Discrete Conformal Structures

Boris Springborn (TUB)
Ulrich Pinkall (TUB)
Peter Schröder (Caltech)

Multi-Res Modeling Group

The Problem

Find "nice" texture maps

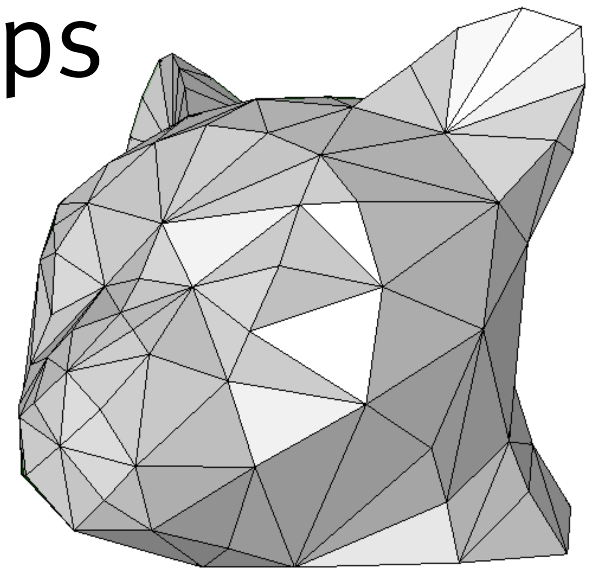
- simplicial surface

$$K = \{V, E, T\}$$

- metric data

satisfy triangle inequality

$$L = \{l_{ij} | e_{ij} \in E, l_{ij} > 0\}$$



$$V = \{v_i | 1 \leq i \leq n\}$$

$$E = \{e_{ij} | i, j \in V\}$$

$$T = \{t_{ijk} | e_{ij}, e_{jk}, e_{ki} \in E\}$$

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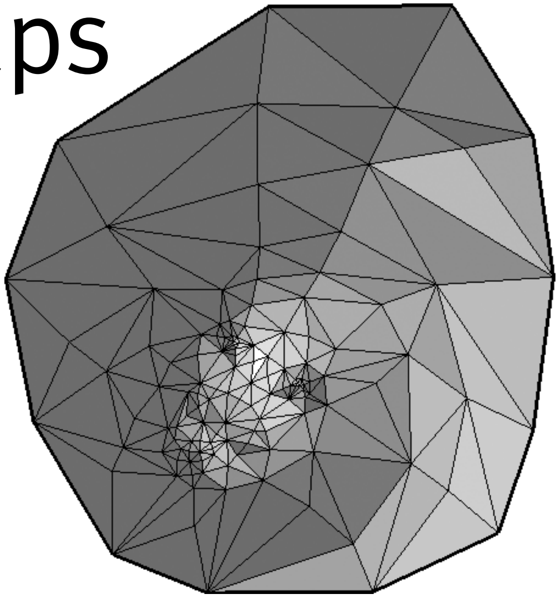
- new (flat) metric $\tilde{L}$



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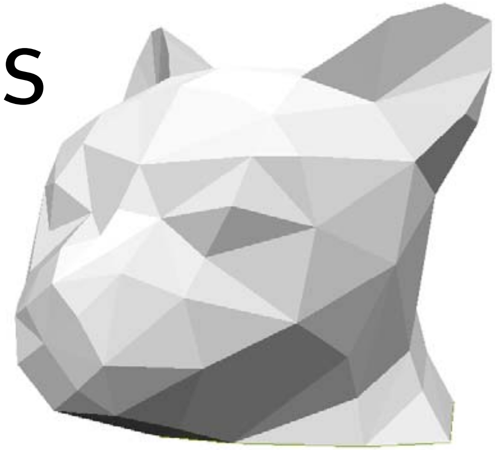

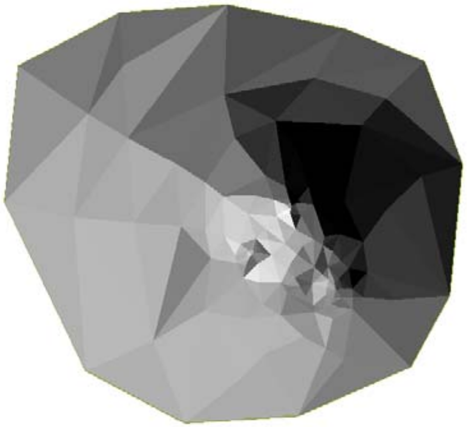

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Ansatz

Seek conformally equivalent metric

- data: simpl. complex & lengths
- output: new metric (i.e., lengths)

$$\tilde{g} = e^{2u} g \qquad K \neq 0 \longrightarrow \tilde{K} = 0$$

new metric

conformal factor

original metric

ignore boundary for the moment

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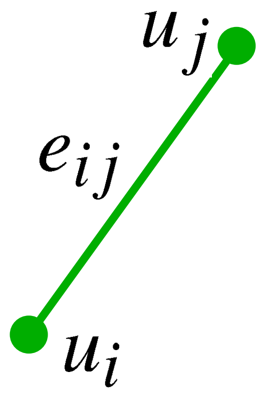
$$\tilde{g} = e^{2u} g \qquad K \neq 0 \longrightarrow \tilde{K} = 0$$

- variables at vertices

$$u(v_i) = u_i \qquad \boxed{\tilde{l}_{ij} = l_{ij} e^{u_i + u_j}}$$



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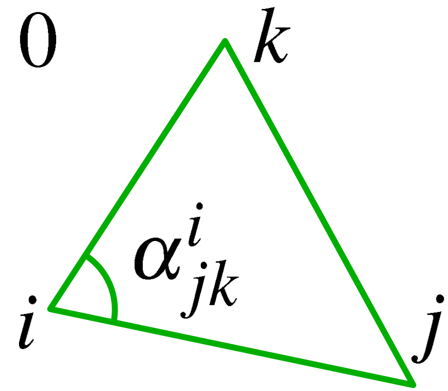


- goal: desired angle sums

$$\Theta(v_i) = \sum_{t_{ijk} \ni v_i} \alpha^i_{jk}(u_i, u_j, u_k) \stackrel{!}{=} \tilde{\Theta}_i$$

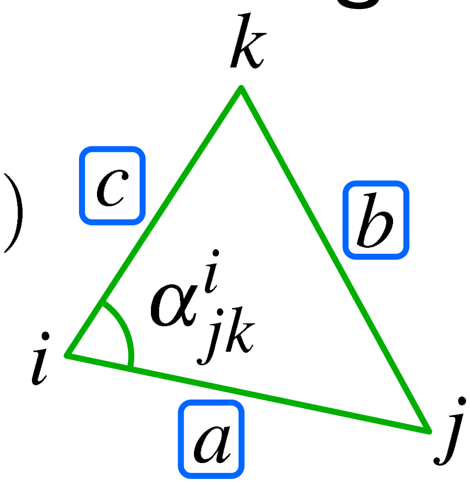
target vertex angle sums

Non-Linear Problem

Find $u(V)$ to satisfy angle sum targets

- from lengths to angles

$$\forall i \in V : \tilde{\Theta}_i = \sum_{t_{ijk} \ni v_i} \alpha^i_{jk}(u_i, u_j, u_k)$$



$$2\tan^{-1}\sqrt{\frac{(-a+b+c)(a+b-c)}{(a-b+c)(a+b+c)}}$$

watch out for triangle inequality!

Non-Linear Problem

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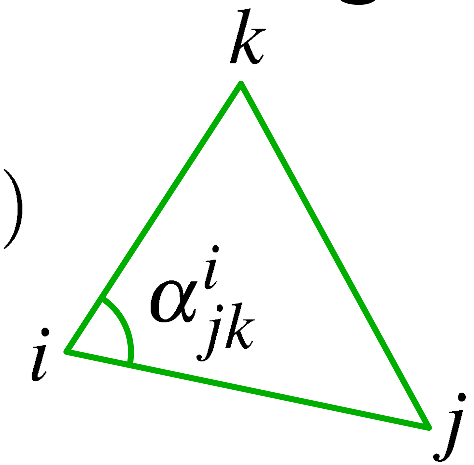


- … a miracle occurs …

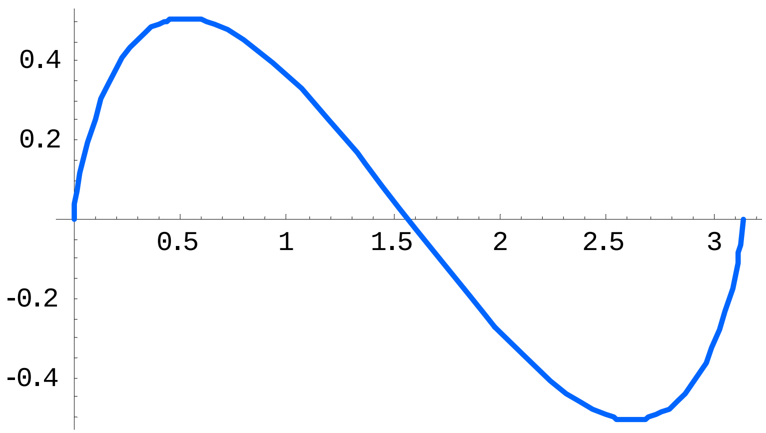
This system of equations can be integrated!

The Energy

Find minimum of a convex energy

- Milnor's Lobachevsky function

$$Л(x) = -\int_0^x \log 2|\sin t|\, dt$$



$$\frac{l_{jk}}{R} = 2\sin\alpha^i_{jk}$$

$$\Downarrow$$

$$\log l_{jk} - \log R = \log 2\sin\alpha^i_{jk}$$

The Energy

Find minimum of a convex energy

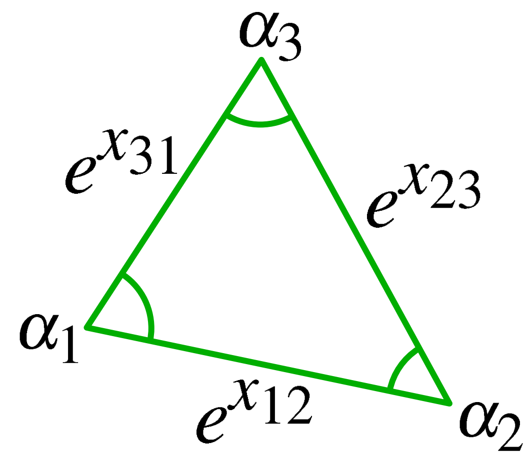
- Milnor's Lobachevsky function
- for each triangle

$$f(\underline{x_{12}, x_{23}, x_{31}}) =$$

$$\alpha_1 x_{23} + \alpha_2 x_{31} + \alpha_3 x_{12} +$$

$$Л(\alpha_1) + Л(\alpha_2) + Л(\alpha_3)$$

logarithmic lengths



$$x_{ij} = \lambda_{ij} + u_i + u_j$$

logarithmic input lengths

The Energy

Find minimum of a convex energy

- Milnor's Lobachevsky function
- for each triangle

$$\frac{d}{du_i} f(x_{ij}, x_{jk}, x_{ki}) = \pi - \alpha_i$$

$$E(u) = \sum_{t_{ijk} \in T} \Big(f(u_i, u_j, u_k) - \pi (u_i + u_j + u_k) \Big) + \sum_{v_i \in V} \tilde{\Theta}_i u_i$$

The Energy

Properties

- convex: Hessian is pos. semi-def.

$$u^T H u = \sum_{e_{ij} \in E} (\cot \alpha^i_{jk} + \cot \alpha^l_{kj})(u_k - u_j)^2$$

only one term for boundary edges

The Energy

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- convex: Hessian is pos. semi-def.

$$u^T H u = \sum_{e_{ij} \in E} (\cot \alpha^i_{jk} + \cot \alpha^l_{kj})(u_k - u_j)^2$$

- solution exists $\Rightarrow$ is unique $\min E(u)$
- gradient flow is curvature flow

$$\frac{d}{dt} u(t) = -\nabla E(u(t)) = \tilde{K} - K(t)$$

target curvature

The Energy

Properties

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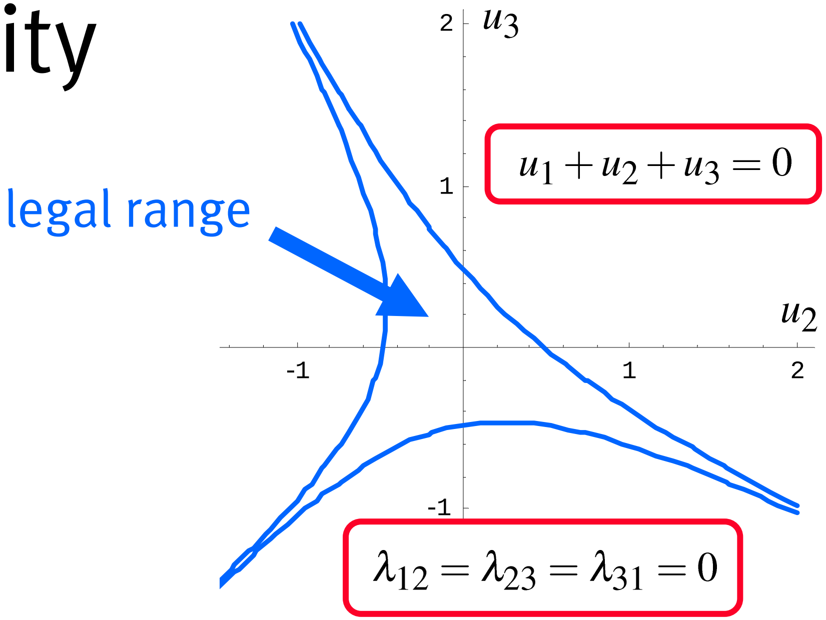
- what about triangle inequality?!

Domain of Definition

Not just any u value is cool...

- triangle inequality



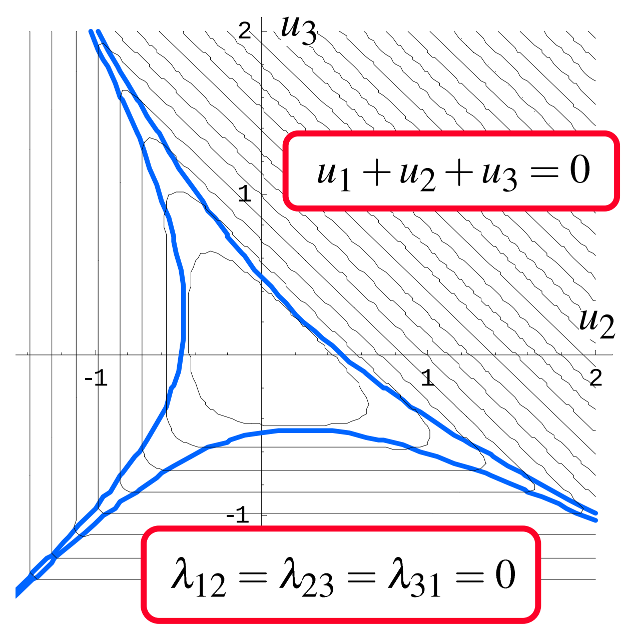
Domain of Definition

Not just any *u* value is cool...

- triangle inequality
- extend definition

$$\alpha = \mathrm{Re}\left(2\tan^{-1}\sqrt{\frac{(-a+b+c)(a+b-c)}{(a-b+c)(a+b+c)}}\right)$$

Real part only



Functional remains C^1 in u

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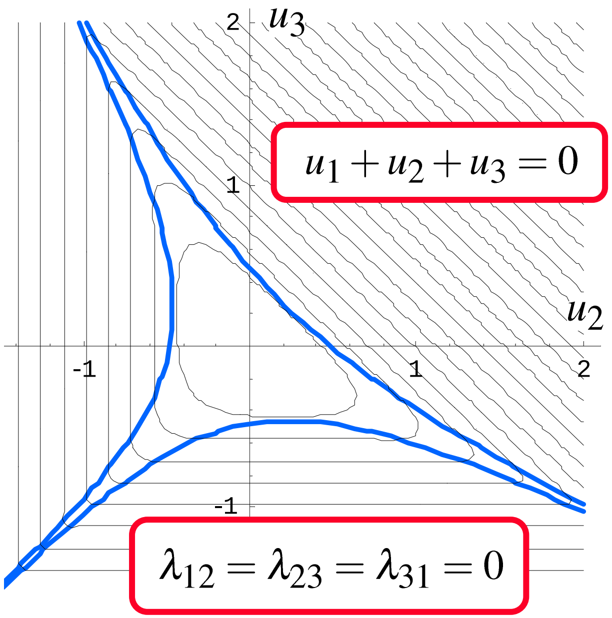
- minimum may occur at illegal values...

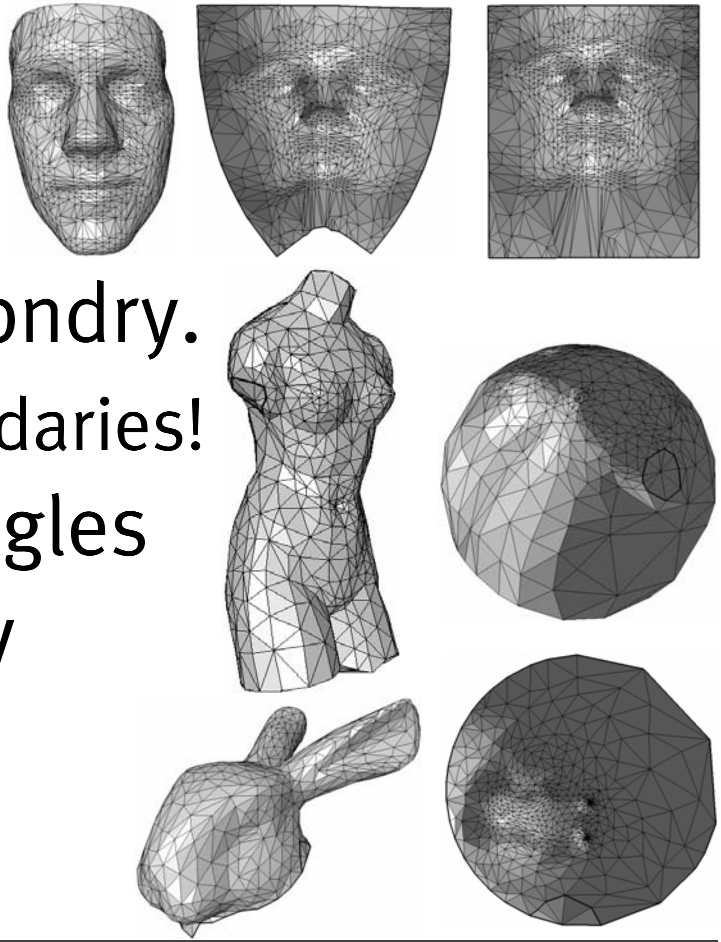


- conditions for existence guarantee?

Boundary Conditions

Fixing variables

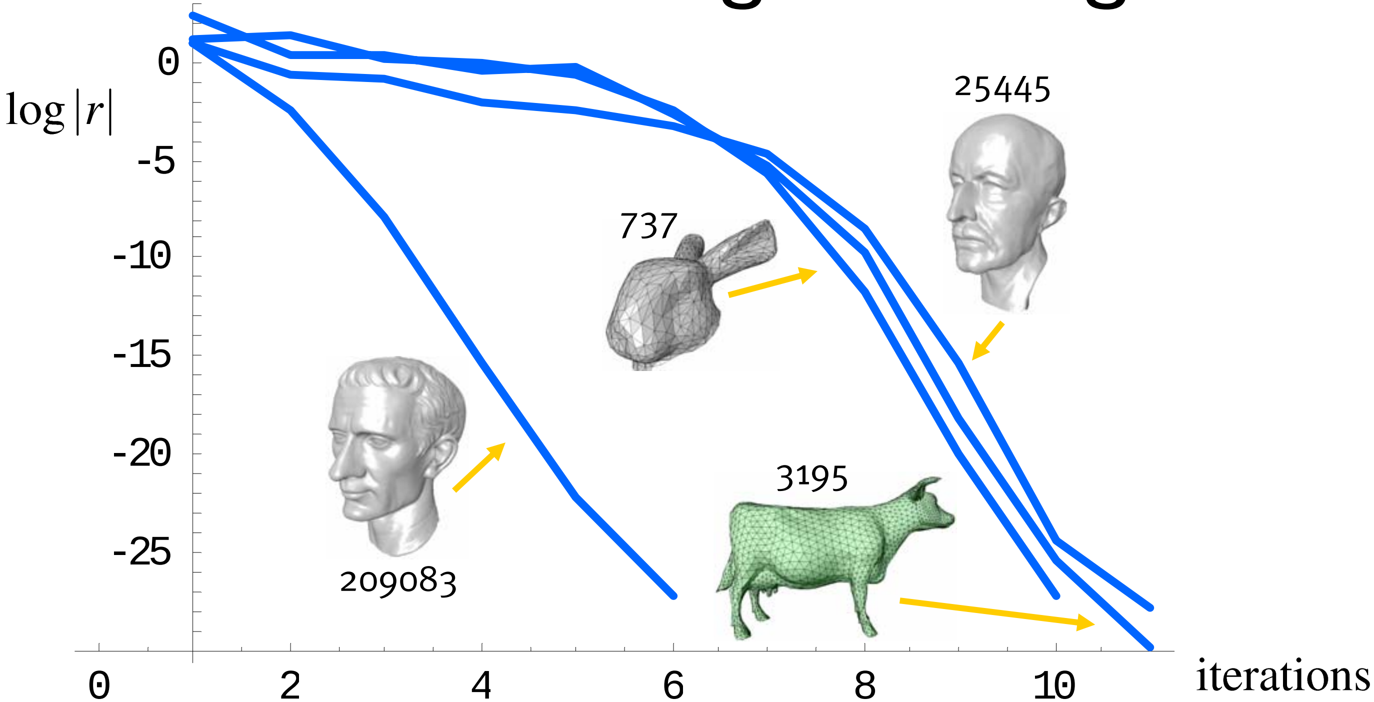
- fix u_i let Θ_i vary
  - $u_i = 0$: isometric bndry.
    - nice for cut-boundaries!
  - unknown cone angles
  - arbitrary topology
- fix Θ_i let u_i vary
  - rectangle, disk...



Practicalities

Convex optimization

- Newton-Steihaug trust region



Multi-Res Modeling Group

Practicalities

Convex optimization

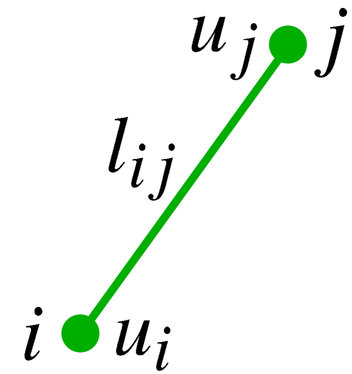
- Newton-Steihaug trust region
- Petsc/TAO library
- SSOR precon for cotan system
- layout: dual spanning tree
  - achieves 10^{-9} to 10^{-13} acc.
  - alternatively: Dirichlet problem

Conformal Equivalence

From continuous to pair of meshes

$$\tilde{g} = e^{2u} g$$

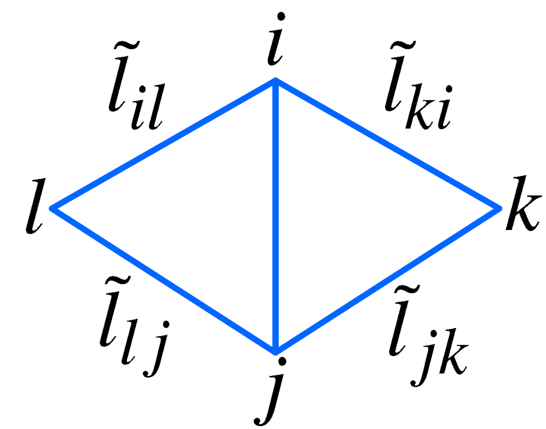
$$\tilde{l}_{ij} = e^{u_i} l_{ij} e^{u_j}$$



- equivalently:

$$\tilde{\mathrm{cr}}_{ij} = \tilde{l}_{il}\tilde{l}_{lj}^{-1}\tilde{l}_{jk}\tilde{l}_{ki}^{-1}$$

e^{-u_j} e^{u_j}



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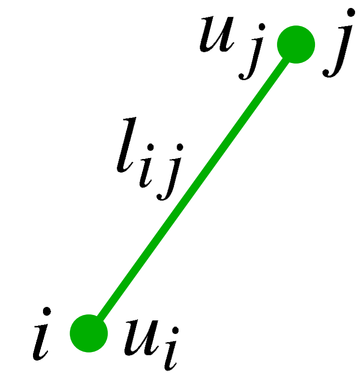
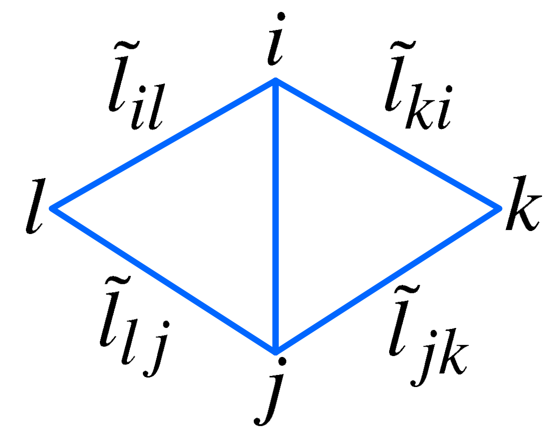


- equivalently:

$$\tilde{\mathrm{cr}}_{ij} = \tilde{l}_{il}\tilde{l}_{lj}^{-1}\tilde{l}_{jk}\tilde{l}_{ki}^{-1} = \mathrm{cr}_{ij}$$

Length cross ratios are preserved



Why $u = 0$ on Boundary?

In the continuous setting

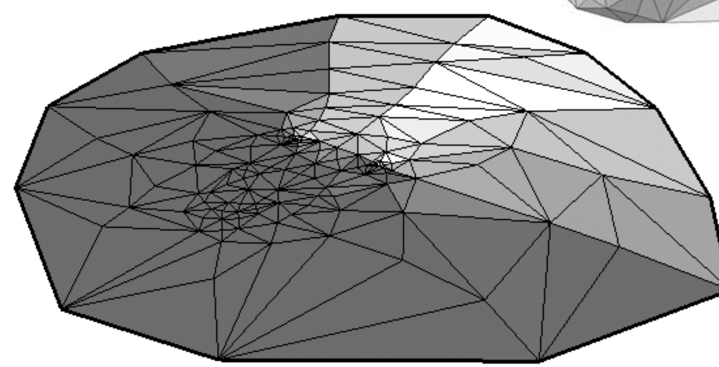
- conformally equivalent metric

$$\tilde{K} = e^{-2u}(K - \Delta u) \quad \text{flat} \quad \Rightarrow \Delta u = K$$

- choose “least varying”:

$$\min_{\text{flat } u} \int |du|^2$$

$$\Leftrightarrow u|_{\partial M} = c$$



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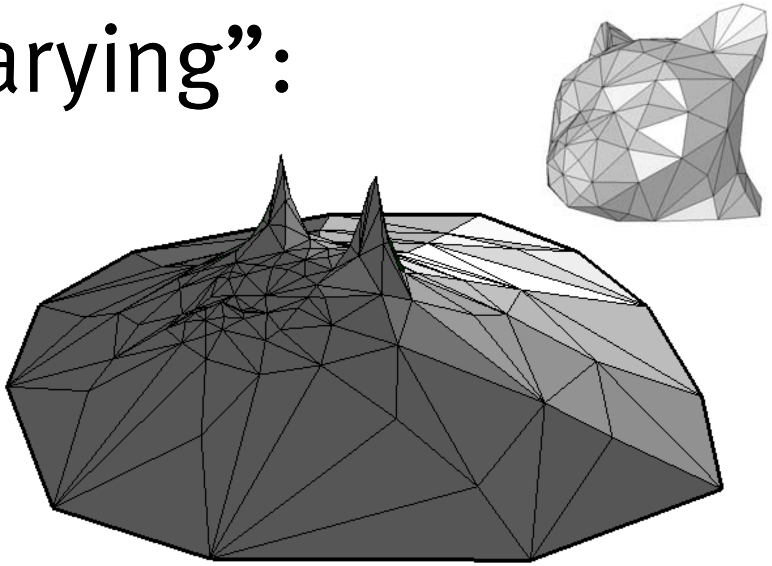

Dual Functional

Angles/lengths are dual variables

- functional in angles

$$f(\alpha_i, \alpha_j, \alpha_k) =$$

$$\lambda_{ij}\alpha_k + \lambda_{jk}\alpha_i + \lambda_{ki}\alpha_j +$$

$$\text{Л}(\alpha_i) + \text{Л}(\alpha_j) + \text{Л}(\alpha_k)$$

original logarithmic lengths

$$E(\alpha) = \sum_{t_{ijk} \in T} f(\alpha_i, \alpha_j, \alpha_k)$$

$$\alpha^i_{jk} > 0$$

$$\alpha^i_{jk} + \alpha^j_{ki} + \alpha^k_{ij} = \pi$$

$$\sum_{t_{ijk} \ni i} \alpha^i_{jk} = \tilde{\Theta}_i$$

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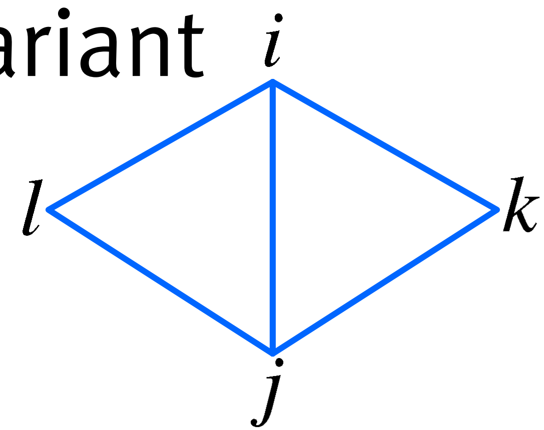
$$\sum_{t_{ijk}\ni i} \alpha^i_{jk} = \tilde{\Theta}_i$$

- length cross ratios invariant

$$\nabla E = 0 \iff \frac{l_{il}l_{jk}}{l_{lj}l_{ki}} = \frac{\tilde{l}_{il}\tilde{l}_{jk}}{\tilde{l}_{lj}\tilde{l}_{ki}}$$



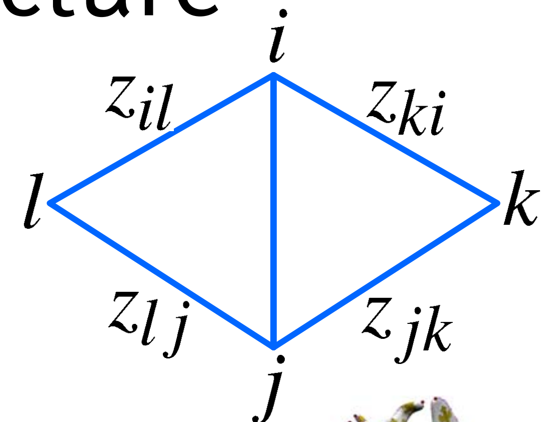
The Big Picture

Discrete conformal structure

- simplicial mesh

  $$z_{il} z_{lj}^{-1} z_{jk} z_{ki}^{-1}$$

- preserve:
  - phase: circle patterns
    - can't read off angles directly...
  - magnitude: new functional
    - CAN read off lengths directly!



TODO List

Future work

- conditions for existence
  - intrinsic Delaunay not required!
- automatic cone singularity placement
  - u provides potential hook

first Newton step → $\Delta u^0 = K$ ← sparse approximation problem