Orthogonality of hat functions in Sobolev spaces

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AND NOW FOR SOMETHING
COMPLETELY DIFFERENT

Outline:

- Recap: quasi interpolation
- Recap: orthogonality of uniform B-splines
- Orthogonality of hat functions in $\mathbb{R}^d$

Quasi interpolation

Consider a univariate spline space $\mathcal{S}$ of order n with B-splines B_j^n.
A quasi interpolant Λ of order ν is a linear operator

$$f \mapsto \Lambda f = \sum_j (\lambda_j f)\, B_j^n \in \mathcal{S}$$

given by functionals λ_j, with

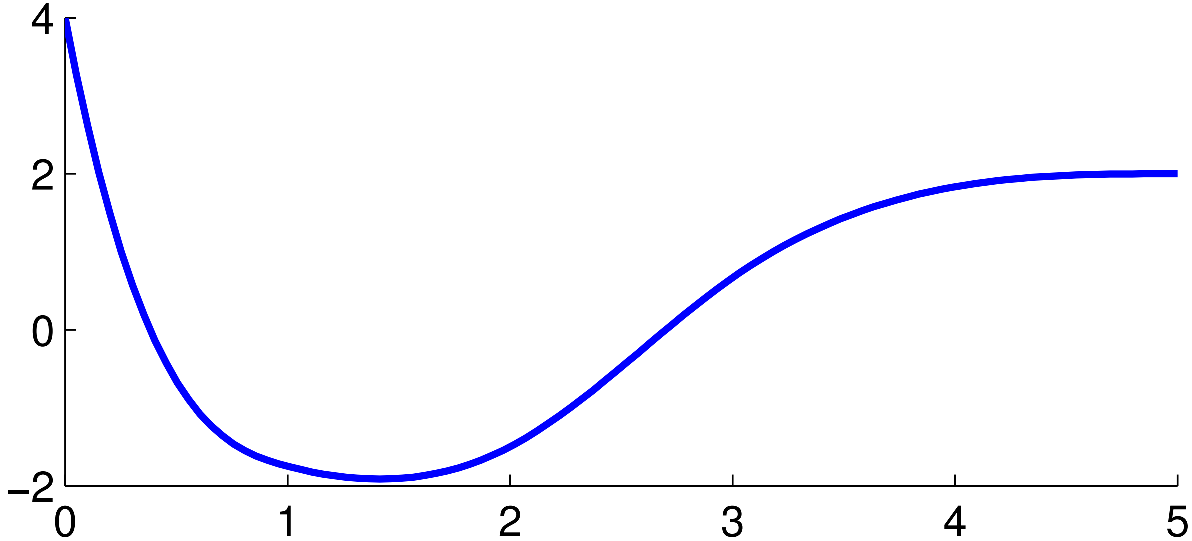
- Locality: $\lambda_j f$ depends only on f restricted to $\operatorname{supp} B_j^n$.
- Boundedness: $\sup_j \|\lambda_j\| < \infty$
- Polynomial precision: $\Lambda p = p$ for all $p \in \mathbb{P}^\nu$, or ideally $p \in \mathbb{P}^n$.

Schoenberg: With μ_j denoting the Greville abscissae,

$$\Lambda_j f = \sum_j f(\mu_j) B_j^n$$

is a QI of second order,

$$\|f - \Lambda f\| = O(h^2).$$



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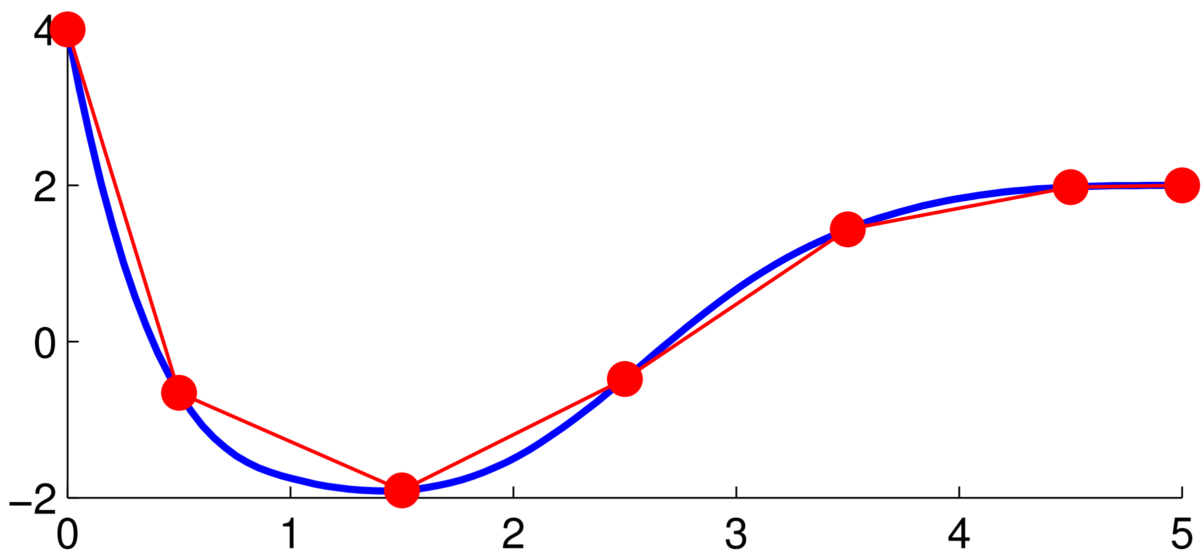

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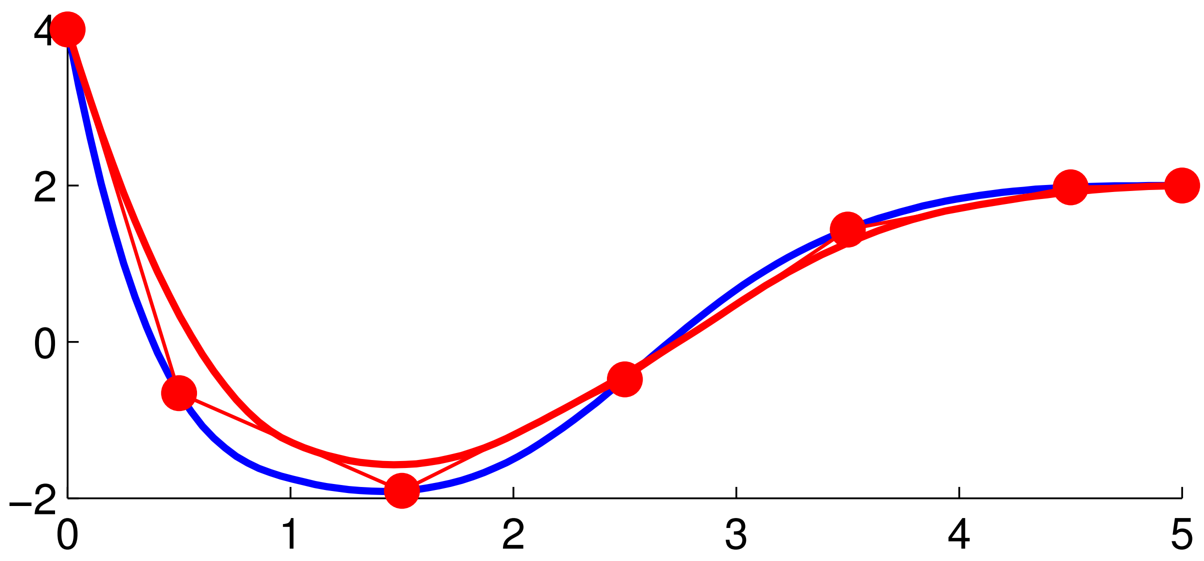

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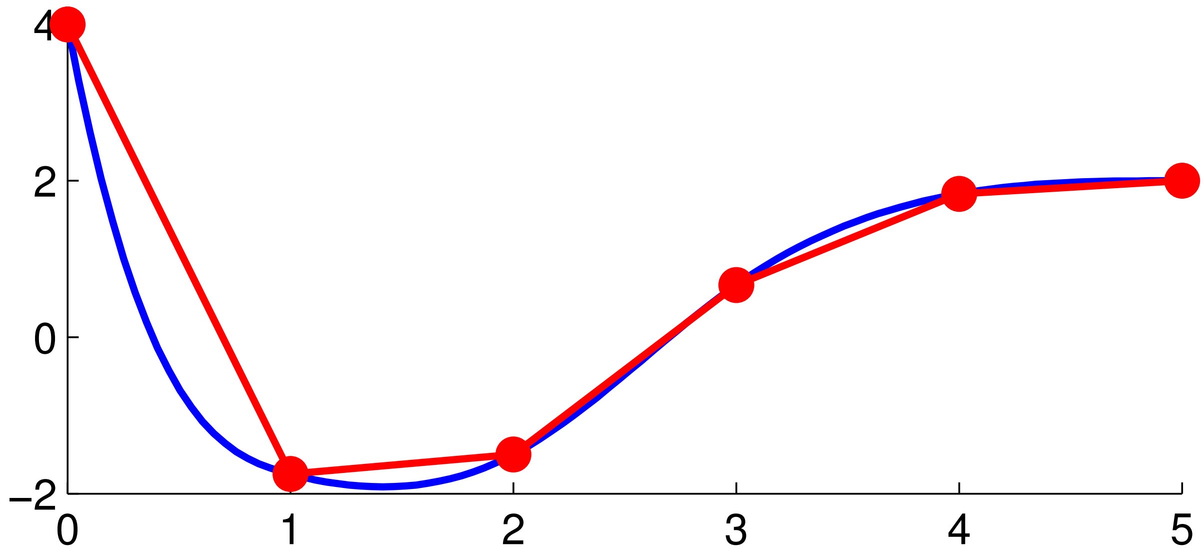
This is already optimal for the piecewise linear case $n = 2$,



de Boor-Fix:

For certain coefficients $\psi_{j,k}$,

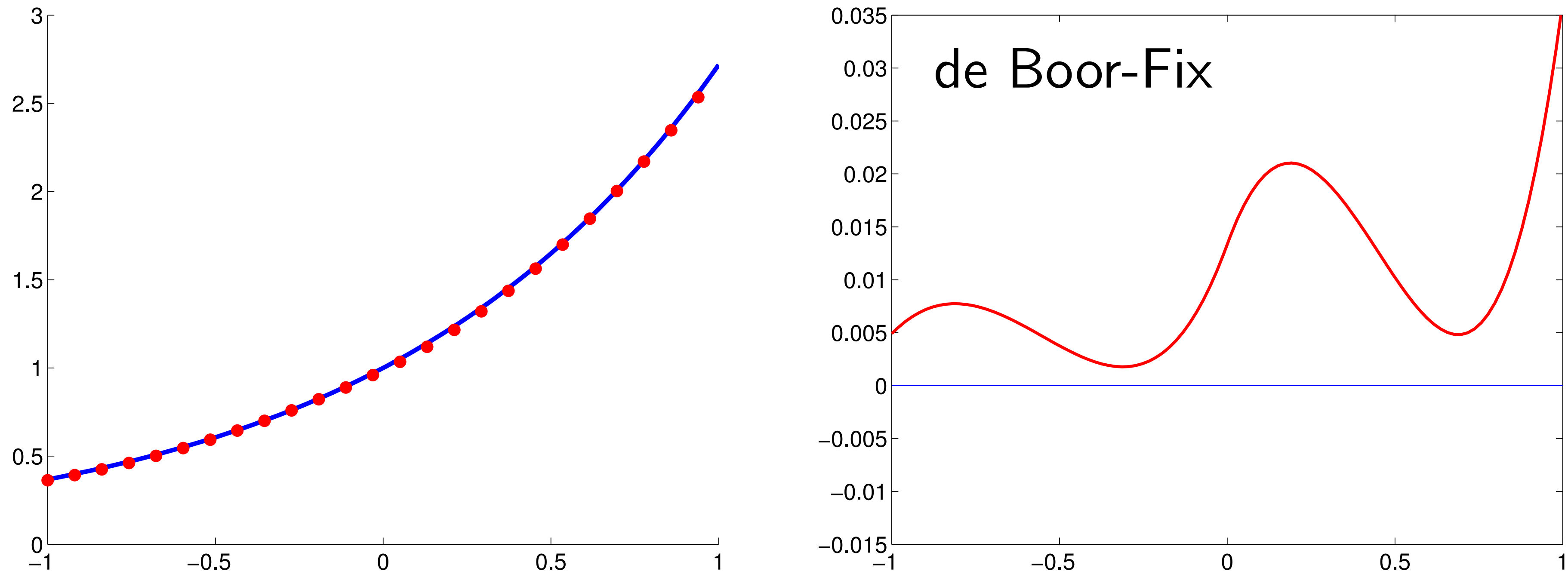
$$\Lambda_j f = \sum_j (\lambda_j f) B_j^n, \quad \lambda_j f = \sum_{k=0}^{n-1} \psi_{j,k} D^k f(\mu_j),$$

is a QI of maximal order n,

$$\|f - \Lambda f\| = O(h^n).$$

Example: quadratic splines with integer knots

$$\lambda_j f = f(j + 3/2) - \frac{1}{8} f''(j + 3/2)$$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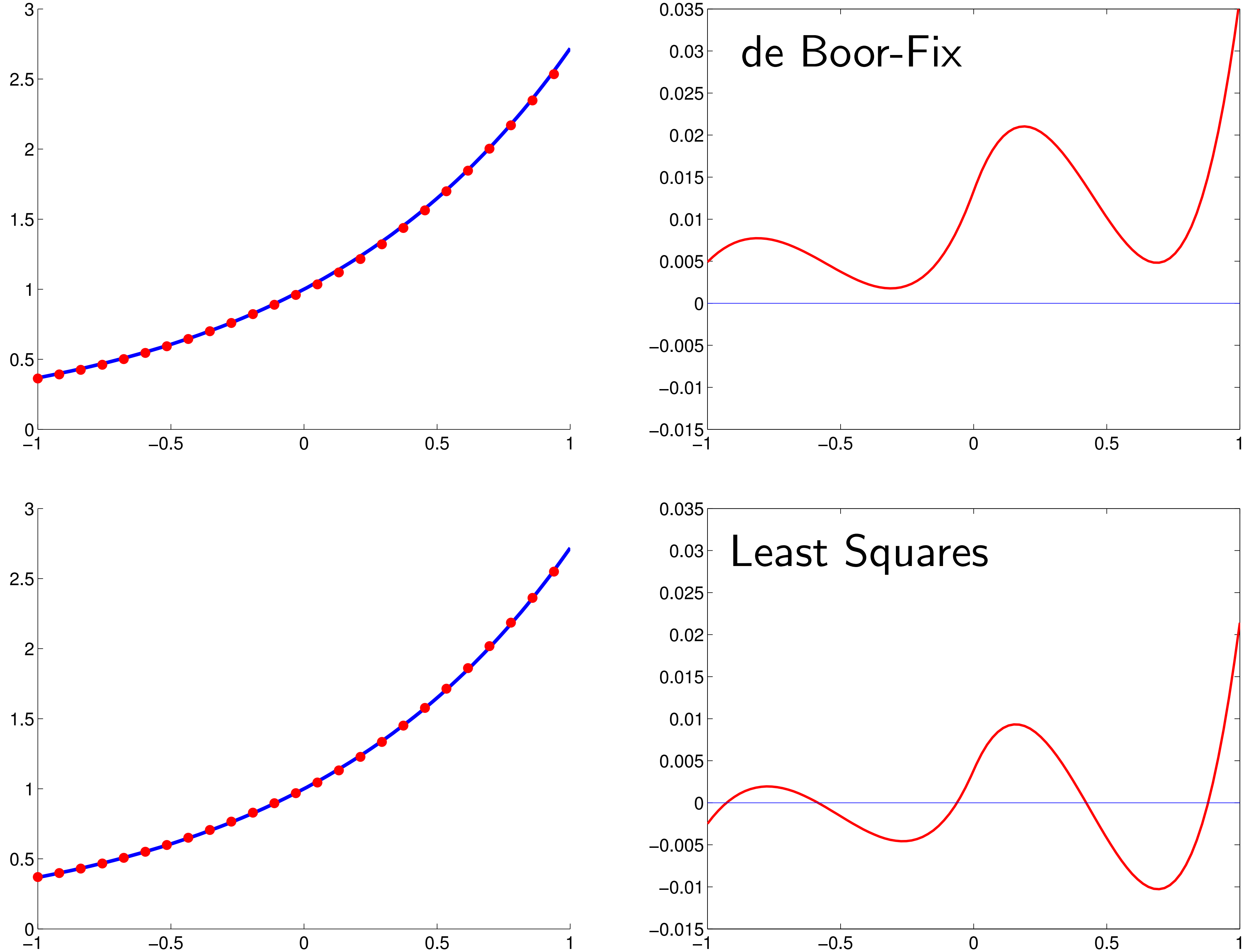
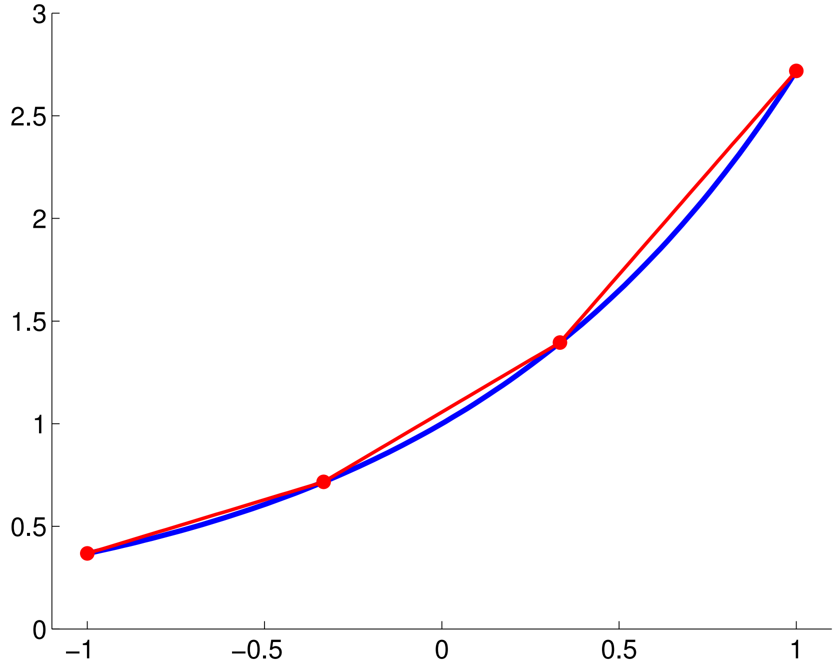
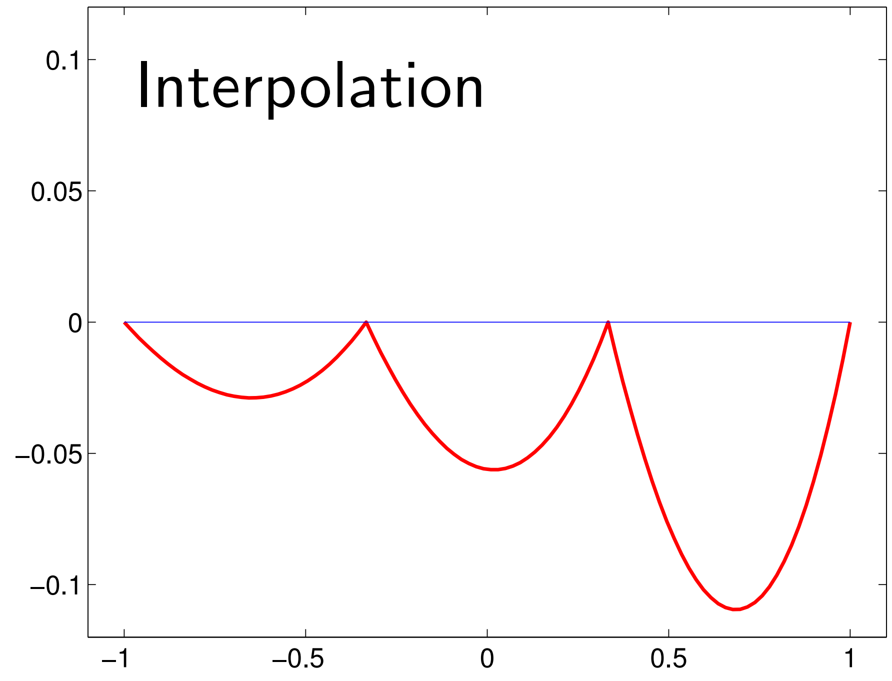
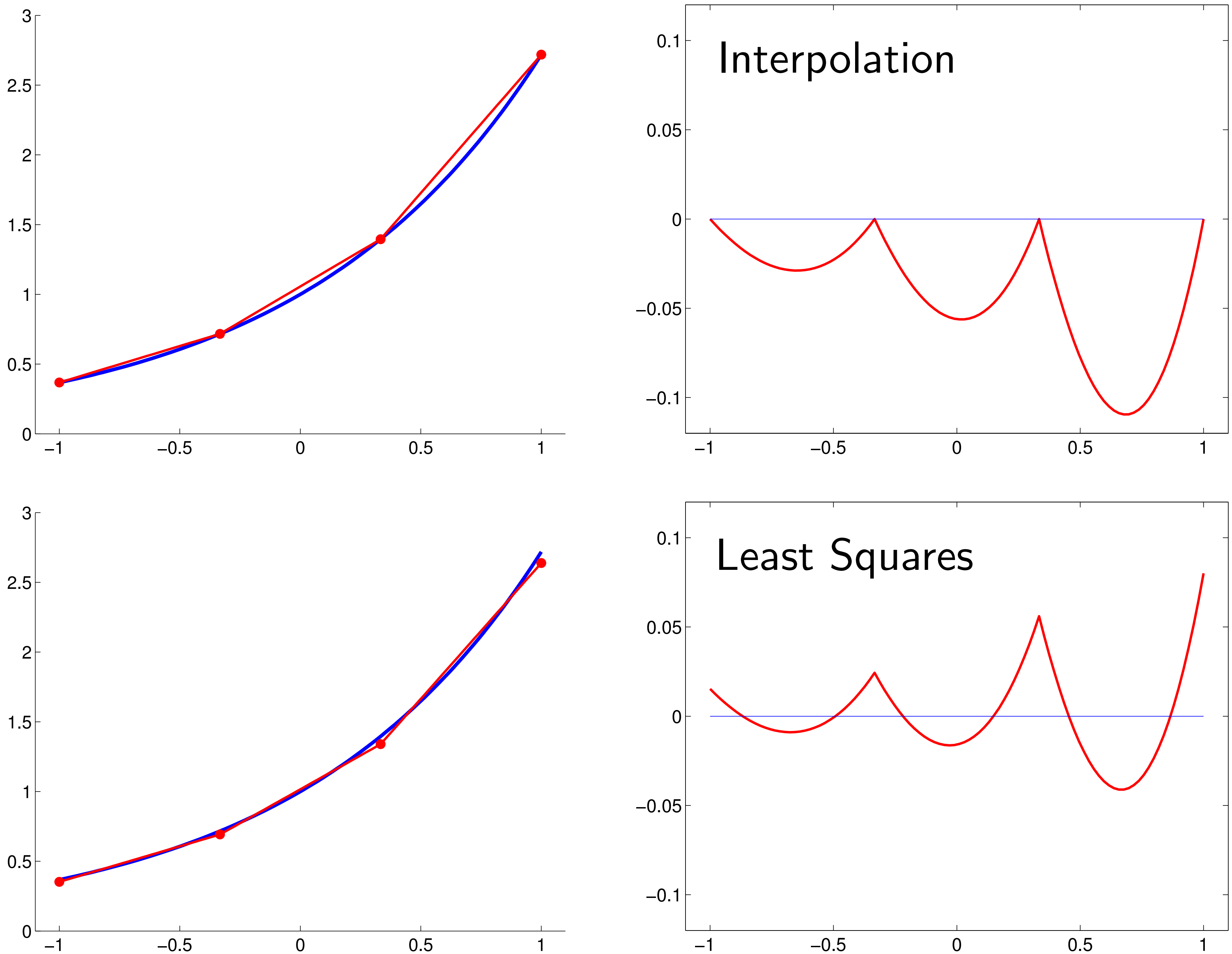

de Boor-Fix
Least Squares


3
2.5
2
1.5
1
0.5
0
–1
–0.5
0
0.5
1


Interpolation
0.1
0.05
0
–0.05
–0.1
–1
–0.5
0
0.5
1


Interpolation
Least Squares

Quasi interpolation vs. Least squares

Quasi interpolation	Least squares
❏ local rules for coefficients	❏ solve global system
❏ asymptotically optimal approximation order h^n	❏ asymptotically optimal approximation order h^n
❏ error not minimal wrt. norm	❏ error minimal wrt. norm

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How to combine the advantages?

- For polynomials: adapt the basis to the inner product.

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For positive weights $\omega := [\omega_0, \ldots, \omega_m]$,

$$(f, g)_\omega := \sum_{\mu=0}^{m} \omega_\mu \langle \partial^\mu f, \partial^\mu g \rangle$$

defines the Sobolev space $H^m_\omega(\mathbb{R})$. The induced norm $\| \cdot \|_\omega$ is equivalent to the standard norm with weights $\omega_0 = \cdots = \omega_m = 1$.

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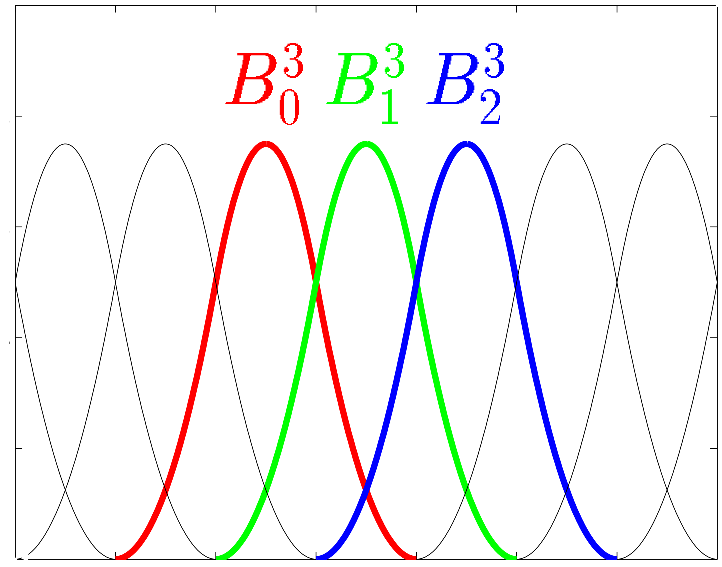
Since $B^n_j \in H^{n-1}(\mathbb{R})$, one can try to determine ω such that

$$(B^n_j, B^n_k)_\omega = \delta_{j,k}.$$

For uniform B-splines, this yields n conditions for the n weights in ω.



$$(B_0^3, B_0^3)_\omega = 1$$

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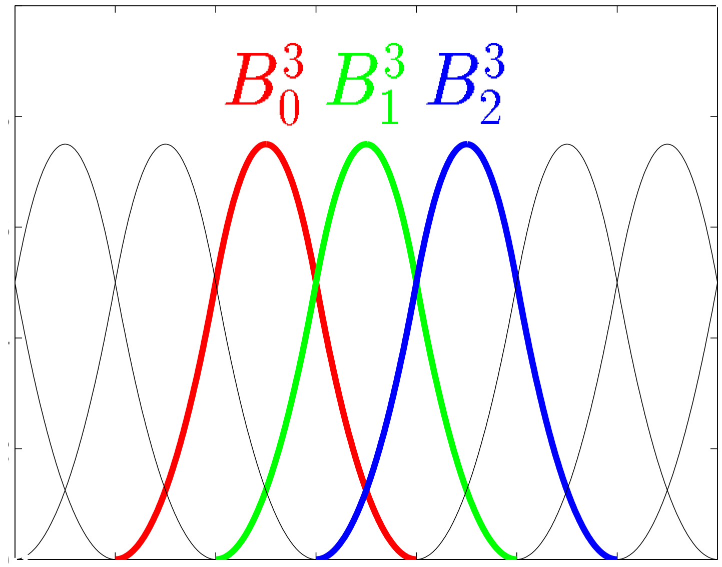

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Example: Quadratic B-splines with integer knots are orthonormal wrt.

$$(f, g)_{\omega(3)} = \langle f, g \rangle + \langle f', g' \rangle / 4 + \langle f'', g'' \rangle / 30\,.$$

More particular cases:

n	$\omega_0(n)$	$\omega_1(n)$	$\omega_2(n)$	$\omega_3(n)$	$\omega_4(n)$	$\omega_5(n)$
1	1					
2	1	$\frac{1}{6}$				
3	1	$\frac{1}{4}$	$\frac{1}{30}$			
4	1	$\frac{1}{3}$	$\frac{7}{120}$	$\frac{1}{140}$		
5	1	$\frac{5}{12}$	$\frac{13}{144}$	$\frac{41}{3024}$	$\frac{1}{630}$	
6	1	$\frac{1}{2}$	$\frac{31}{240}$	$\frac{139}{6048}$	$\frac{479}{151200}$	$\frac{1}{2772}$

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Under mild regularity assumptions, Fourier series and transforms

$$\hat{\varphi}(y) := \int_{\mathbb{R}} \varphi(x) \exp(-ixy)\, dx$$

$$\bar{\varphi}(y) := \sum_{j \in \mathbb{Z}} \varphi(j) \exp(-ijy)$$

are related by the Poisson summation formula

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With B^n the centered B-spline of order n, let

$$\varphi(x) := \big(B^n, B^n(\cdot - x)\big)_\omega.$$

Then orthonormality is equivalent to

$$\varphi(j) = \delta_{j,0} \quad \Leftrightarrow \quad \bar{\varphi}(y) \equiv 1 \quad \Leftrightarrow \quad \bar{\varphi}(y) \equiv 1 + \mathcal{O}(y^{2n})\,.$$

By the convolution property of B-splines,

$$\varphi(x) := (B^n, B^n(\cdot - x))_\omega$$

$$= \sum_{m=0}^{n-1} \omega_m (-1)^m \partial^{2m} B^{2n}(x)$$

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$$\begin{aligned}\bar{\varphi}(y) &= \sum_{k\in\mathbb{Z}} \hat{\varphi}(y + 2k\pi) \\ &= \sum_{m=0}^{n-1} \sum_{k\in\mathbb{Z}} \omega_m y^{2m} \operatorname{sinc}^{2n}(y/2 + k\pi) \\ &= \Big(\sum_{m=0}^{n-1} \omega_m y^{2m}\Big)\Big(\operatorname{sinc}^{2n}(y/2) + \mathcal{O}(y^{2n})\Big) \stackrel{!}{=} 1 + \mathcal{O}(y^{2n})\,.\end{aligned}$$

Theorem [R. '95]

- The sequence $\{B_j^n, j \in \mathbb{Z}\}$ of cardinal B-splines is orthonormal in $H_\omega^{n-1}(\mathbb{R})$ for weights $\omega = \omega(n)$ defined by

$$1/\operatorname{sinc}^{2n}(y/2) = \sum_{m=0}^{n-1} \omega_m(n) y^{2m} + \mathcal{O}(y^{2n}) \,.$$

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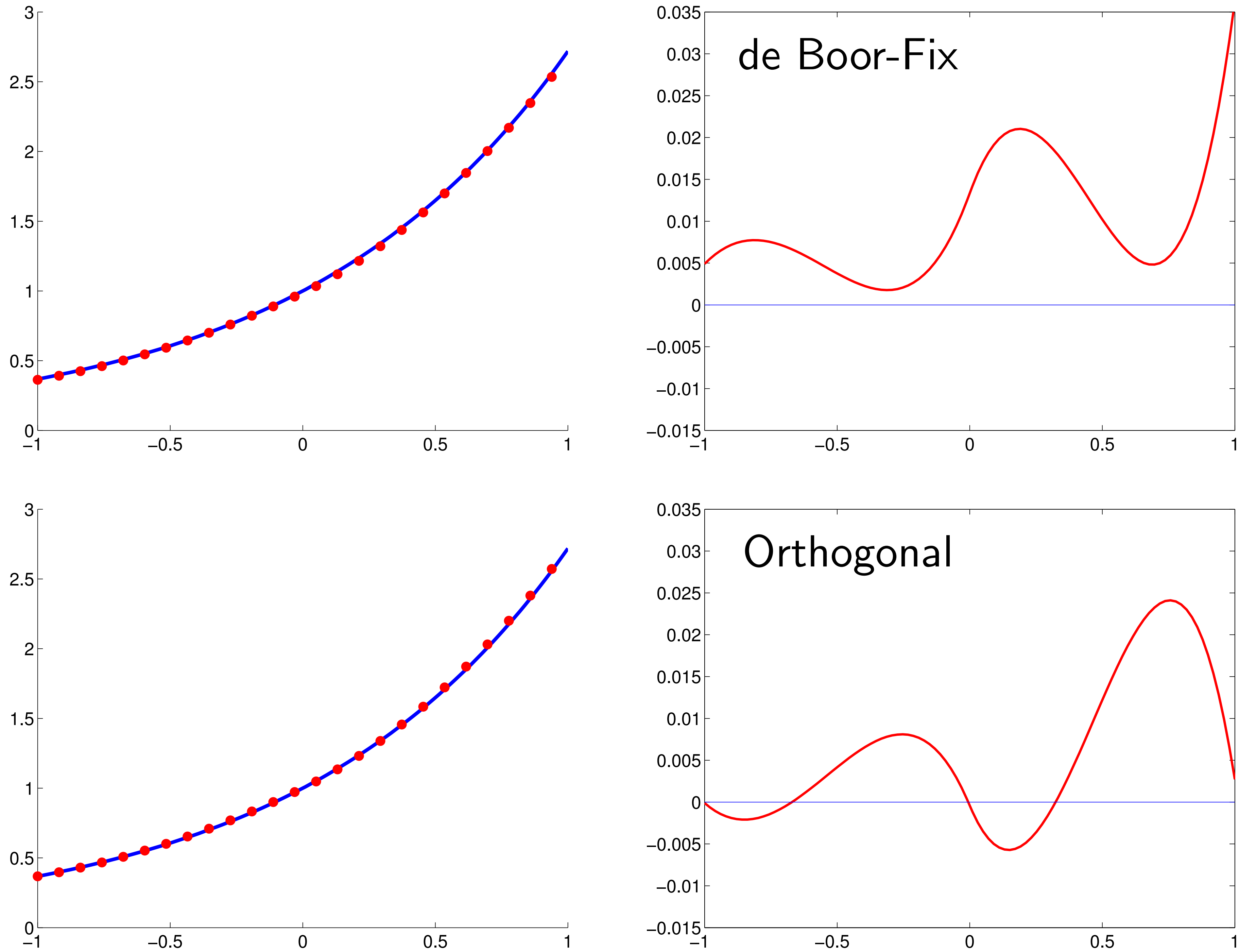
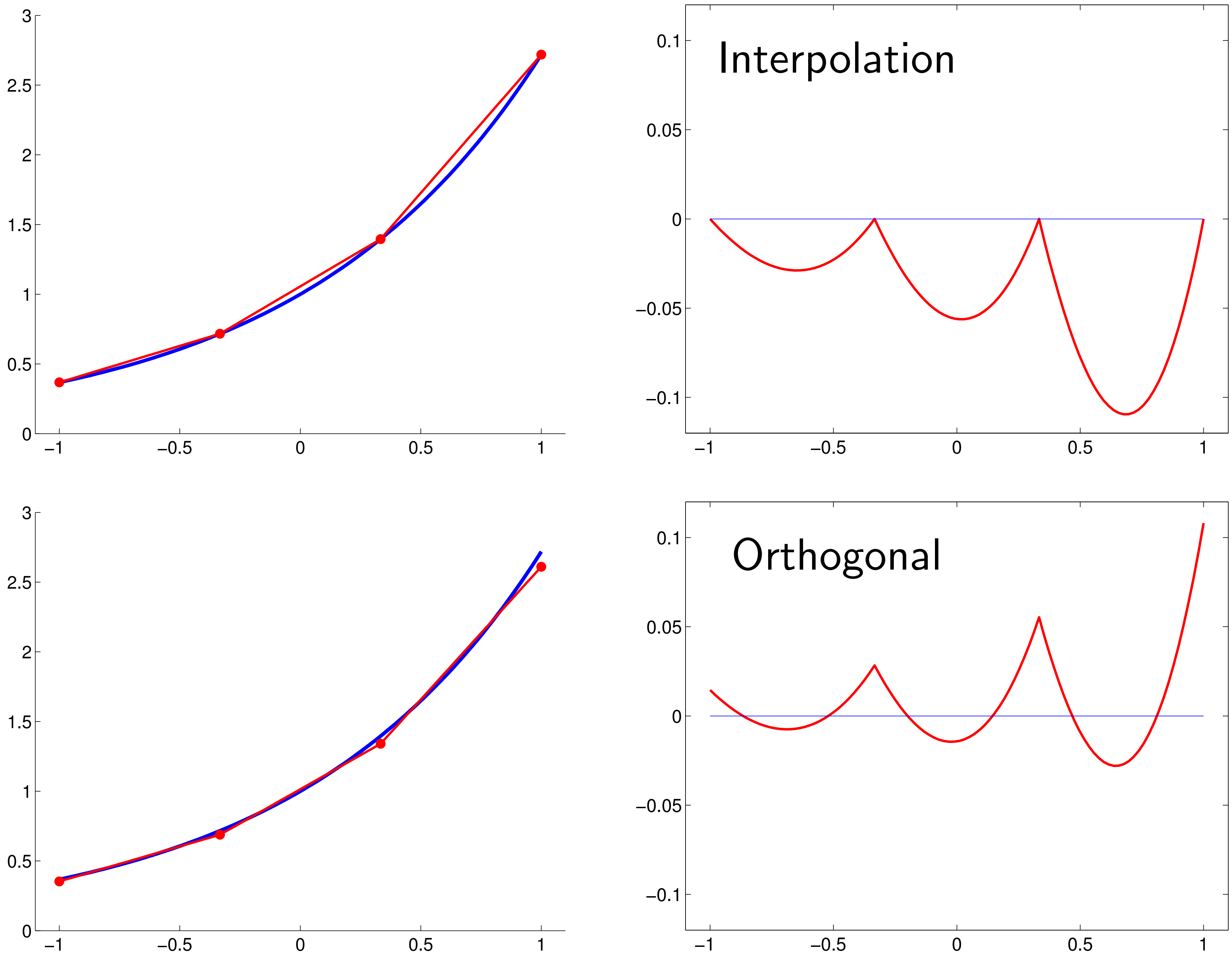
- The solution $g := \Lambda_h^n f$ to the least squares approximation problem of f in $H_{\omega(n,h)}^{n-1}(\mathbb{R})$ is given by the quasi interpolant Λ_h^n with

$$\lambda_h^n f := (B_h^n, f)_{\omega(n,h)}.$$


de Boor-Fix
Orthogonal



Interpolation
Orthogonal

Generalizations:

- approximation on bounded intervals for $n \leq 10$
- non-uniform B-splines
  - obvious for $n = 2$
  - tricky for $n = 3$
  - open for $n \geq 4$
- uniform tensor product B-splines
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- **new:** hat functions on arbitrary triangulations in $\mathbb{R}^d$

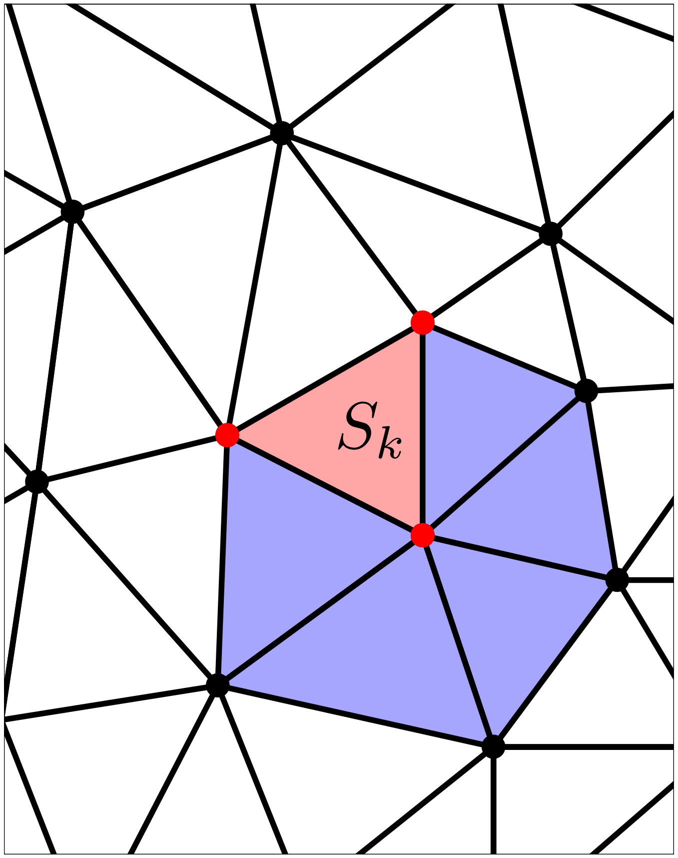
Hat functions:

The triangulation of a set $\Omega \subset \mathbb{R}^d$ consists of

- simplices $S_k, k \in K$, and
- vertices $V_i, i \in I$.

The hat functions $B_i : \Omega \to \mathbb{R}$ are defined by

$$B_i(V_j) = \delta_{i,j}\,, \quad i,j \in I$$
$$B_{i|S_k} \in \mathbb{P}_2(S_k)\,, \; k \in K\,.$$



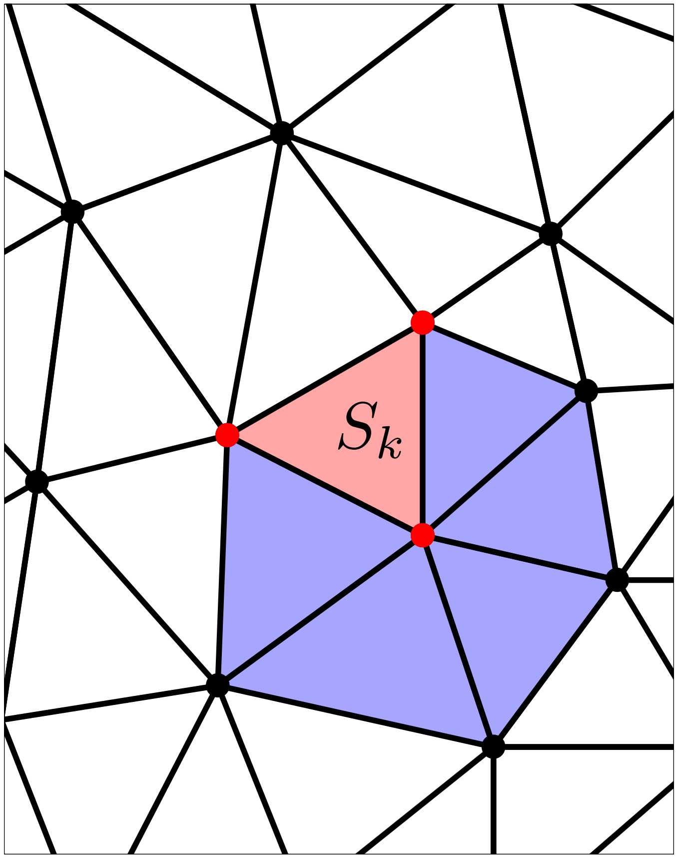
Hat functions:

The orthogonality conditions

$$(B_i, B_j)_\Omega = \delta_{i,j}\,, \quad i,j \in I$$

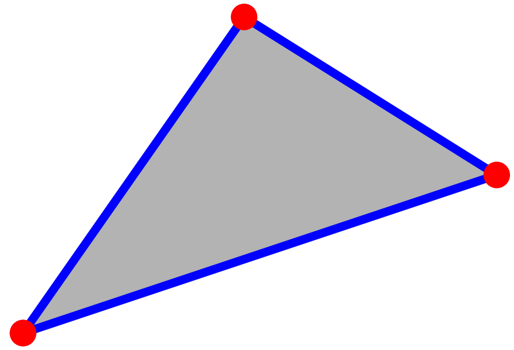
are globally coupled. A stronger, but local condition requires orthogonality on each simplex,

$$(B_i, B_j)_k = \delta_{i,j}\,, \quad i,j \in I\,,\ k \in K\,.$$



In $\mathbb{R}^2$: For each triangle S_k, there exist 3 non-vanishing hat functions, yielding

3 homogeneous conditions



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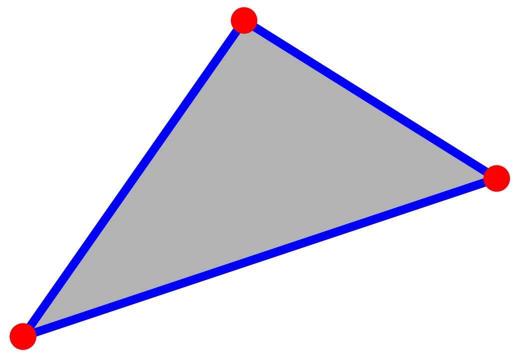


for orthogonality. The ansatz

$$(f, g)_k := \int_{S_k} \left(\omega_k f g + \nabla f \, W_k \, \nabla g^{\mathrm{t}}\right)$$

$$\omega_k > 0, \quad W_k \text{ spd } (2 \times 2)\text{-matrix},$$

provides the appropriate number of degrees of freedom.

Theorem [App, R.] On the unit triangle

$$S := \{\mathbf{x} \in \mathbb{R}^2_{\geq 0} : x_1 + x_2 \leq 1\},$$

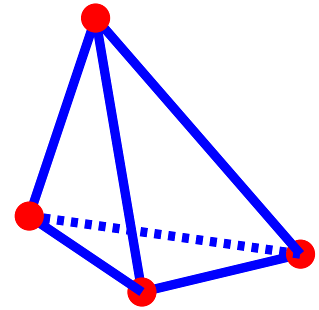
the weights

$$\omega := 6 \ , \quad W := \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}$$

yield orthonormality.

In $\mathbb{R}^d$: For each simplex S_k, there exist $d+1$ non-vanishing hat functions, yielding

$$\frac{d^2+d}{2} \quad \text{homogeneous conditions}$$



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the weights

$$\omega := (d+1)! \ , \quad W := \frac{d!}{d+2} \begin{bmatrix} d & -1 & -1 & \cdots & -1 \\ -1 & d & -1 & \cdots & -1 \\ & \vdots & & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & d \end{bmatrix}$$

yield orthonormality.

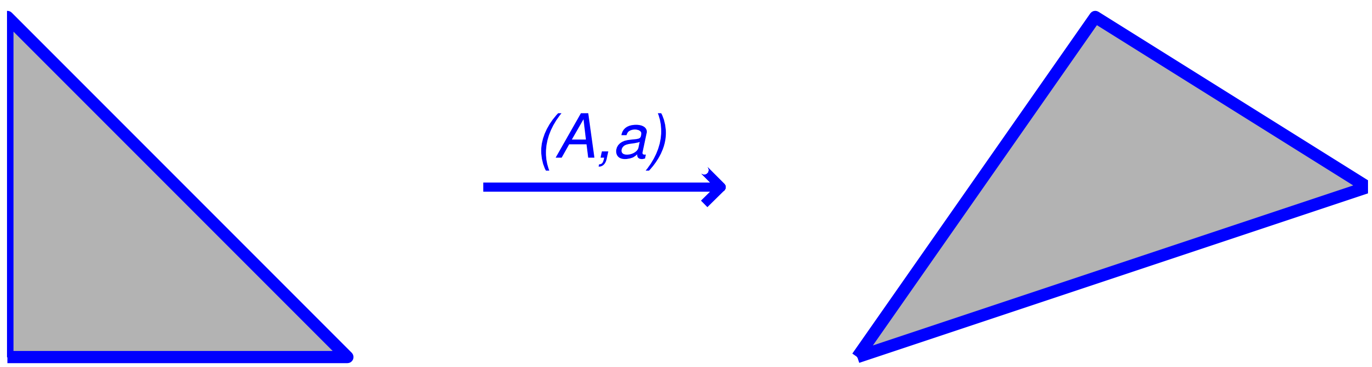
For an arbitrary simplex S_k, obtained from S by an affine map

$$S_k = AS + a,$$

the weights

$$\omega_k := \frac{\omega}{|\det A|}, \quad W_k := \frac{AWA^{\mathrm{t}}}{|\det A|}$$

yield orthogonality.



Orthogonality: Define the Sobolev type inner product

$$(f,g)_\Omega := \sum_{k\in K}(f,g)_k = \sum_{k\in K}\int_{S_k}\big(\omega_k f g + \nabla f\, W_k\, \nabla g^{\mathrm{t}}\big),$$

then, with $\#B_i$ the number of simplices in $\operatorname{supp} B_i$,

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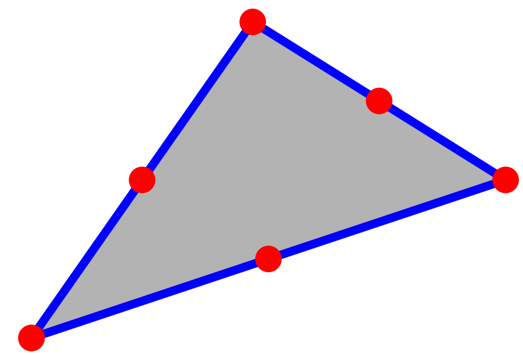
Approximation: The solution $g = \Lambda f$ to the least squares approximation problem of f wrt. the inner product $(\cdot,\cdot)_\Omega$ is given by

$$\Lambda f = \sum_i \frac{(f,B_i)_\Omega}{\#B_i}\,B_i.$$

Discrete Variants: Denote vertices and edge midpoints of S_k by

$$P_k = \begin{bmatrix} p_k^1 \\ \vdots \\ p_k^m \end{bmatrix}$$



and the quadratic polynomial interpolating the function f at P_k by

$$q_k = Q_k f.$$

Then

$$[f, g]_\Omega := \sum_{k \in K} (Q_k f, Q_k g)_k$$

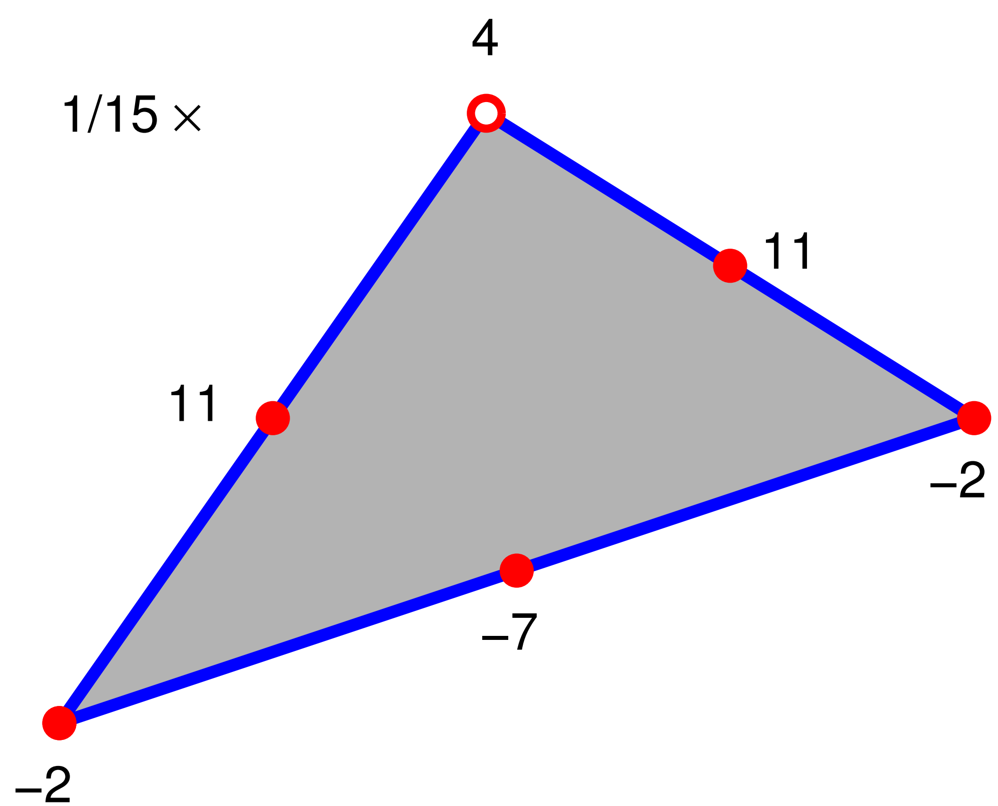
is a discrete inner product with

$$[B_i, B_j]_\Omega = \sum_{k \in K} (Q_k B_i, Q_k B_j)_k = \sum_{k \in K} (B_i, B_j)_k = (B_i, B_j)_\Omega.$$

Evaluation of $[f, B_i]_\Omega$ is cheap,

$$[f, B_i]_k = (Q_k f, B_i)_k = V \cdot F_k, \quad F_k = f(P_k),$$

where V is independent of k. In the $2d$-case,



Average savings in 2d: For similar accuracy wrt.

- max norm, interpolation requires $\approx 40\%$ more triangles.
- L^2-norm, interpolation requires $\approx 100\%$ more triangles.

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Applications:

- Computer graphics?
- FEM
- Nonlinear optimization
- . . .

Conclusion:

- Sobolev orthogonality available for uniform B-splines
- Sobolev orthogonality available for hat functions in $\mathbb{R}^d$
- Cheaper than L^2
- Better than QI