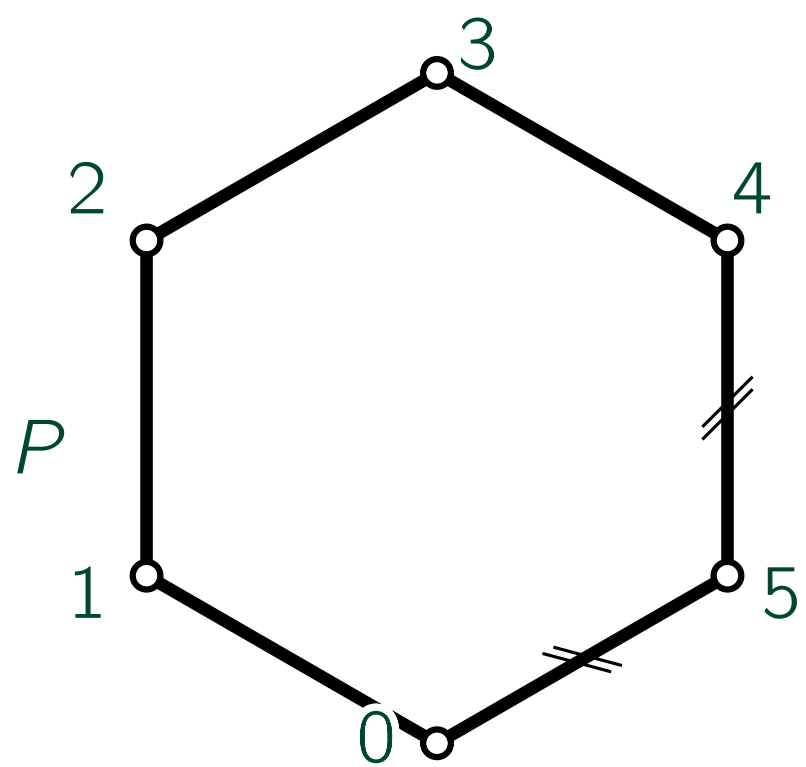
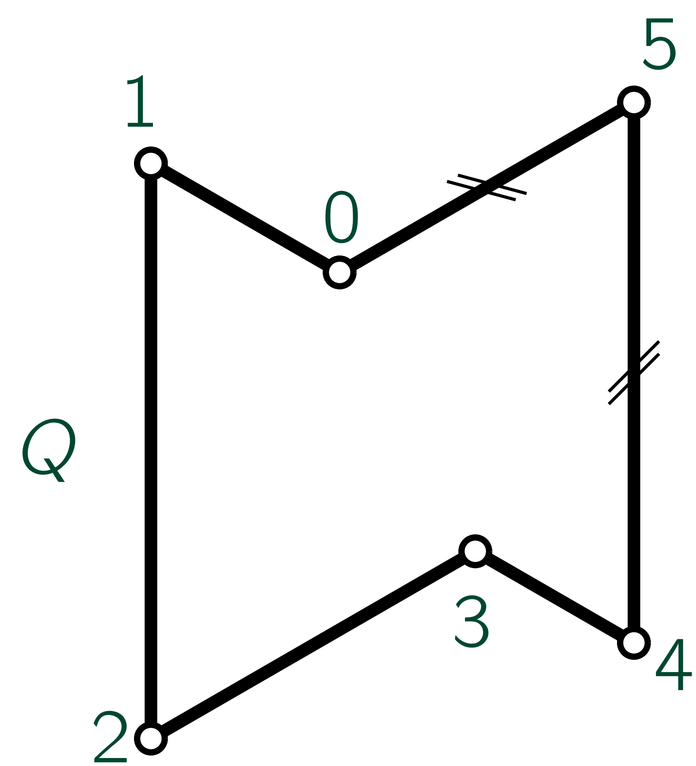


Vanishing mixed area

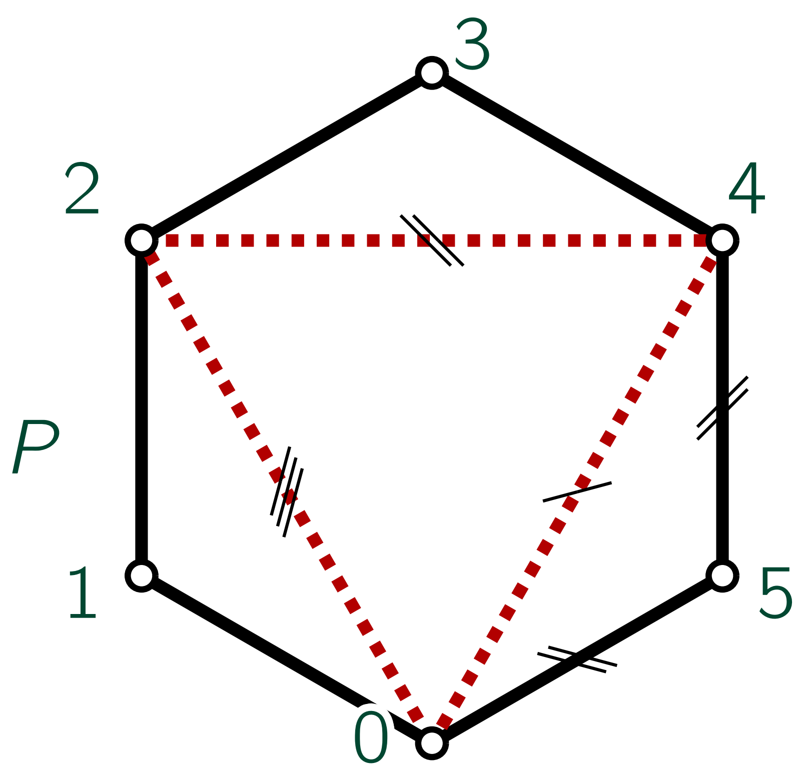
For two parallel hexagons P and Q, the mixed area $\text{area}(P, Q)$ is zero, if and only if $\{L(q_j, p_{j-1} - p_{j+1}) \mid j \text{ odd}\}$ and $\{L(q_j, p_{j-1} - p_{j+1}) \mid j \text{ even}\}$ are concurrent.

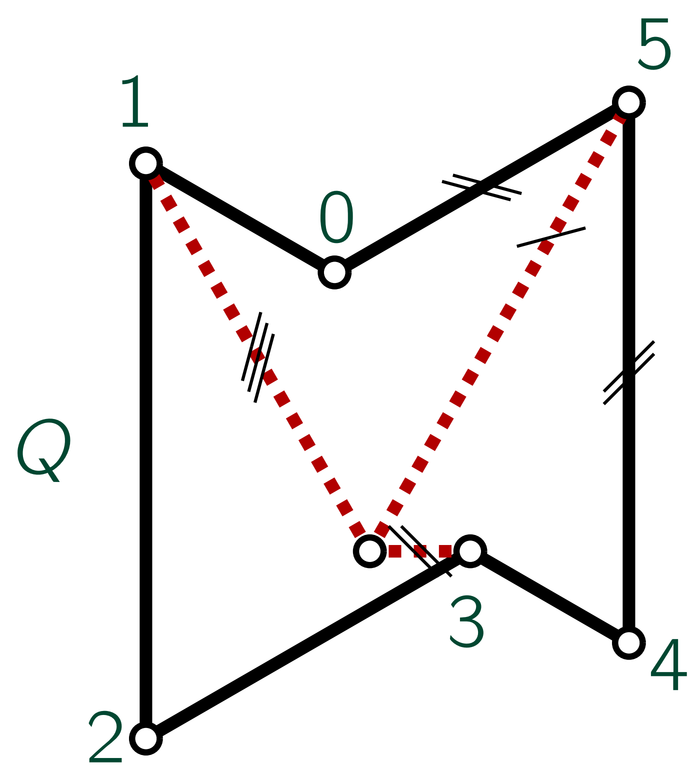




Vanishing mixed area

For two parallel hexagons P and Q, the mixed area area(P, Q) is zero, if and only if $\{L(q_j, p_{j-1} - p_{j+1}) \mid j \text{ odd}\}$ and $\{L(q_j, p_{j-1} - p_{j+1}) \mid j \text{ even}\}$ are concurrent.





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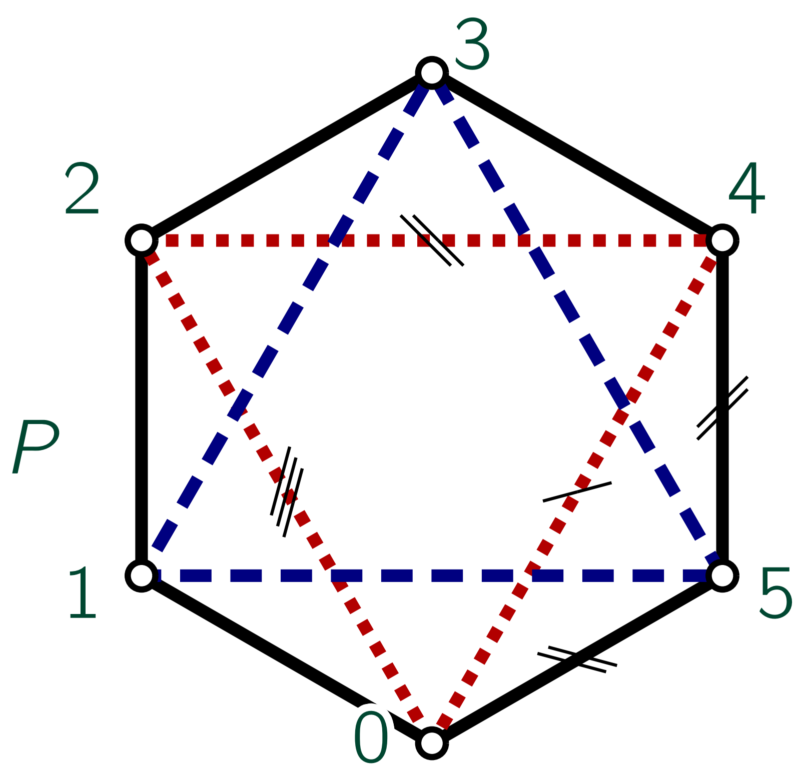

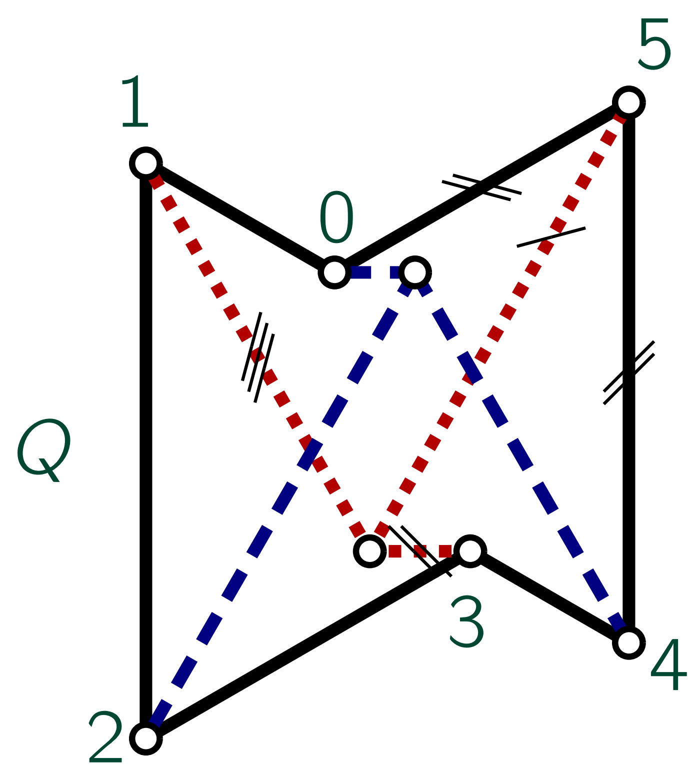

Vanishing mixed area

Proof:

The ‘only if’ part follows from a Lemma before.

In order to show the ‘if’ part we assume area$(P,Q)=0$ and show that the lines $L(q_1, p_0-p_2)$, $L(q_3, p_2-p_4)$, and $L(q_5, p_4-p_0)$ are concurrent. The proof that $L(q_0, p_5-p_1)$, $L(q_2, p_1-p_3)$, and $L(q_4, p_3-p_5)$ are concurrent works analogously.
The considered lines are concurrent if and only if

$$\det\left[\begin{pmatrix} q_1 \\ 1 \end{pmatrix} \times \begin{pmatrix} p_0-p_2 \\ 0 \end{pmatrix}, \begin{pmatrix} q_3 \\ 1 \end{pmatrix} \times \begin{pmatrix} p_2-p_4 \\ 0 \end{pmatrix}, \begin{pmatrix} q_5 \\ 1 \end{pmatrix} \times \begin{pmatrix} p_4-p_0 \\ 0 \end{pmatrix}\right] = 0$$

(this follows imediately from the formulae for the span of two points when using homogeneous coordinates). Because vanishing mixed area is affinely invariant, we can assume without loss of generality that $p_0=(0,0)$, $p_2=(1,0)$, and $p_4=(1,0)$. Then the determinant simplifies to

$$\begin{aligned} &\det\begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ \det q_1 p_0-p_2 & \det q_3 p_2-p_4 & \det q_5 p_4-p_0 \end{pmatrix} = \\ &= \det q_1 p_0-p_2+\det q_3 p_2-p_4+\det q_5 p_4-p_0. \end{aligned} \tag{1}$$

We have already shown that area$(P,Q)=0 \iff \sum_{i=0}^{5}\det(p_i, q_{i+1}-q_{i-1})=0 \iff \sum_{i=0}^{5}\det(q_i, p_{i+1}-p_{i-1})=0$. In view of these equations, (1) equals

$$-\left(\det q_0 p_5-p_1+\det q_2 p_1-p_3+\det q_4 p_3-p_5\right). \tag{2}$$

The expression in (1) also equals

$$\begin{aligned} &\det q_1 p_0-p_1+\det q_1 p_1-p_2+\det q_3 p_2-p_3+ \\ &+\det q_3 p_3-p_4+\det q_5 p_4-p_5+\det q_5 p_5-p_0. \end{aligned}$$

Now we use the parallelity of the edges which means $\det q_i-q_{i+1} p_i-p_{i+1}=0$ and $\det -q_i+q_{i+1} p_i-p_{i+1}=0$, respectively, and get

$$\begin{aligned} &\det q_0 p_0-p_1+\det q_2 p_1-p_2+\det q_2 p_2-p_3+ \\ &\quad+\det q_4 p_3-p_4+\det q_4 p_4-p_5+\det q_0 p_5-p_0= \\ &=\det q_0 p_5-p_1+\det q_2 p_1-p_3+\det q_4 p_3-p_5 \end{aligned} \tag{3}$$

We know that (2) equals (3) which is only possible when it is 0. This shows that the determinant considered above equals 0, which proves the propsition.

Vanishing mixed area

For two parallel hexagons P, Q:

$\text{area}(P, Q) = 0$ is equivalent to the concurrence of the following sets of lines:

1. $\{L(q_0, p_5 - p_1), L(q_2, p_1 - p_3), L(q_4, p_3 - p_5)\}$; or

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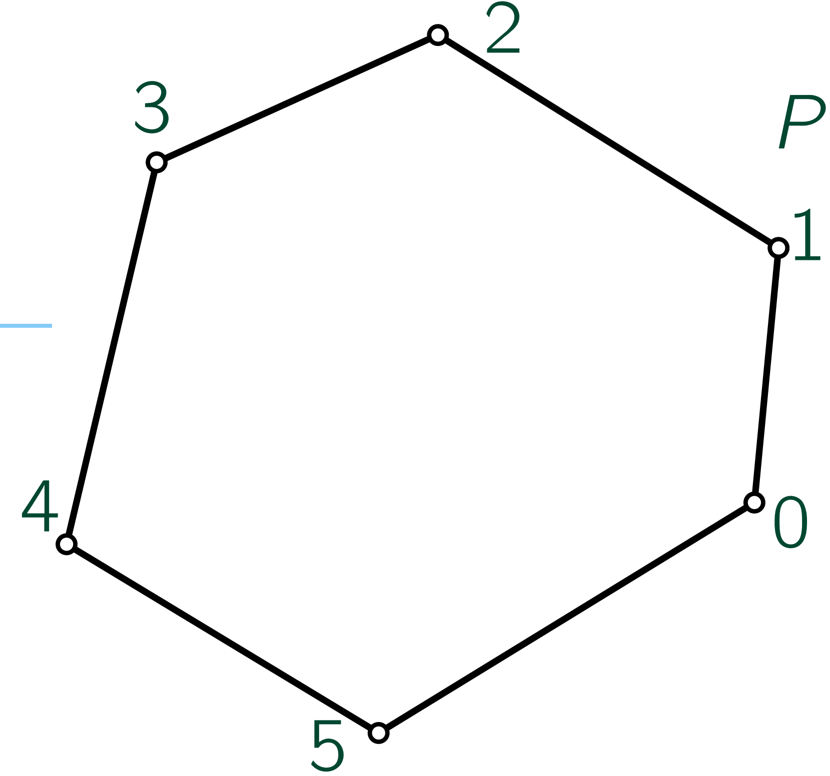
2. $\{L(q_1, p_0 - p_2), L(q_3, p_2 - p_4), L(q_5, p_4 - p_0)\}$; or
3. both $\{L(q_1, p_0 - p_2), L(q_3, p_2 - p_4), L(q_5, p_4 - p_0)\}$, $\{L(q_0, p_5 - p_1), L(q_2, p_1 - p_3), L(q_4, p_3 - p_5)\}$

Construction

Given: hexagon P

Look for a parallel hexagon Q

with $\text{area}(P, Q) = 0$.



Construction

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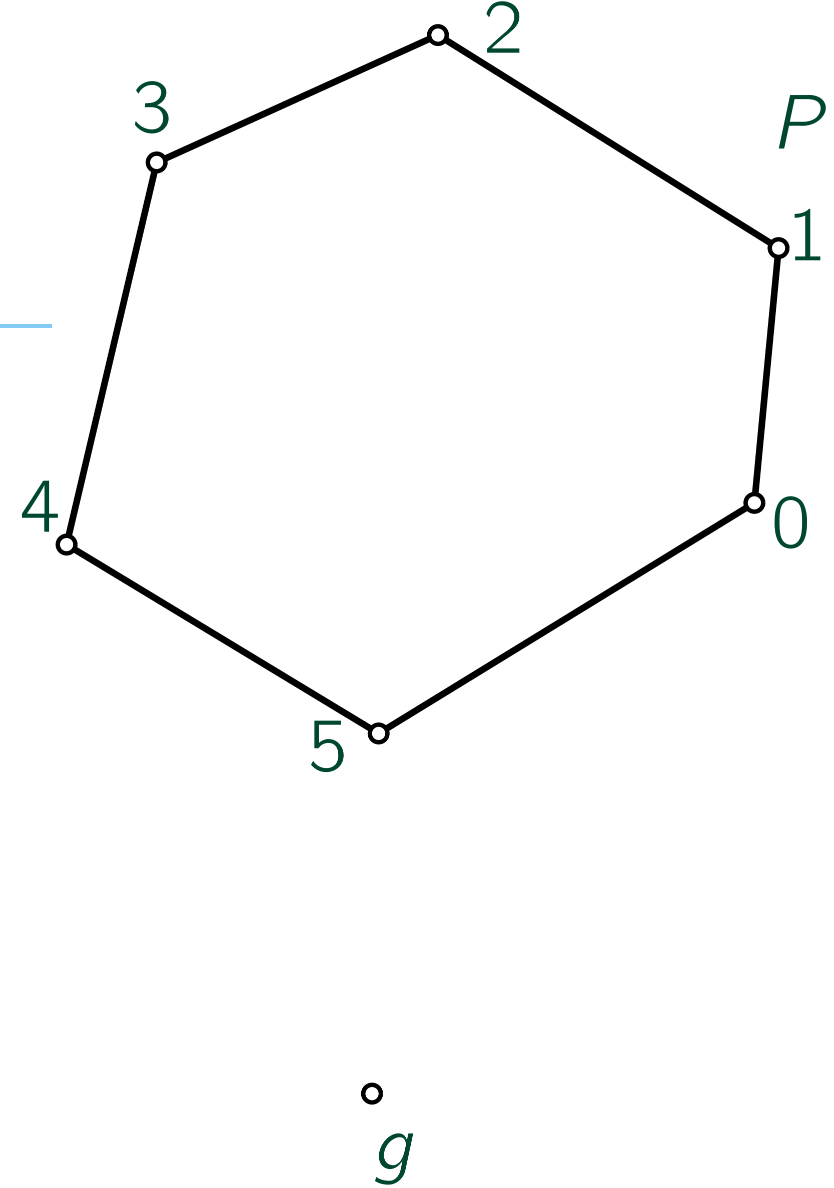
- choose one arbitrary point g



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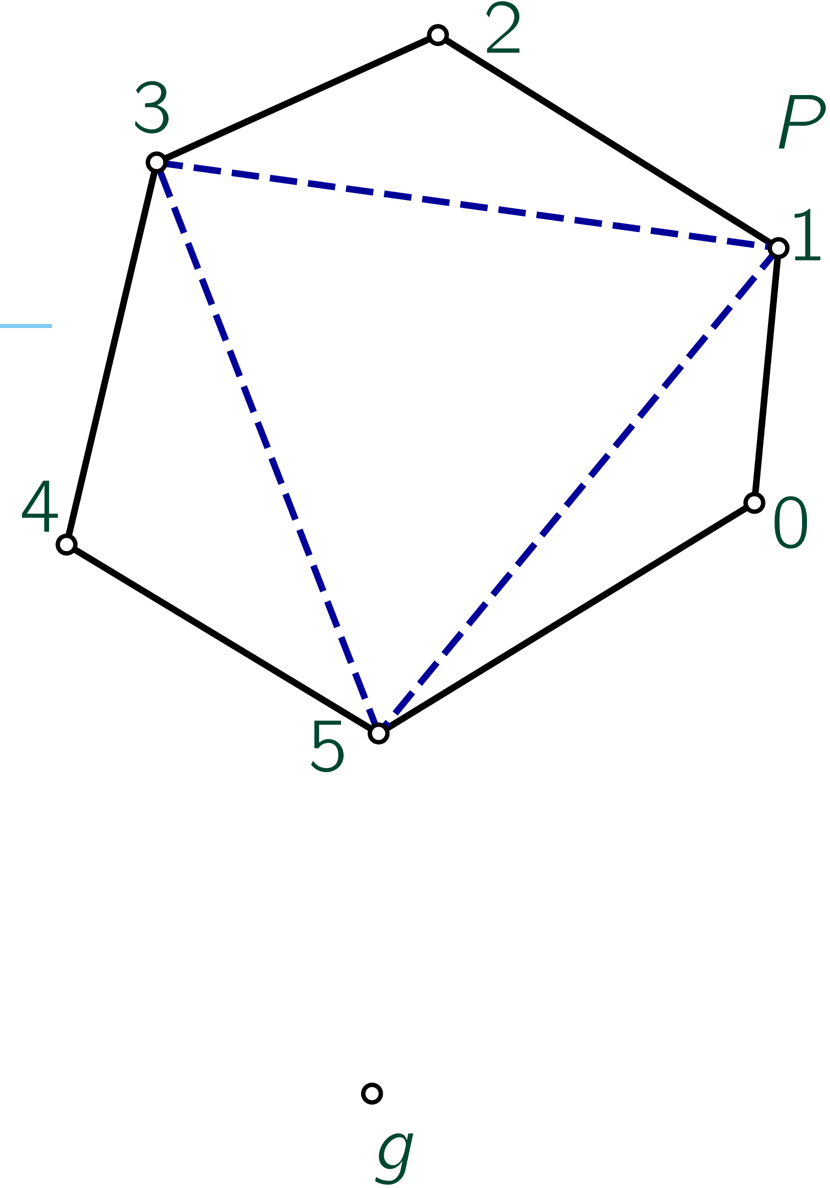


Construction

Given: hexagon P

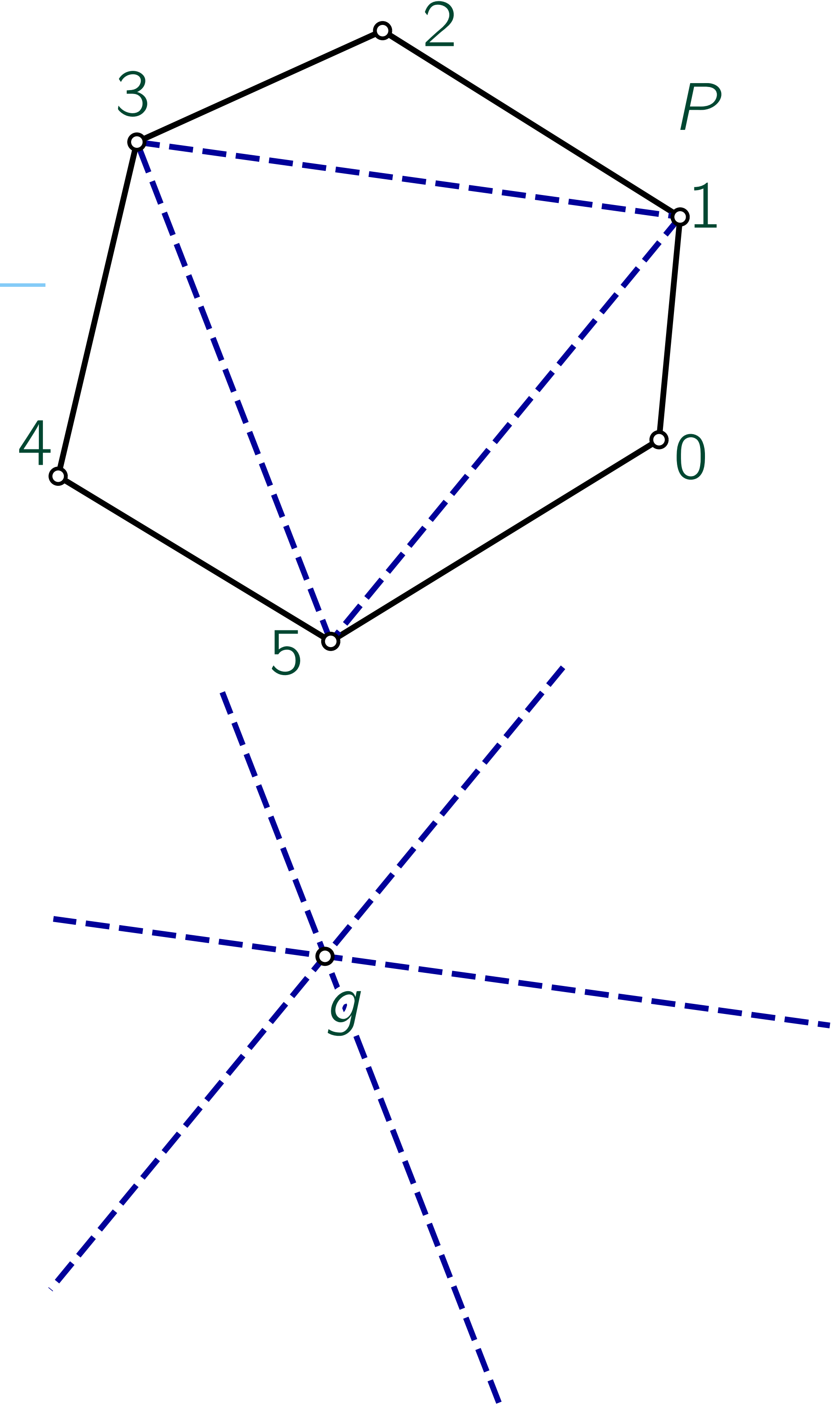
Look for a parallel hexagon Q

with $\text{area}(P, Q) = 0$.

- choose one arbitrary point g
- draw straight lines parallel to

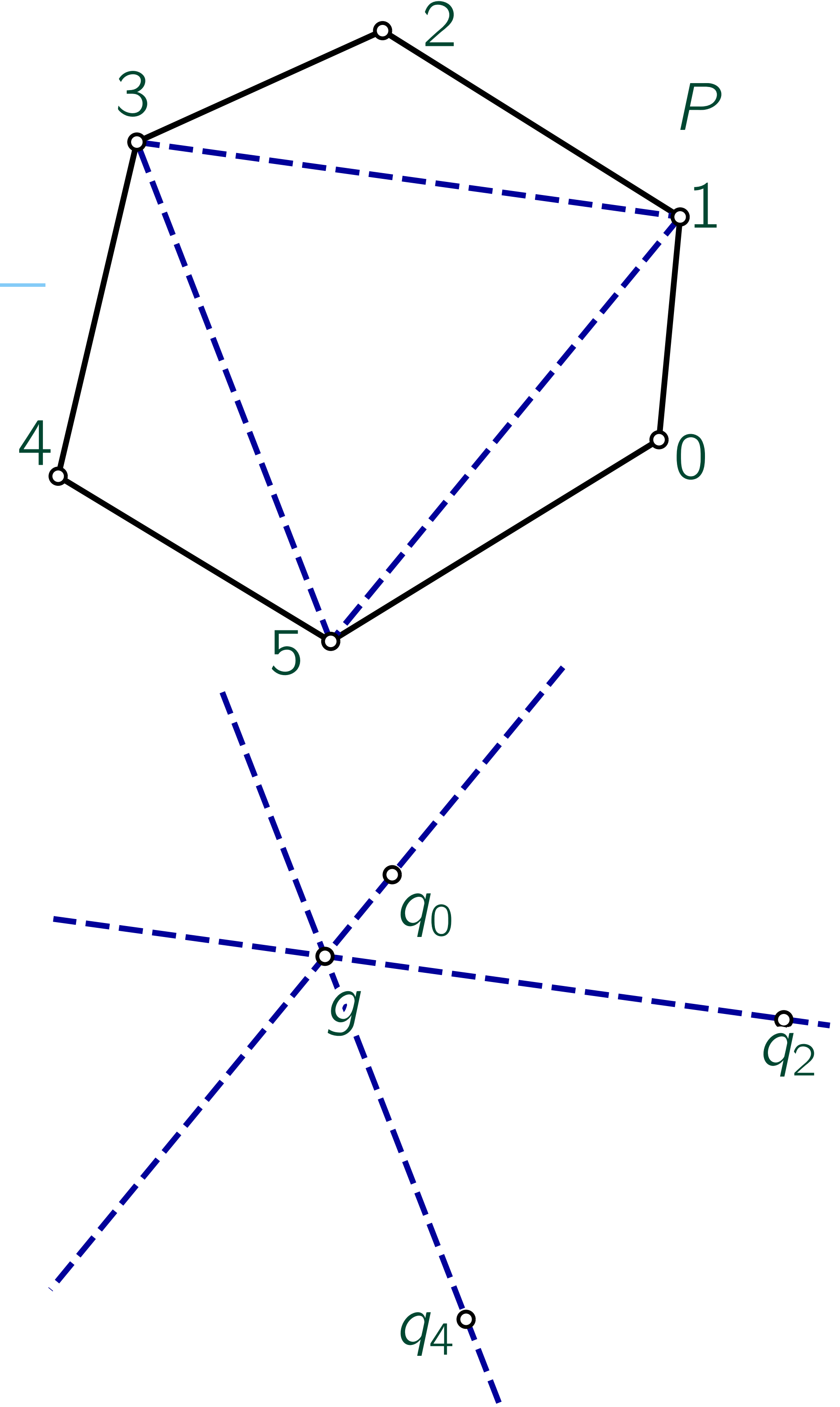
  $p_5 - p_1$, $p_1 - p_3$, $p_3 - p_5$

  through g



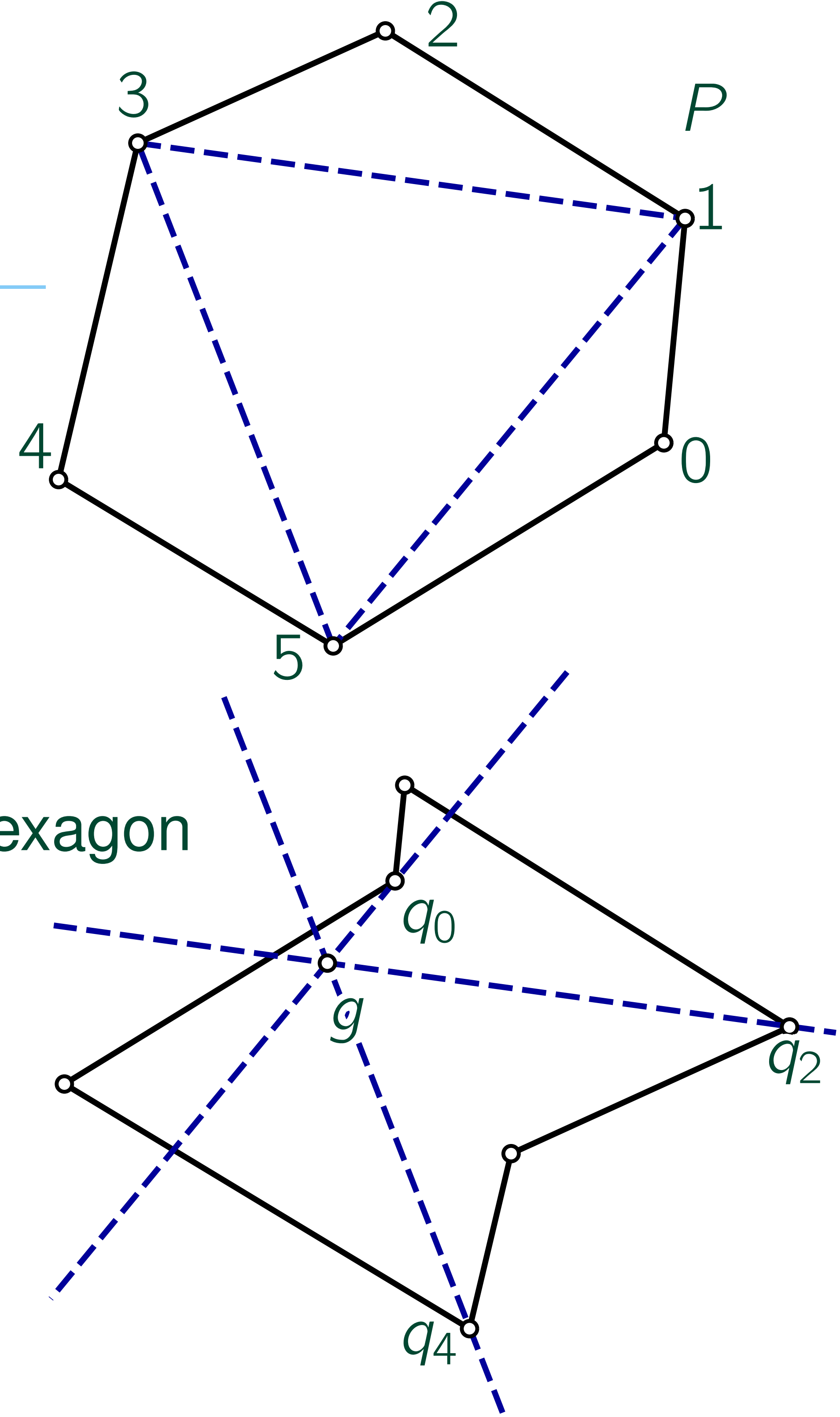
Construction

- choose points q_0, q_2, q_4



Construction

- choose points q_0, q_2, q_4
- Get hexagon Q by drawing edges parallel to the given hexagon



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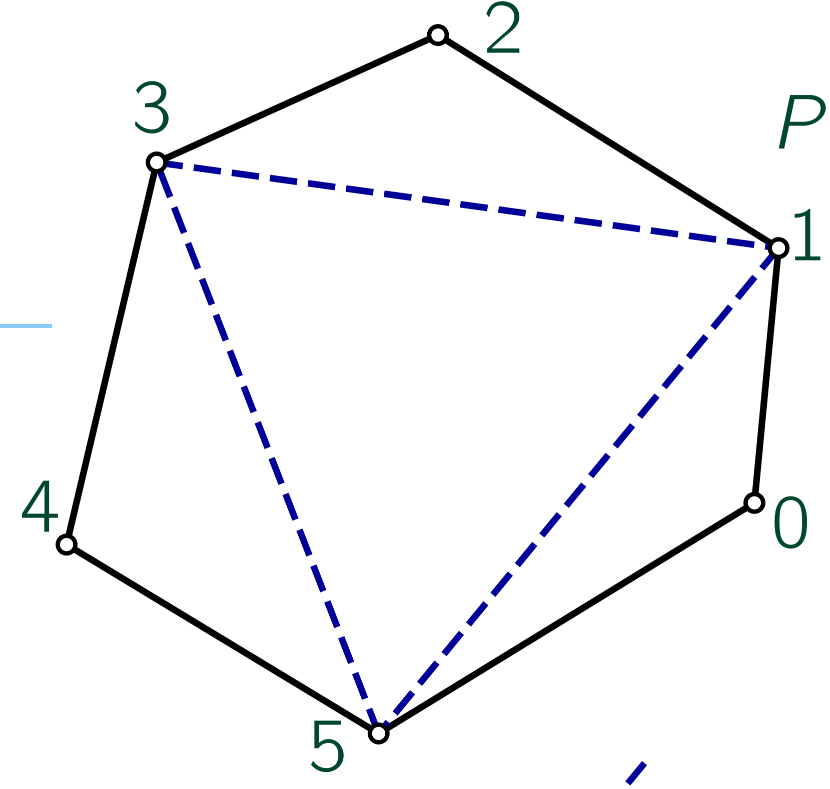
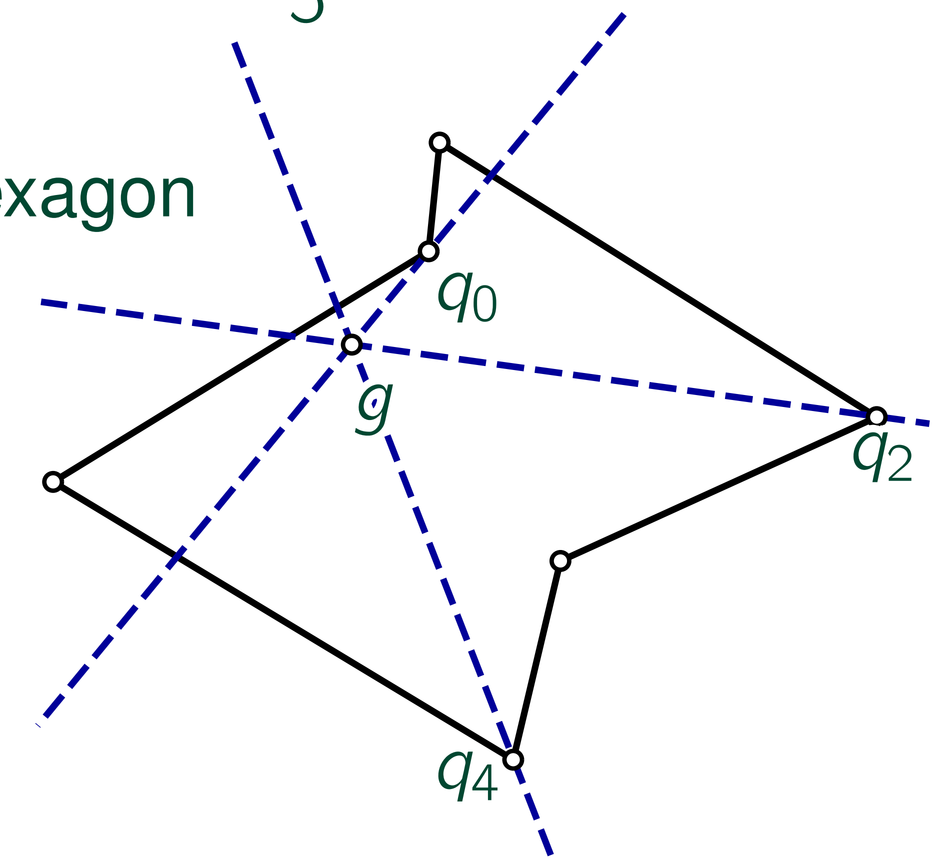


- choose points q_0, q_2, q_4
- Get hexagon Q by drawing edges parallel to the given hexagon

$\Longrightarrow \text{area}(P, Q) = 0$



Construction

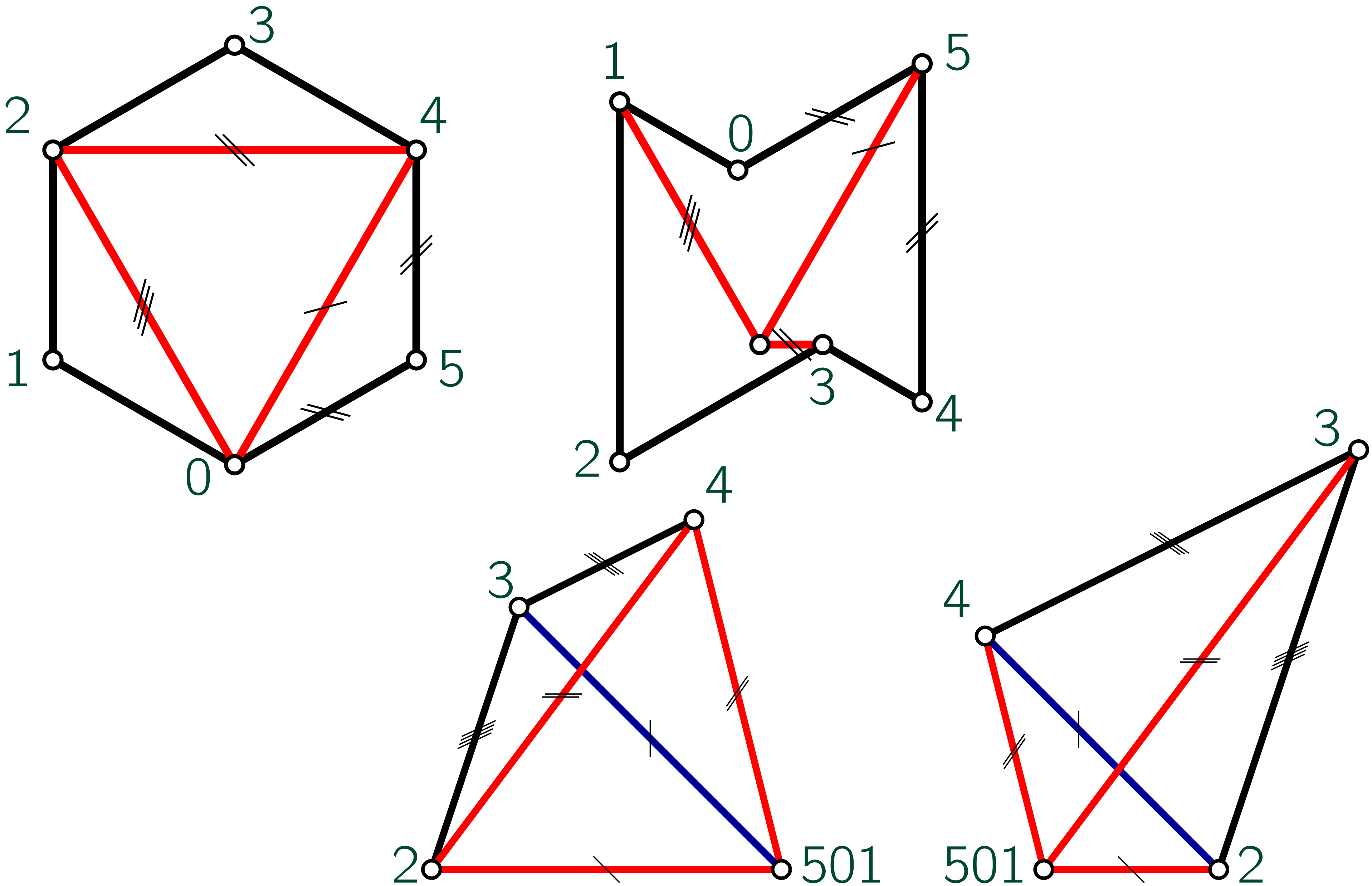
- The vector space of all hexagons parallel to a given one has dimension 4.

Construction

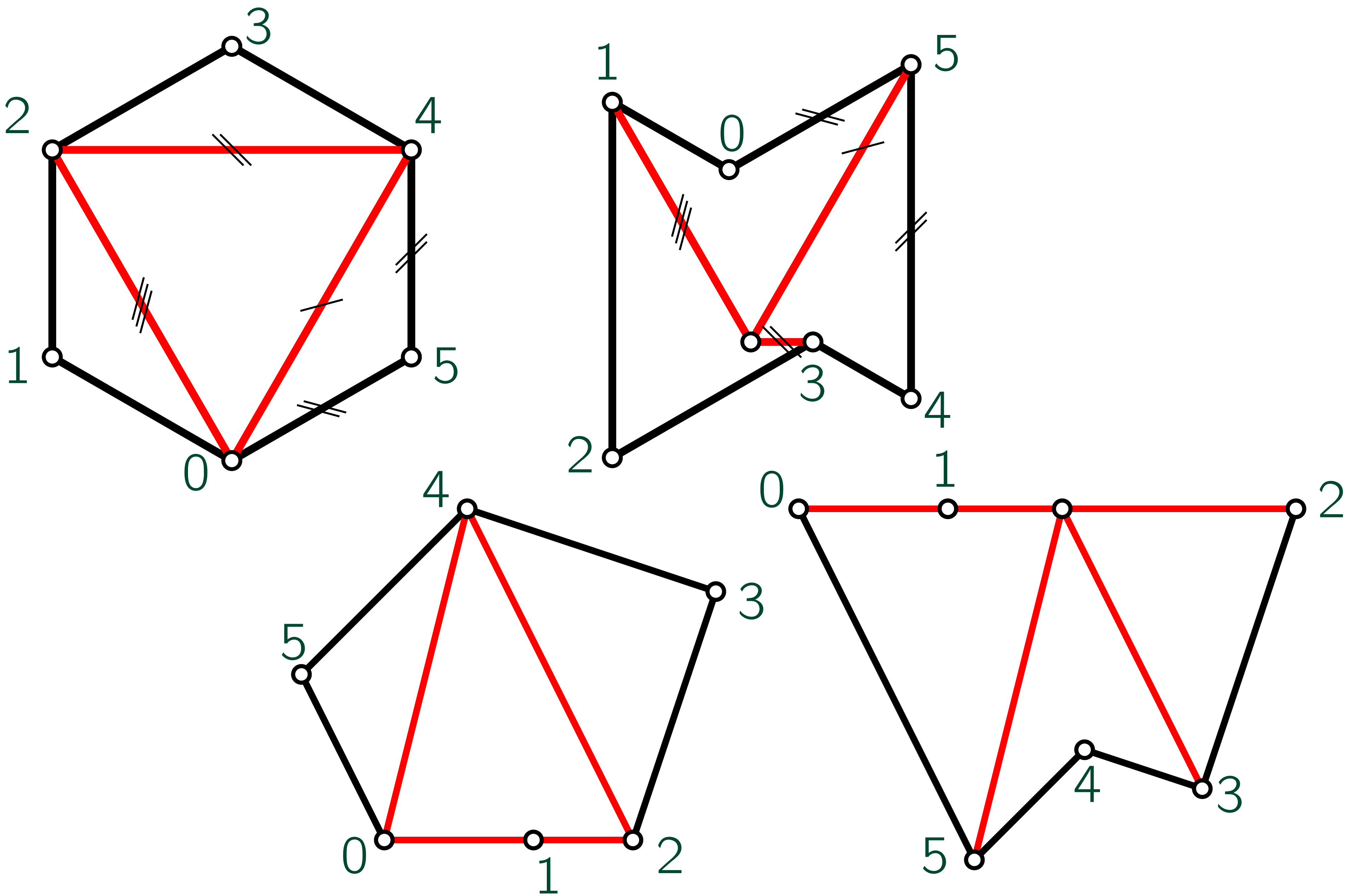
- The vector space of all hexagons parallel to a given one has dimension 4.
- Having vanishing mixed area is a linear condition.

  So these hexagons form a 3 dimensional subspace.

Quads as hexagons

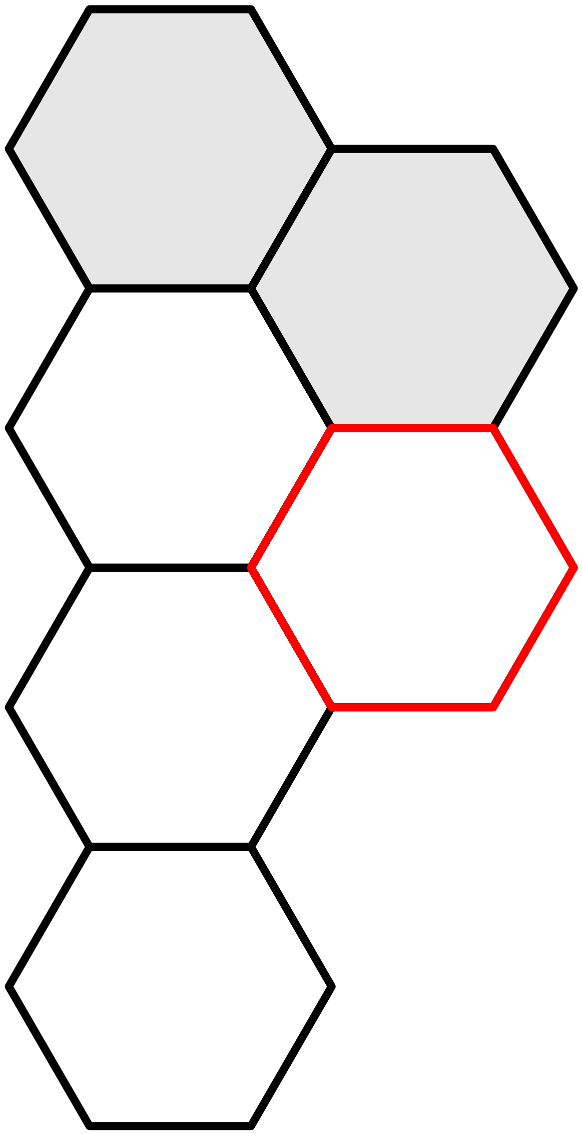


Pentagons as hexagons

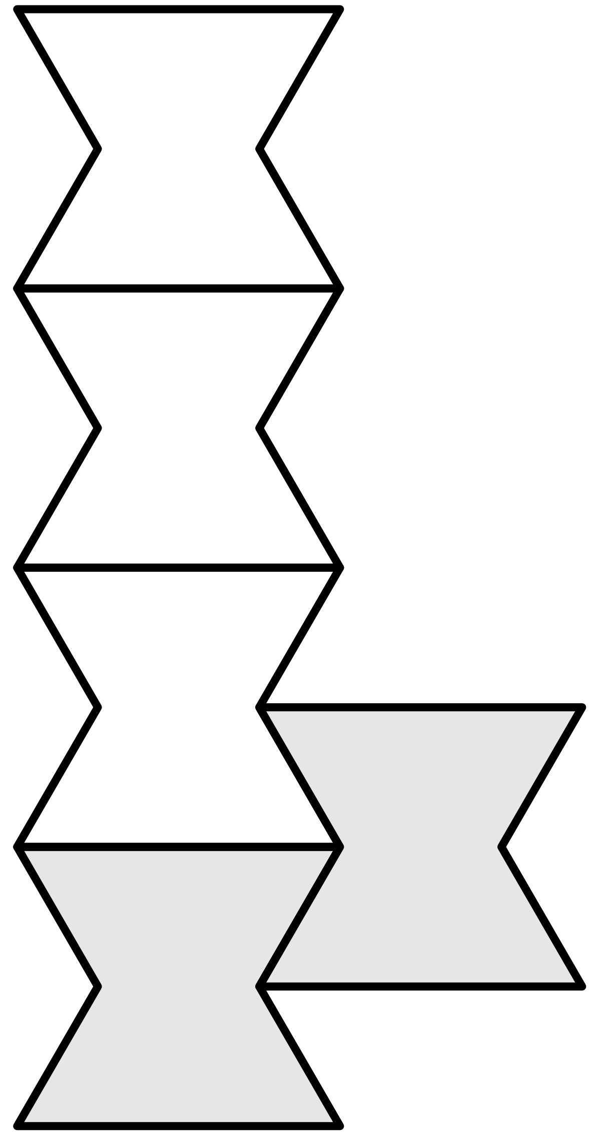


Discrete minimal surfaces

given mesh



dual mesh



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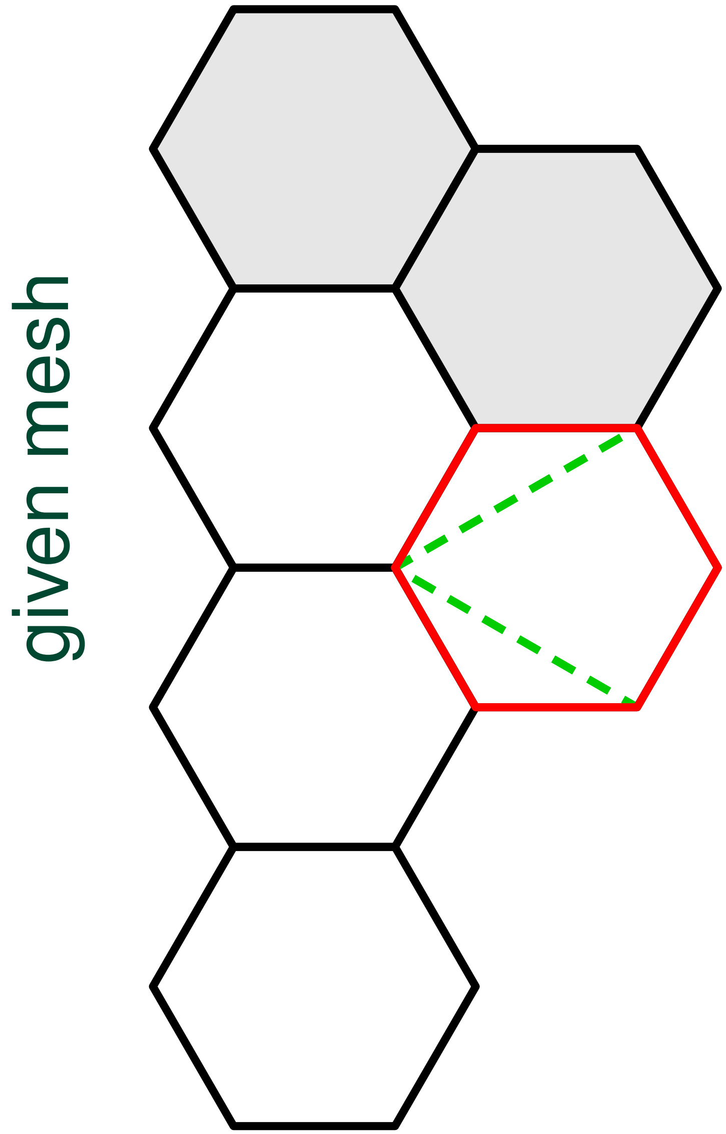
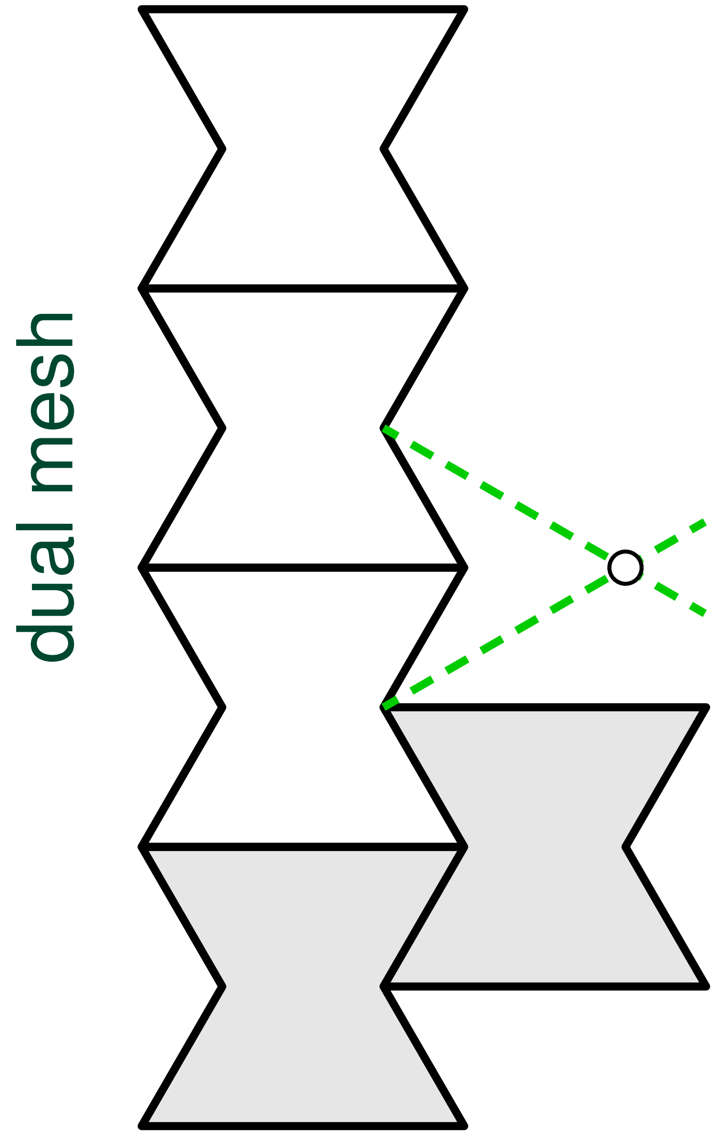
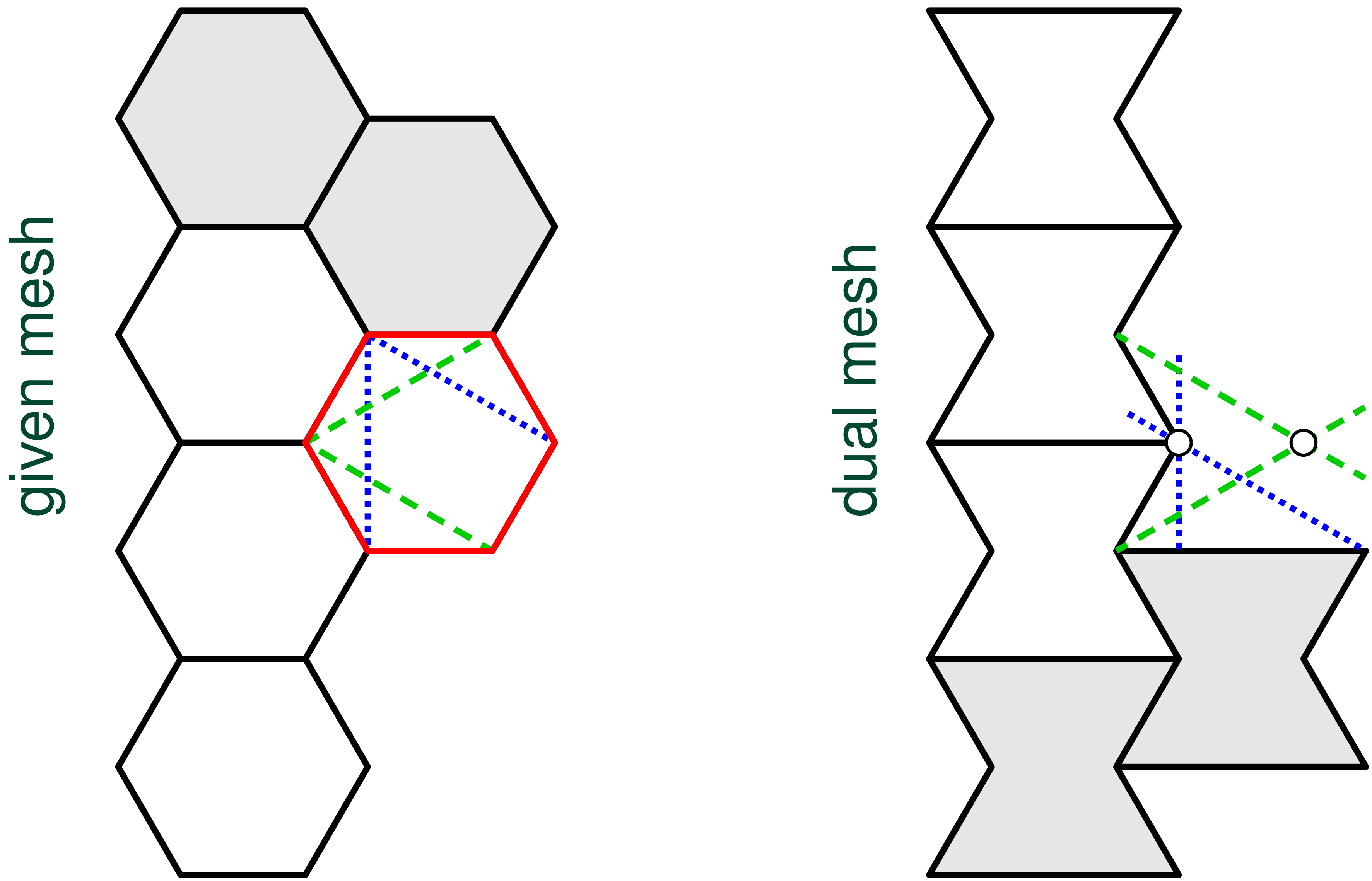
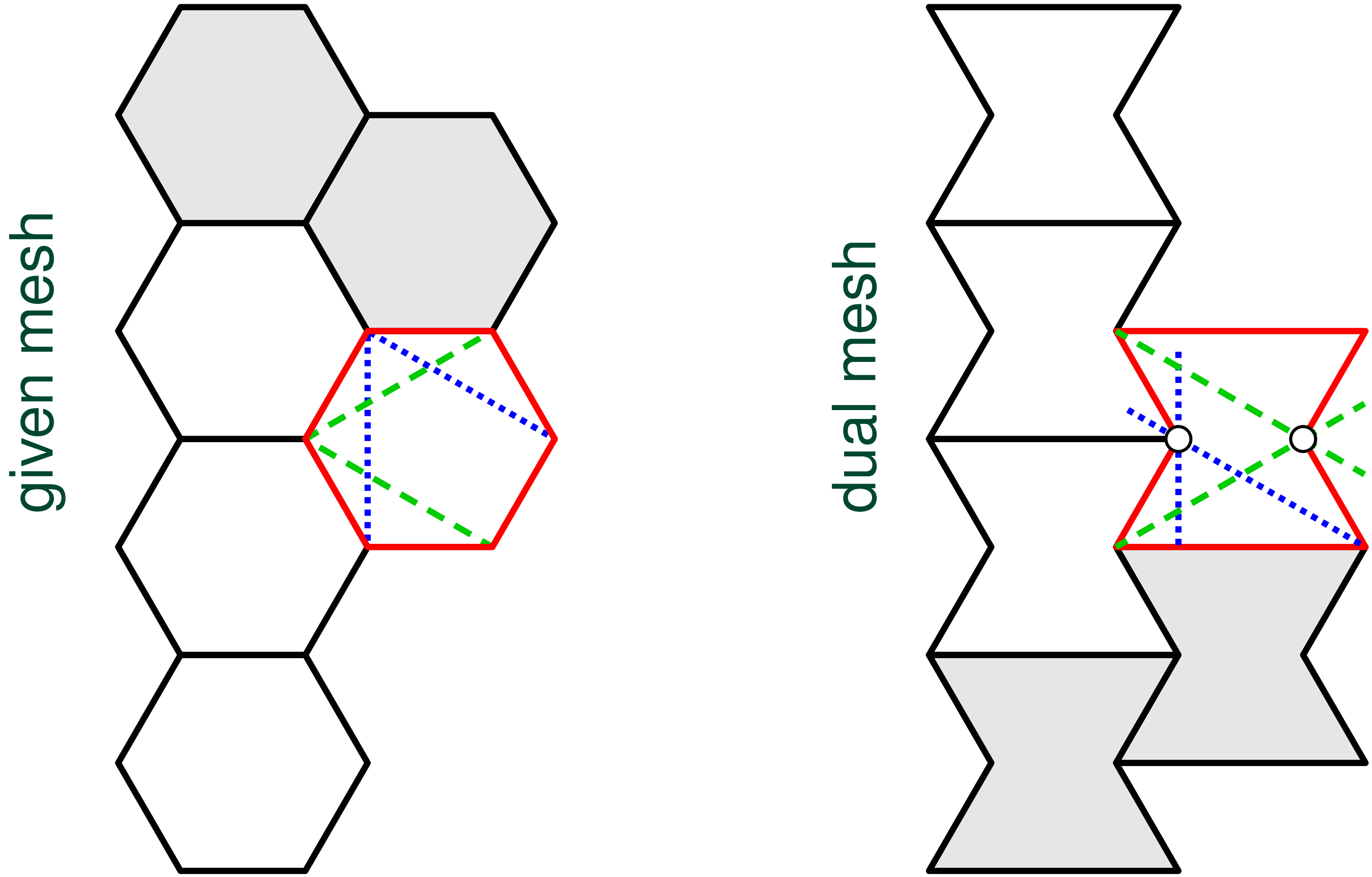




Discrete minimal surfaces



Discrete minimal surfaces



Discrete minimal surfaces

Christoffel dual construction

- f isothermic parametrisation

Discrete minimal surfaces

Christoffel dual construction

- f isothermic parametrisation
- construction of the dual surface f^*

$$f_u^* = \frac{f_u}{||f_u||^2} \qquad f_v^* = -\frac{f_v}{||f_v||^2}$$

Discrete minimal surfaces

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- sphere: $\mathrm{I} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \quad \mathrm{II} = \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix}$

Discrete minimal surfaces

Christoffel dual construction

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$$f_u^* = \frac{f_u}{\|f_u\|^2} \qquad f_v^* = -\frac{f_v}{\|f_v\|^2}$$

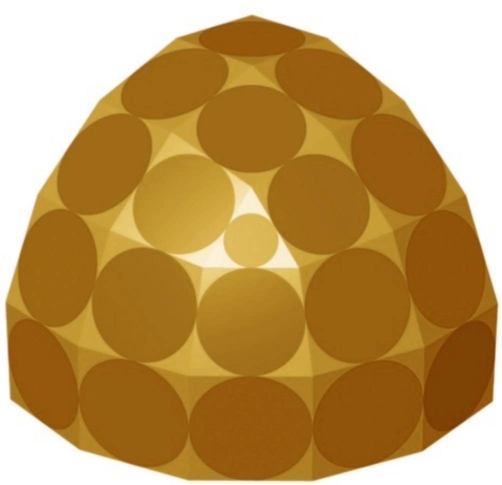
- sphere: $\mathrm{I} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \quad \mathrm{II} = \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix}$
- $\mathrm{I}^* = \begin{pmatrix} 1/\alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \quad \mathrm{II}^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow$ mean curvature $H = 0$

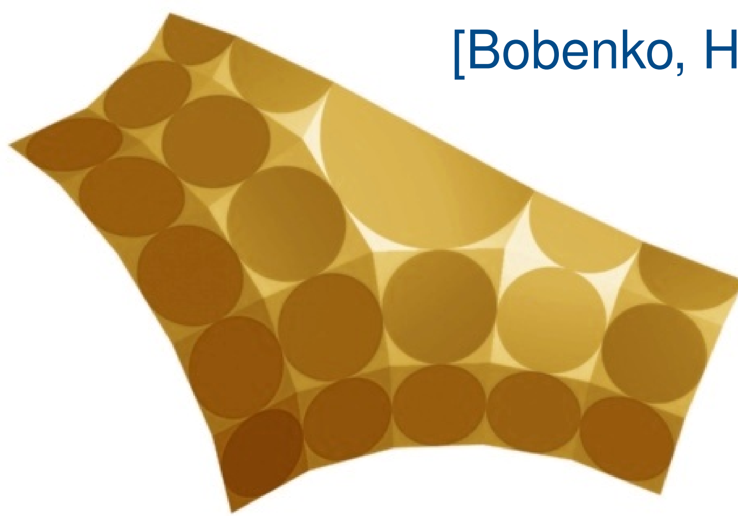
Discrete minimal surfaces

Discrete Christoffel dual construction

[Bobenko, Pinkall 96]

[Bobenko, Hoffmann, Springborn 06]

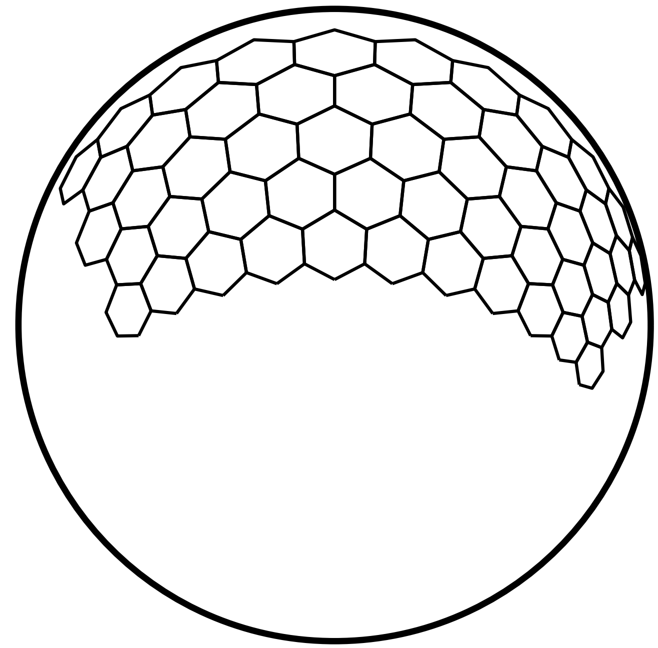




[Bobenko, Hoffmann, Springborn 06]

Discrete minimal surfaces

- hexagonal mesh with vertices on the unit sphere

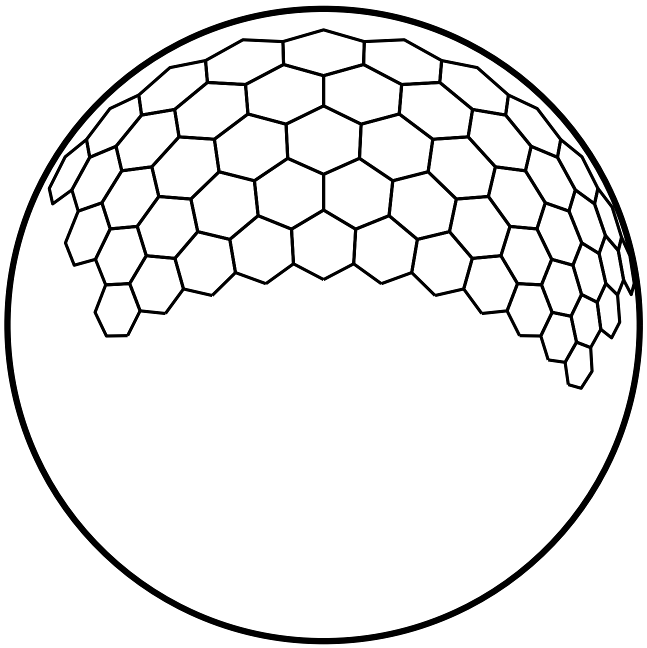


Discrete minimal surfaces

- hexagonal mesh with vertices

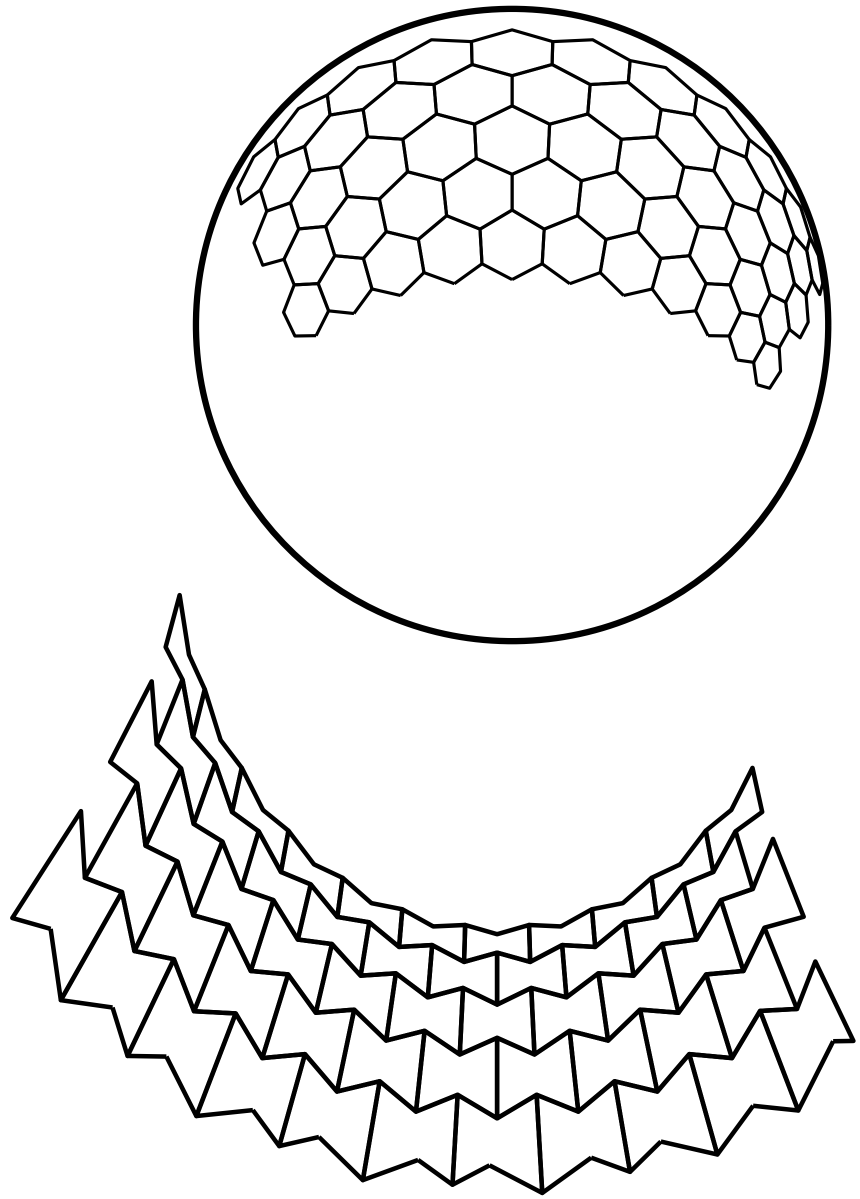
  on the unit sphere

- Christoffel dual construction

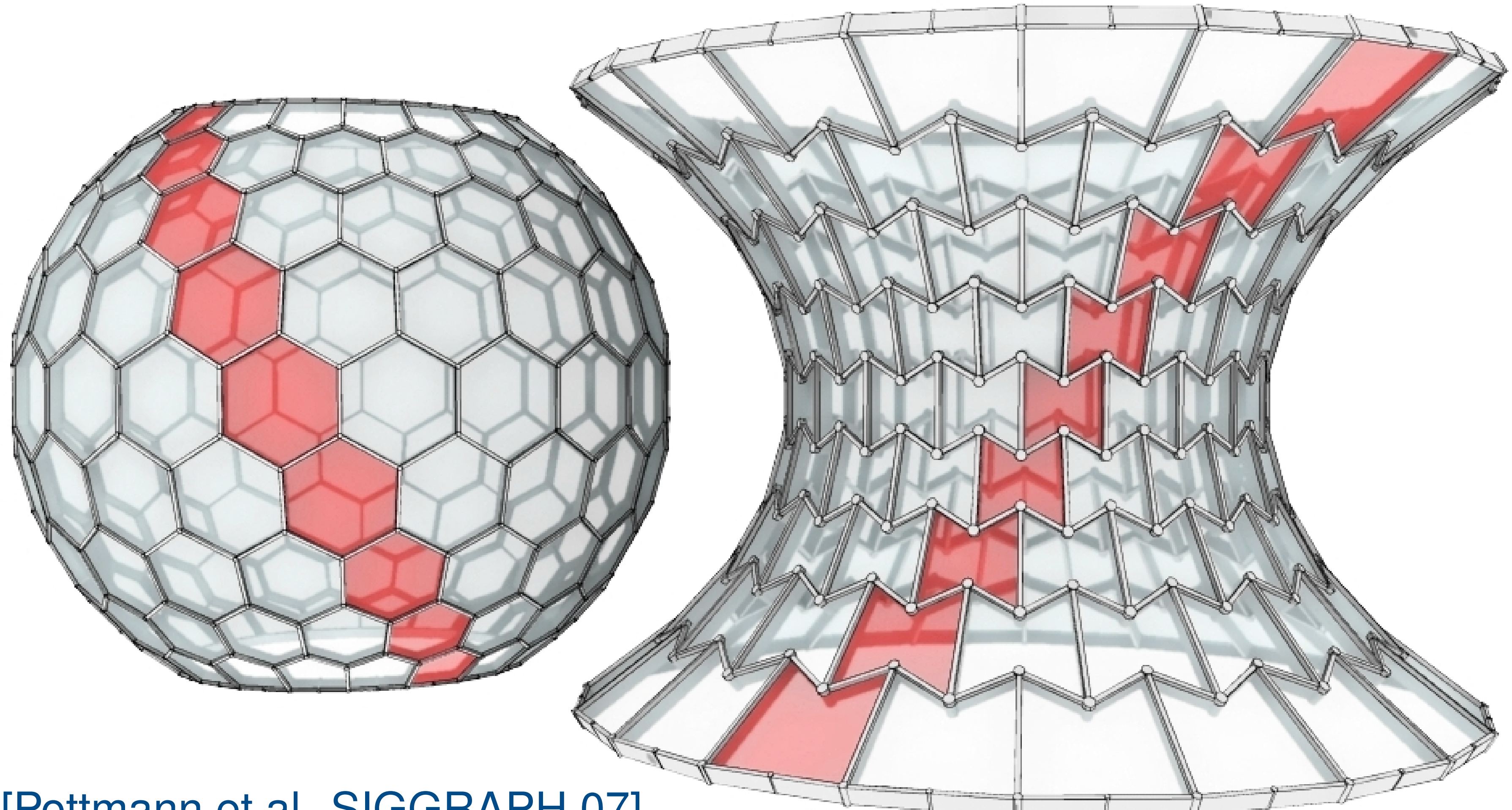


Discrete minimal surfaces

- hexagonal mesh with vertices on the unit sphere
- Christoffel dual construction
- $\Longrightarrow$ discrete minimal surface



Discrete minimal surfaces



[Pottmann et al. SIGGRAPH 07]

Summary

- relation between discrete minimal surfaces and mixed area

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- pairs of hexagons with vanishing mixed area

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