Geometry and Computation of Mesh Surfaces with Planar Hexagonal Faces

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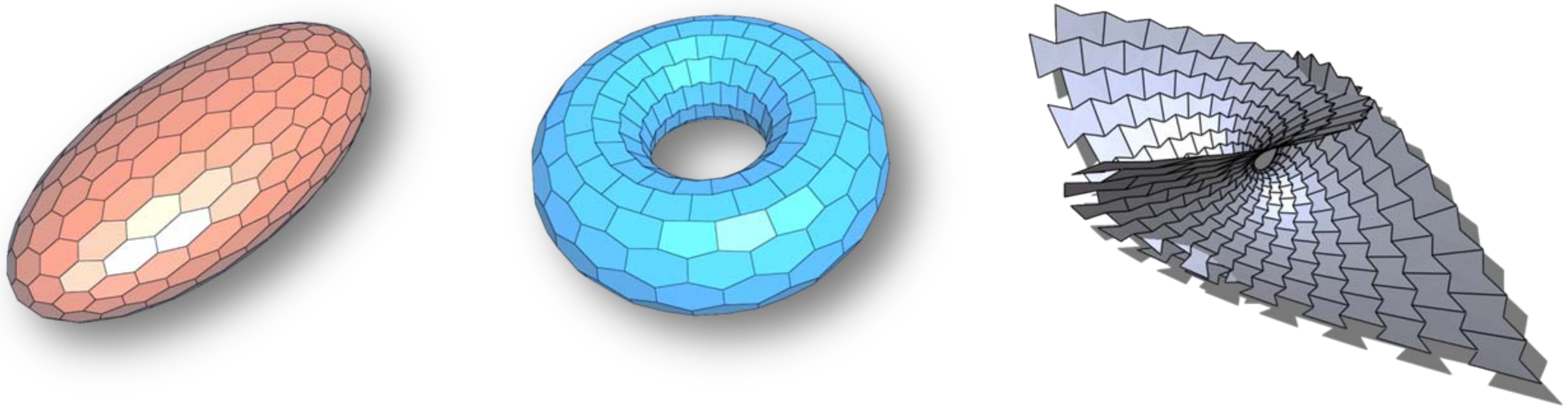
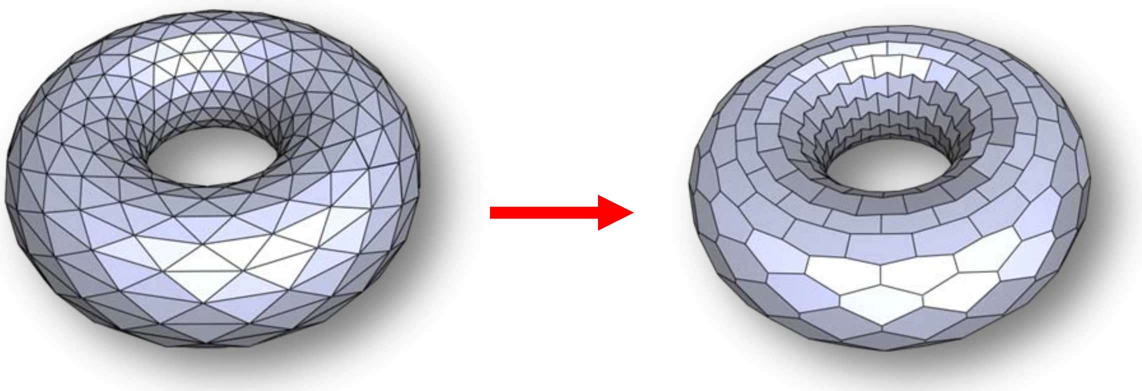
Problem Formulation

- We want to tile a free-form surface using planar hexagonal mesh -- **P-Hex mesh**.
- Wish to have regular titling with every vertex valence = 3, (which is not possible for closed surface if genus g ≠ 1).



Approach proposed

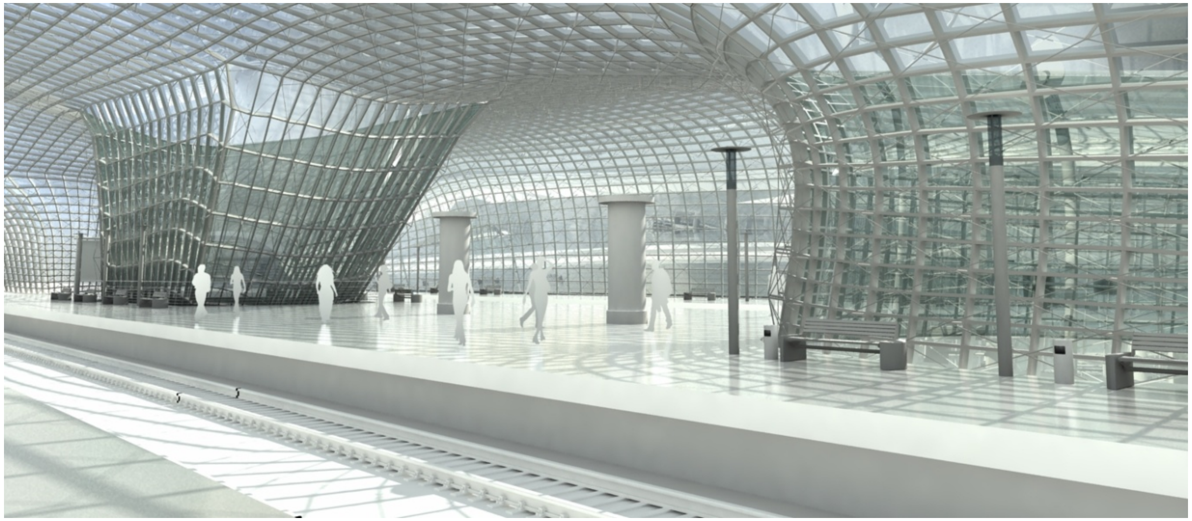
Computing P-Hex mesh from regular triangulation of smooth surface.



Introduction

Applications in architectural design

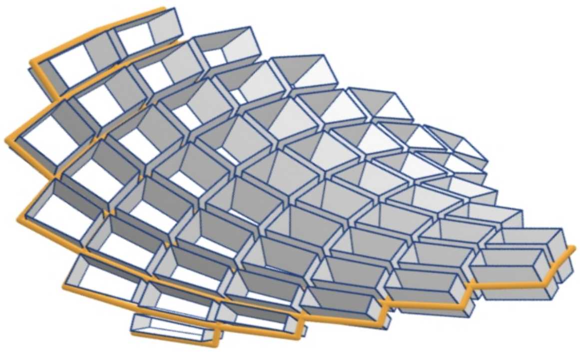
-- glass/metal panels

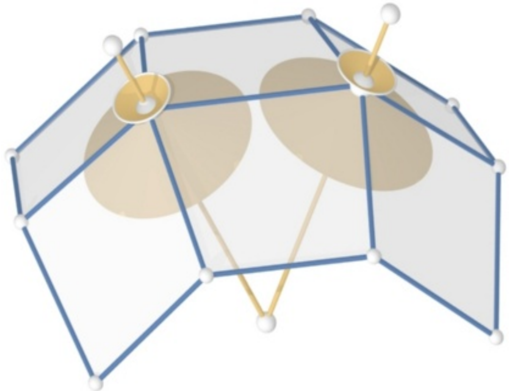


[Liu et al, 2006]

P-Quad Meshes

- P-Quad meshes, related to conjugate curve networks [SAUER 1970, Bobenko and Suris 2005]
- Conical P-Quad meshes, related to curvature lines [Liu et al, 2006]





Beyond Quad Meshes ..

P-Hex Mesh for Quadrics via Power Diagram [Diaz et al, 2006]

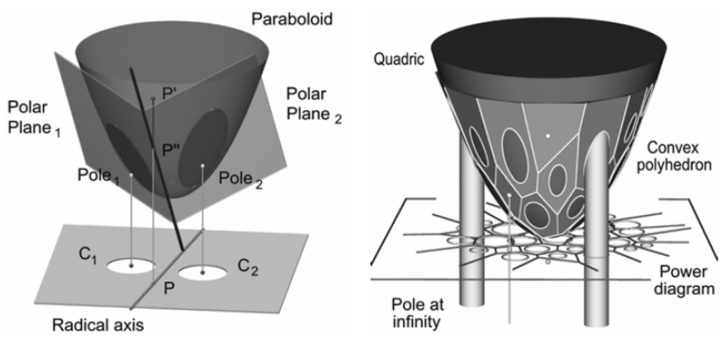


Fig. 1. Spatial interpretation of the chordal as the intersection of polar planes (*left*). Polyhedron that materializes the paraboloid of revolution through cylindrical projection of a power diagram (*right*).

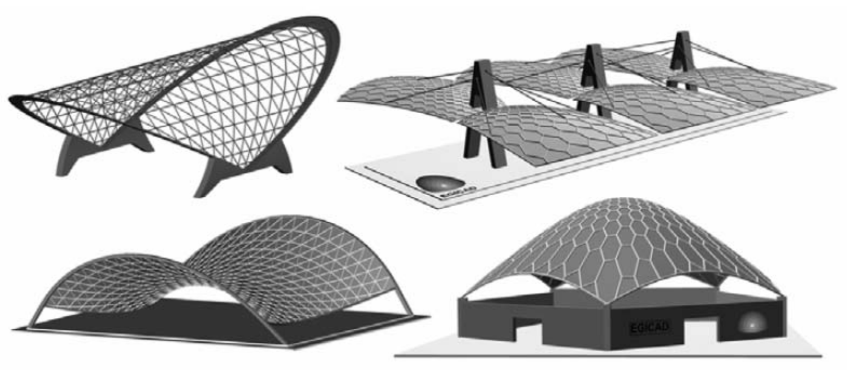
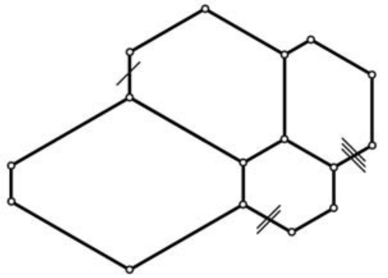
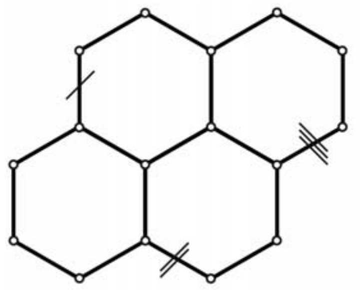
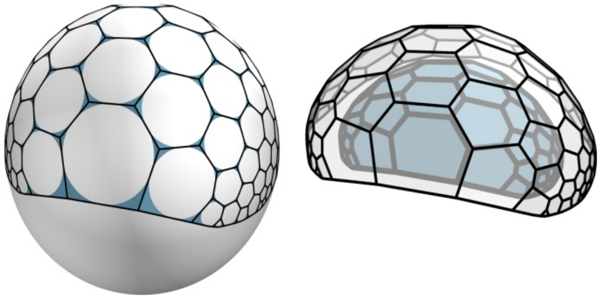


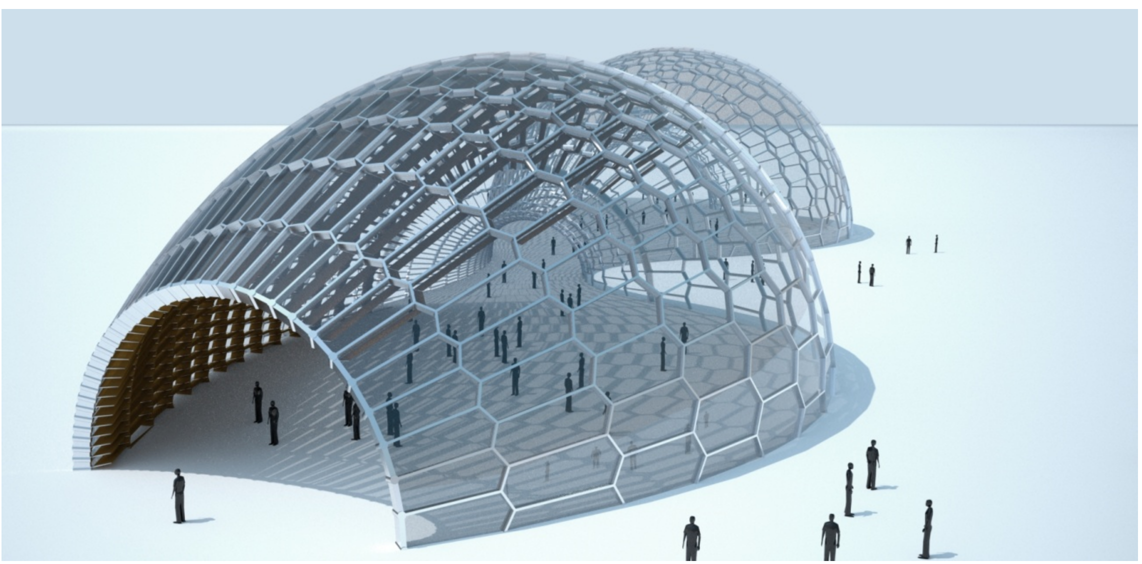
Fig. 16. Portions of quadrics used as space enclosures

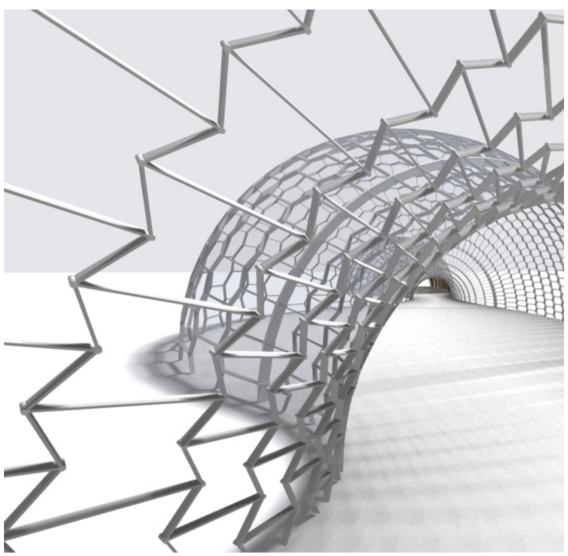
Parallel Meshes [Pottmann et al, 2007]





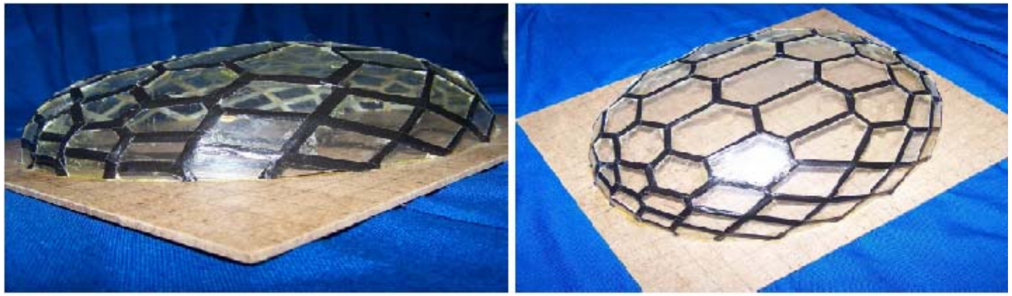




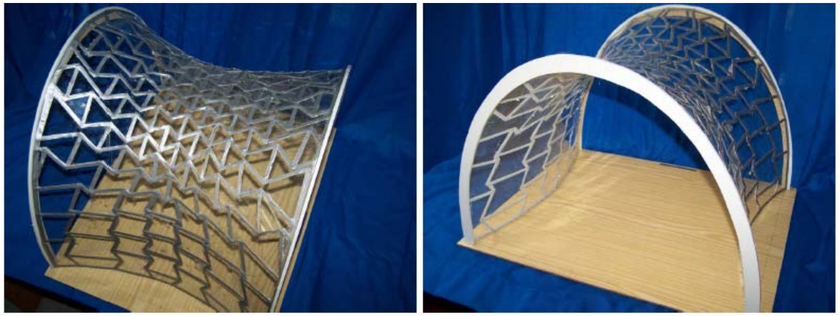


Support Functions [Almegaard et al, 07]

- P-Hex mesh from piecewise linear support function over triangulation of Gaussian sphere.



Courtesy of Bert Juettler



Planar Clustering [Cutler & Whiting, 2007] (based on [Cohen-Steiner et al, 2004])

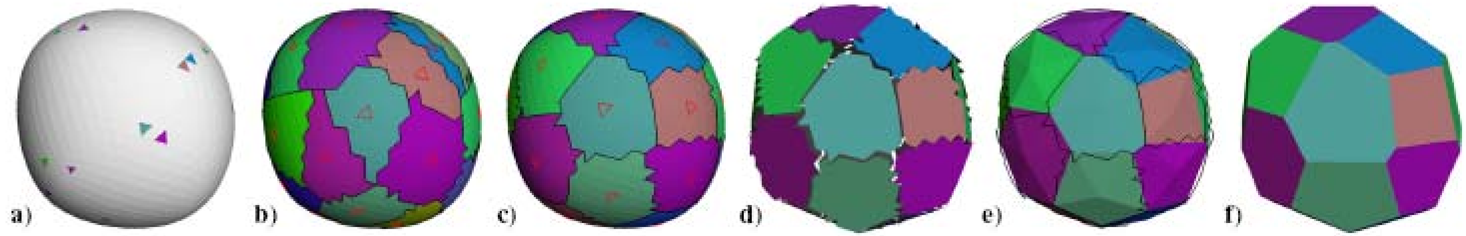
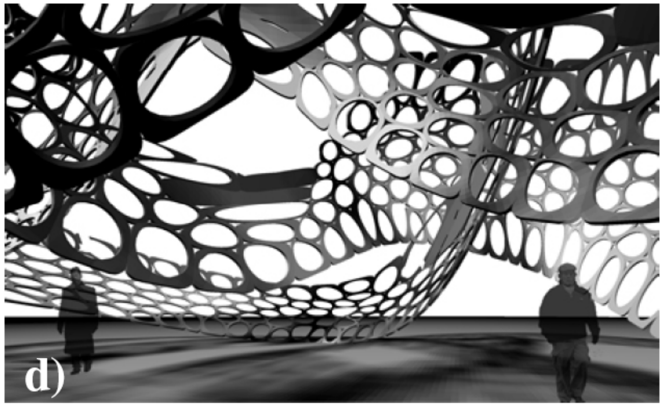
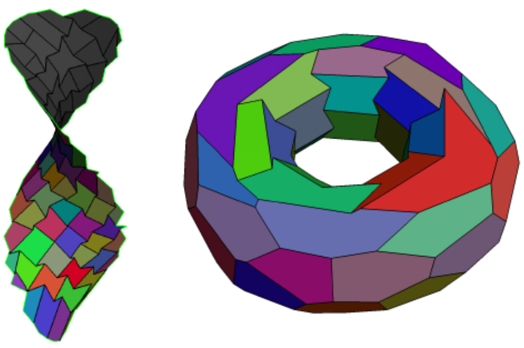


Figure 5: The basic algorithm begins by a) selecting random seed triangles from the original mesh. Next the mesh is b&c) iteratively clustered about these seeds and the seeds are repositioned; d) shows a visualization of the original mesh triangles projected onto the corresponding proxy planes; e) using the vertices and neighbors from the clusters we render non-planar polygons, similar to Cohen-Steiner et al. [6]; and f) the proper intersection of the neighboring planes.



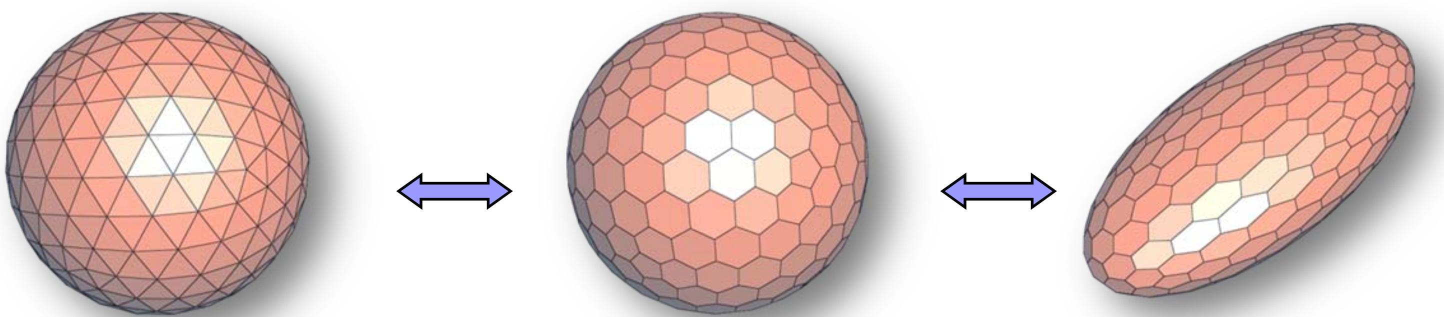


Projective Duality [Karahawada & Sugihara, 2006]

Projective duality: correspondence between planes and points:

plane $ax + by + cz - 1 = 0$ ⟺ point (a, b, c)

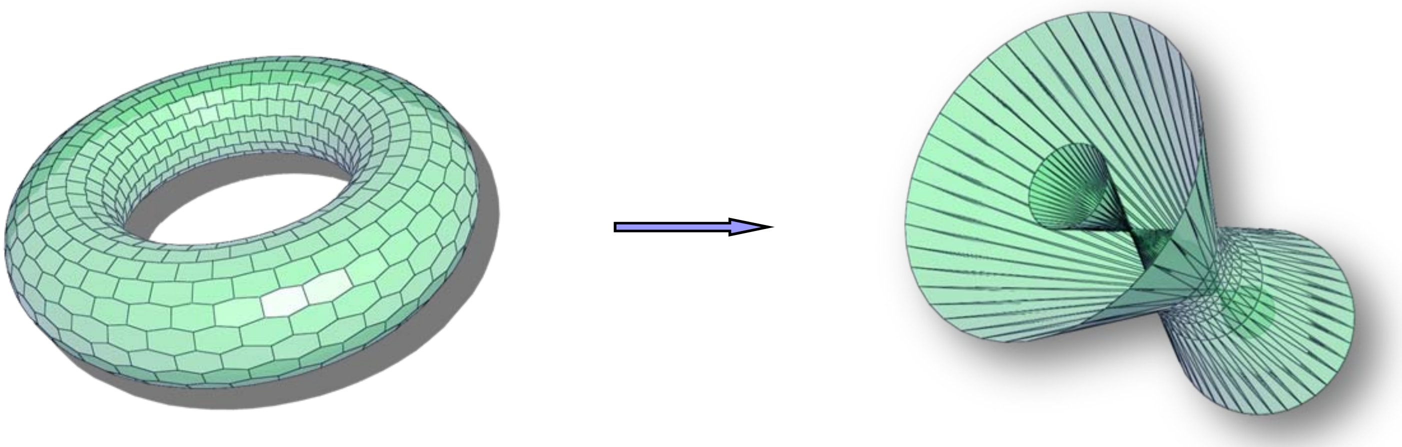
in prime space ***P*** ⟺ in dual space ***D***



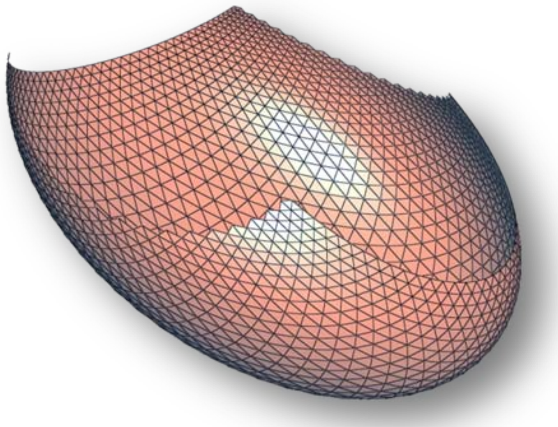
in ***D*** ⟺ in ***P*** ⟺ by affine trans.

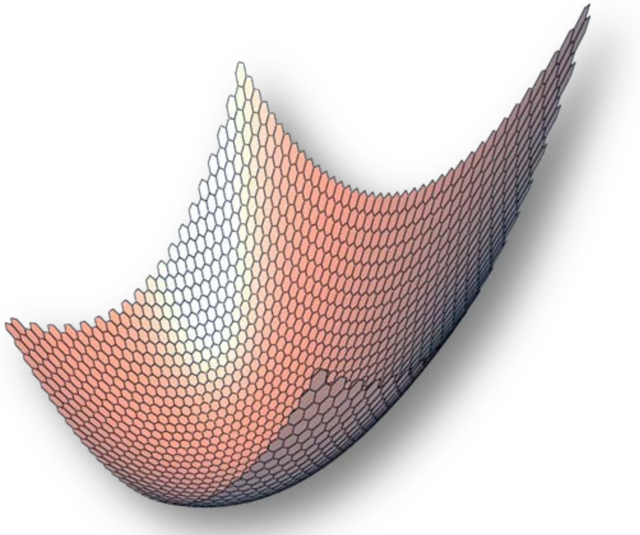
Anomalies of Projective Duality
-- not a one-to-one mapping in many cases

- A developable in ***P*** yields a curve ***D***
- Parabolic lines on surface in ***P*** correspond to singularity on surface in ***D***
- High metric distortion



What is a good triangulation in dual space?

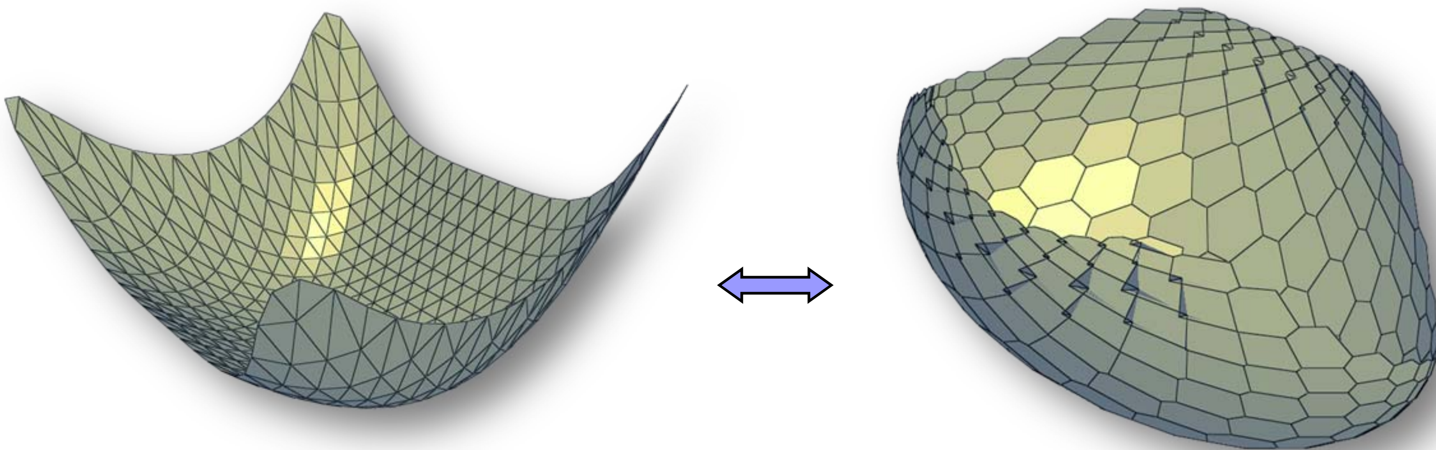




Triangle mesh in ***D***

P-Hex mesh in ***P***

Self-intersecting P-Hex Mesh



Triangle mesh in ***D***　　　　P-Hex mesh in ***P***

Main Results

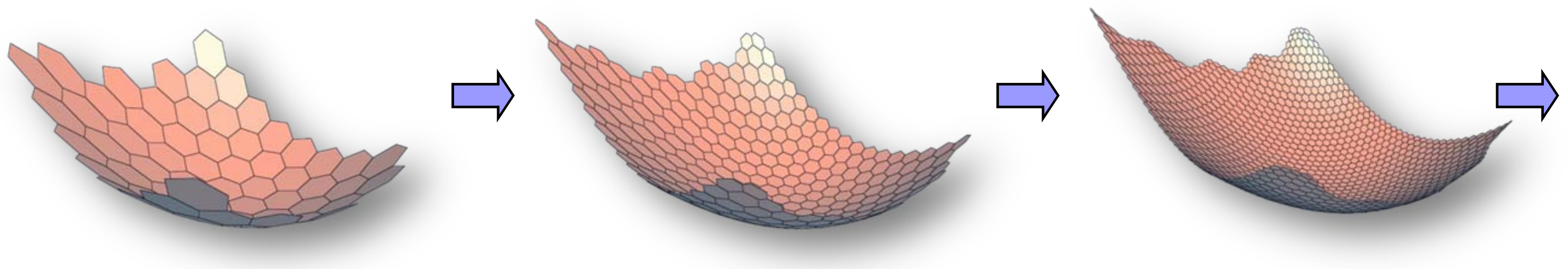
1. A new method for computing P-Hex meshes from regular triangle meshes using *Dupin duality*, a new concept to be introduced.

2. Conditions on P-Hex meshes thus computed to be free of self-intersecting faces

In the limit …

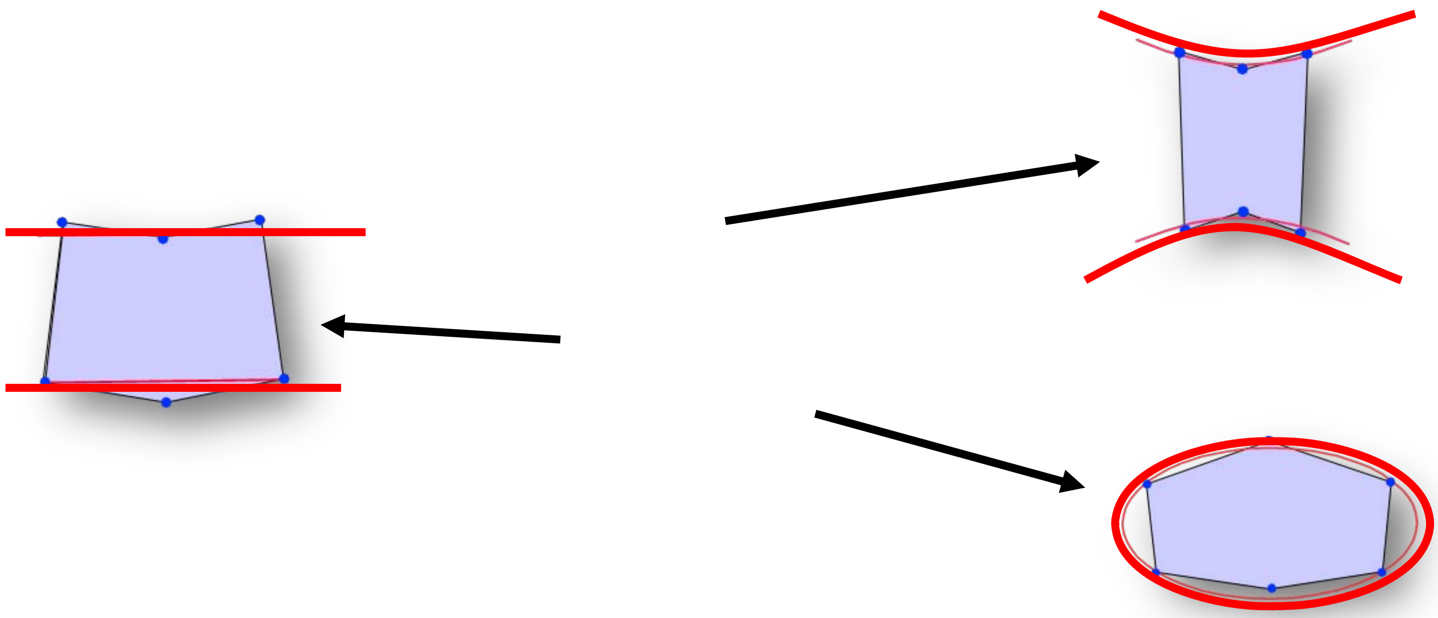
*A*ssume a sequence of P-hex meshes converging to a given smooth surface.

----- d*iscrete differential geometry.*

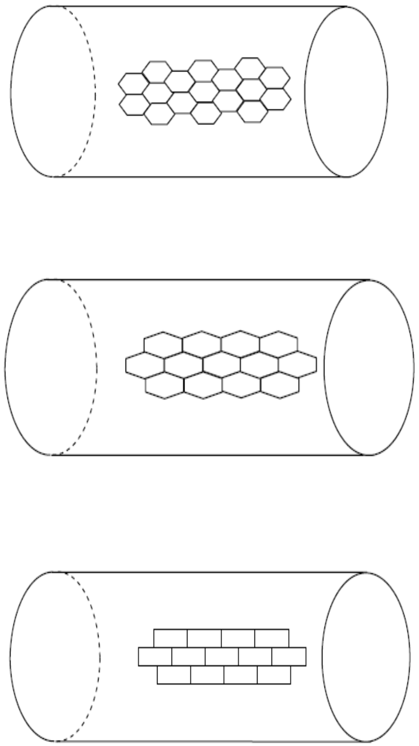


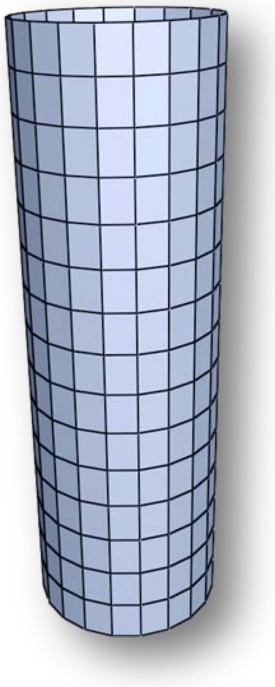
Shape of P-Hex Face on Surface

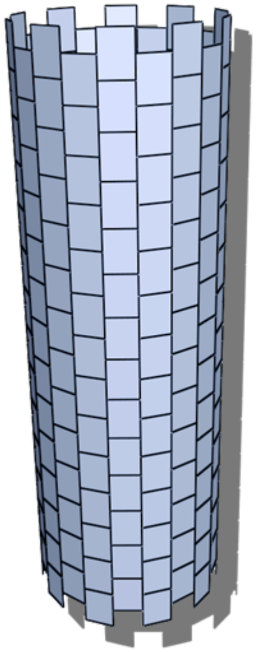
Theorem: *Suppose that a P-Hex mesh* M *approximates a surface* S. *In the limit, the six vertices of P-Hex face of* M *at a point* v *of* S *lie on a homothetic copy of Dupin conic of* S *at* v.



Which P-Hex mesh is possible?

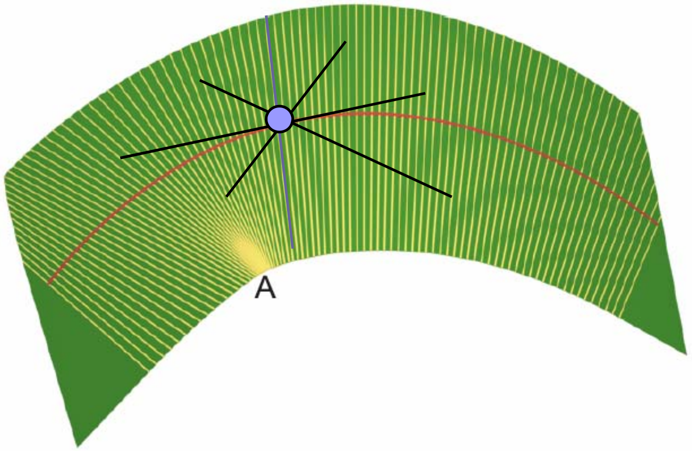






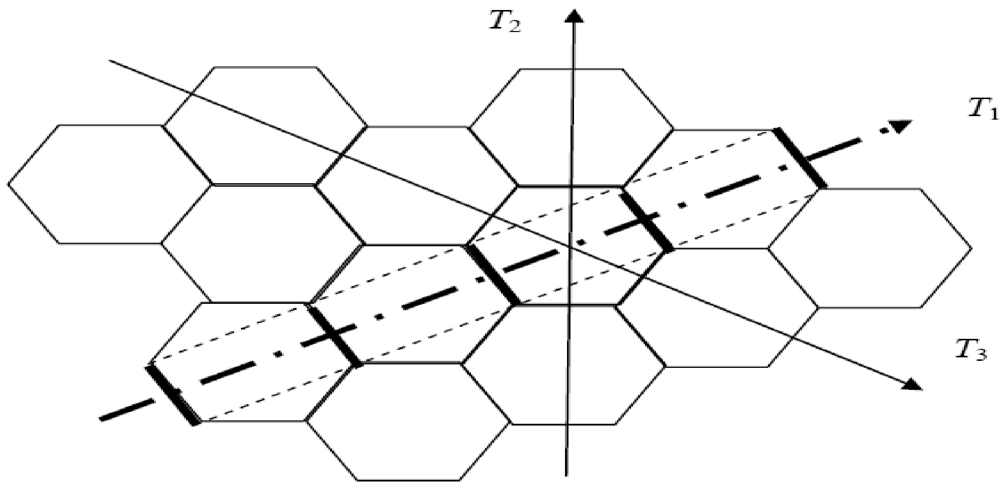
Conjugate directions on a developable

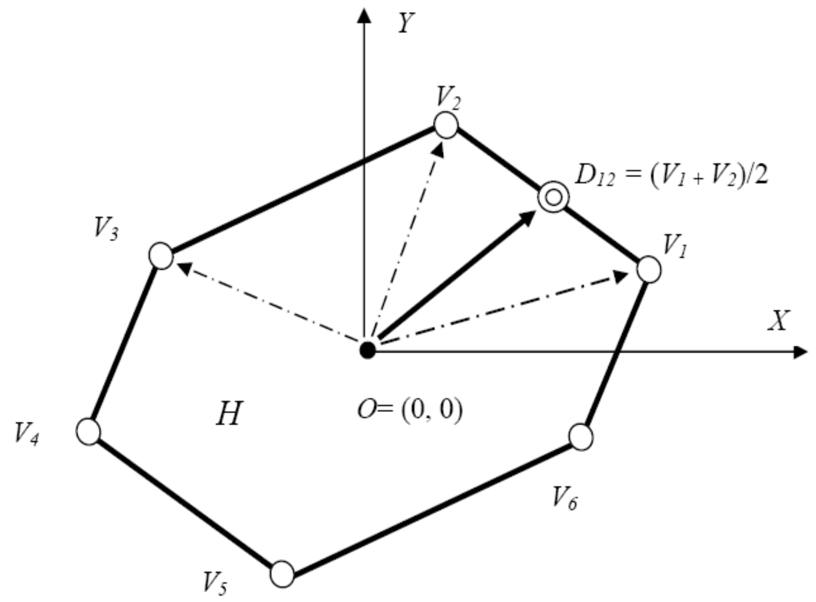
-- Any direction is conjugate to ruling direction on a developable.



Discrete Developable Strip

Strip direction and rulings are conjugate on a developable strip of P-Hex faces





A centrically symmetric hex is uniquely determined by V_1, V_2, V_3.

Denote $V_i = (\ell_i \cos\theta_i, \ell_i \sin\theta_i)^T$, $i = 1, 2, 3$.

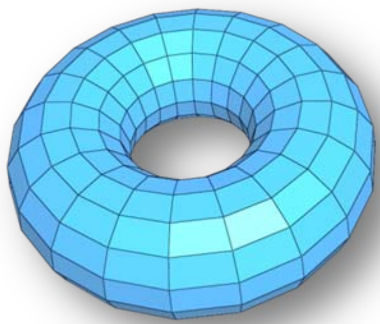
Since $D_{12} = (V_1 + V_2)/2$ is conjugate to the ruling $V_2 - V_1$,

$$\kappa_1(\ell_1^2 \cos^2\theta_1 - \ell_2^2 \cos^2\theta_2) + \kappa_2(\ell_1^2 \sin^2\theta_1 - \ell_2^2 \sin^2\theta_2) = 0.$$

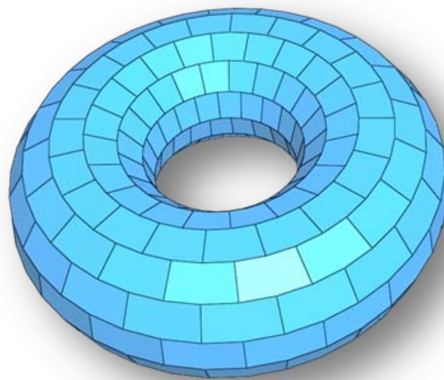
Therefore,

$$\kappa(\theta_1)\ell_1^2 - \kappa(\theta_2)\ell_2^2 = 0,$$

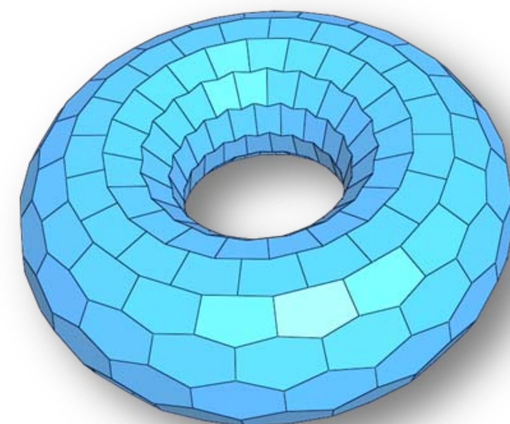
Construction of P-Hex mesh using developable strips



Step 1:
conjugate network



Step 2:
brick-wall



Step 3: Optimize: P-Hex

Optimization

Objective function:

- Constraint: face planarity
- Minimize distances to target surface

Solver:

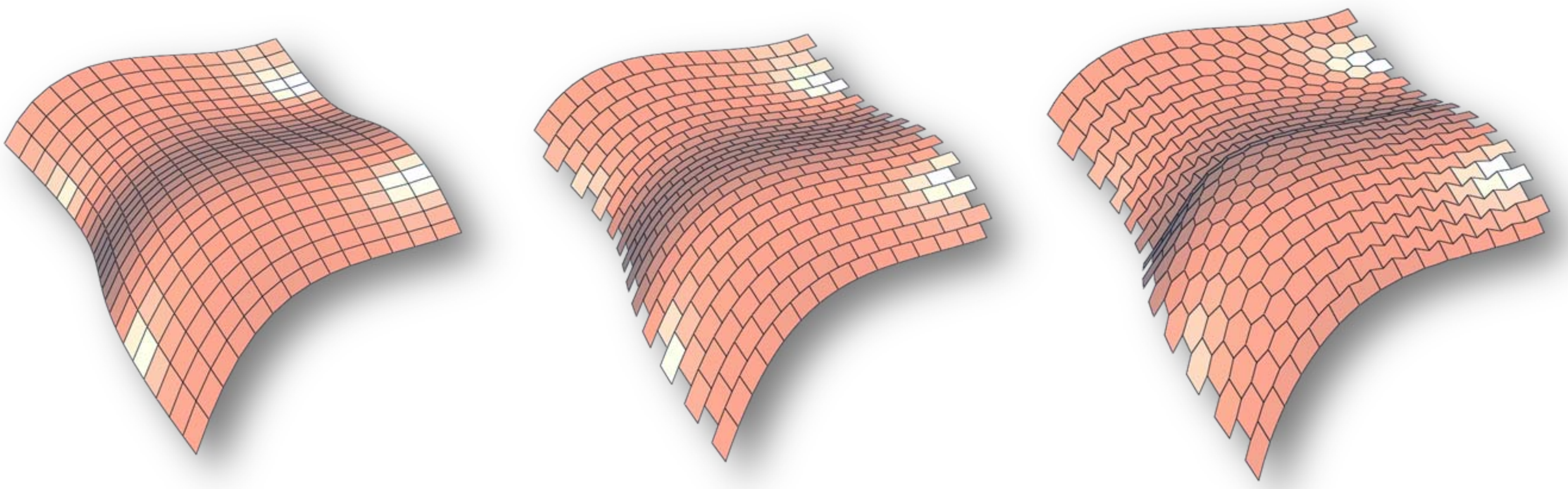
- Lagrange-Newton method, or
- Penalty method

Initialization is key!

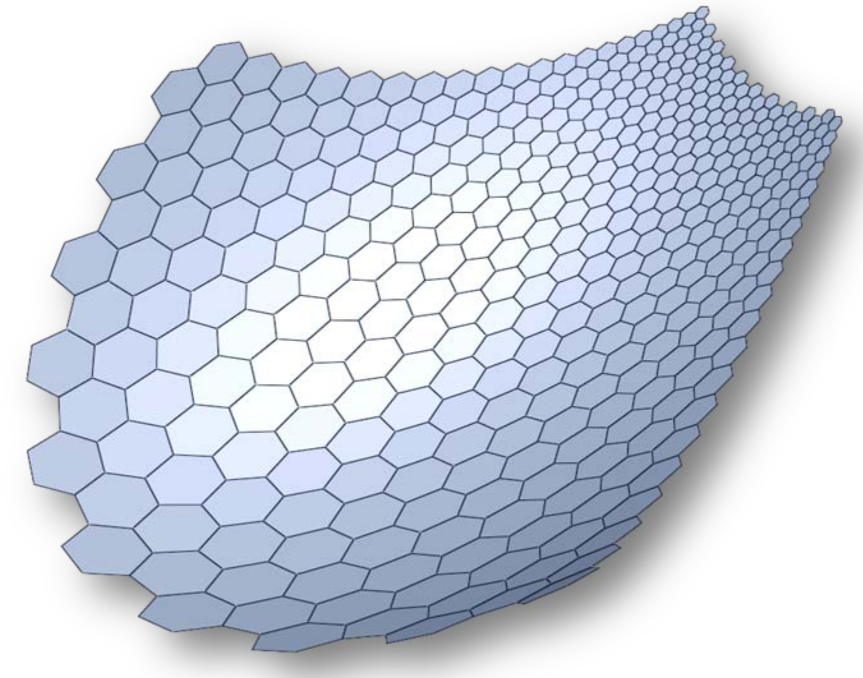
Example of translational surface

$$P(s,t) = (\sin(s)+2\cos(t/2),\ \sin(s/4)+t,\ \ s+\sin(t/2))$$

$$0 <= s <= 2Pi\ ,\quad 0 <= t <= 2Pi$$

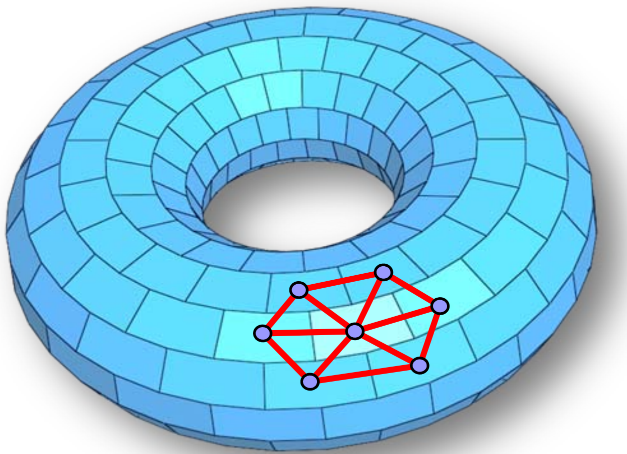


Trapezoidal P-Hex Mesh



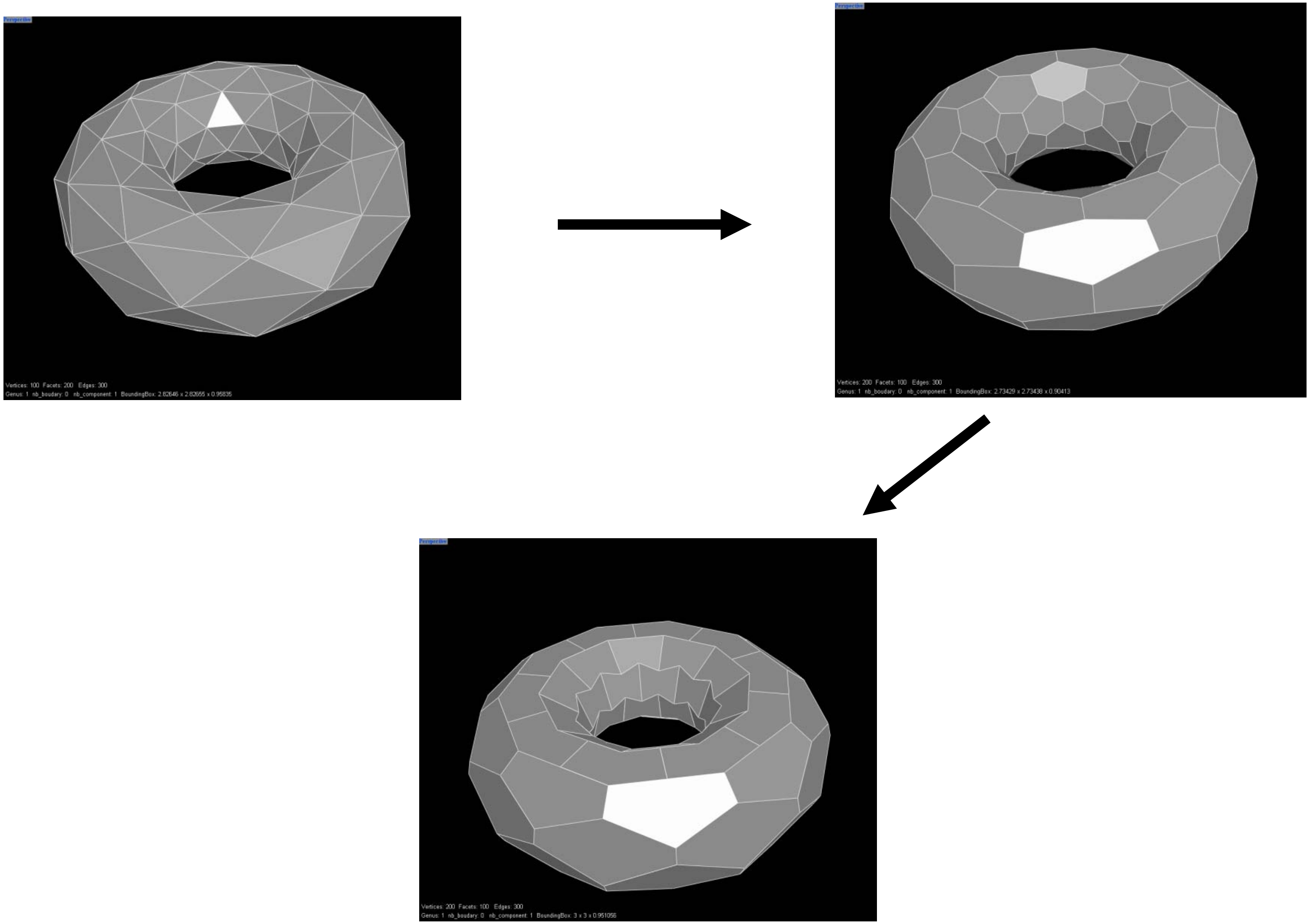
Does brick-wall initialization always work?

Correspondence between brick wall and triangulation



This leads us to consider triangulation as a means of initialization.

A possible scheme -- center duality



Does center duality always work?

Connecting centers of adjacent triangles yields a hex mesh, which is not necessarily planar.

1) Can such a hex mesh always be 'pressed' into a good P-Hex mesh? Or,

2) what kind of regular triangle mesh corresponds to a good P-Hex mesh?

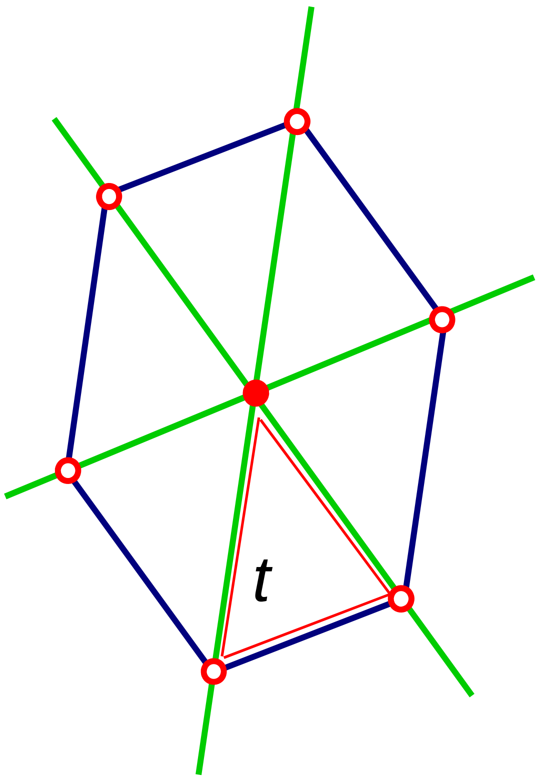
Good P-Hex mesh = all P-Hex faces have no self-intersection

P-Hex Mesh from Regular Triangle Mesh

Consider computing P-Hex mesh from regular triangle mesh of surface *S*.

Regular triangle mesh -- valence is 6, locally composed of congruent triangles, and characterized by three ***principal line directions*** (in green).

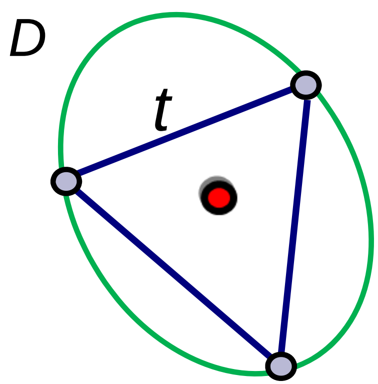
Any of the six congruent triangles is called a ***fundamental triangle, t.***

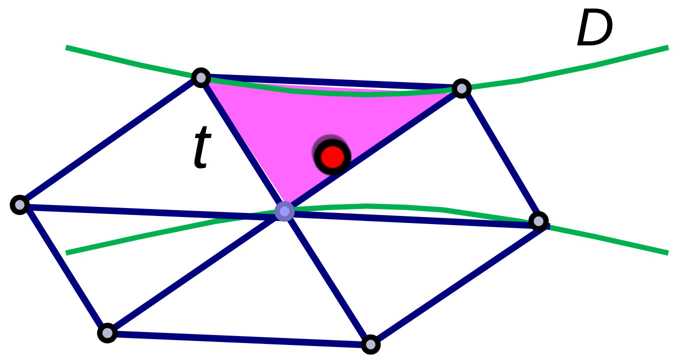


Dupin Duality

Let *D* denote Dupin conic of surface *S* at *v*. Suppose that *D* is either elliptic or hyperbolic.

Dupin center of triangle *t* is the center of the (unique) circumscribing Dupin conic of *t*.





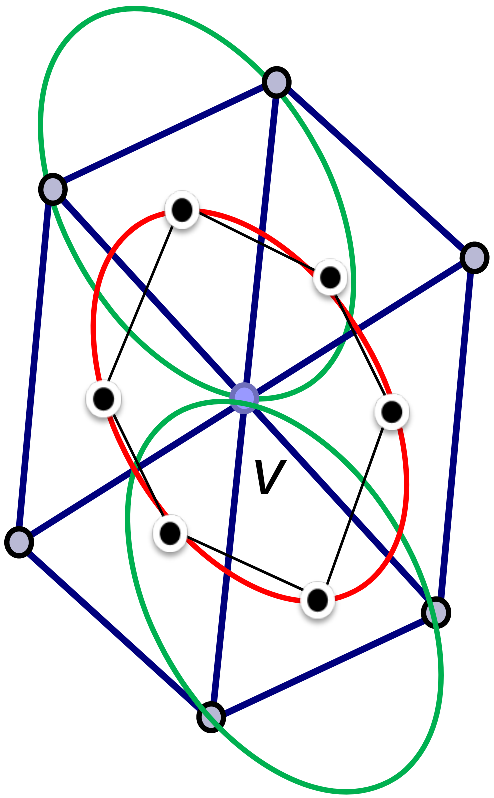
Dupin Dual of Triangle Mesh

Given a regular triangle mesh *T* approximating surface *S*.

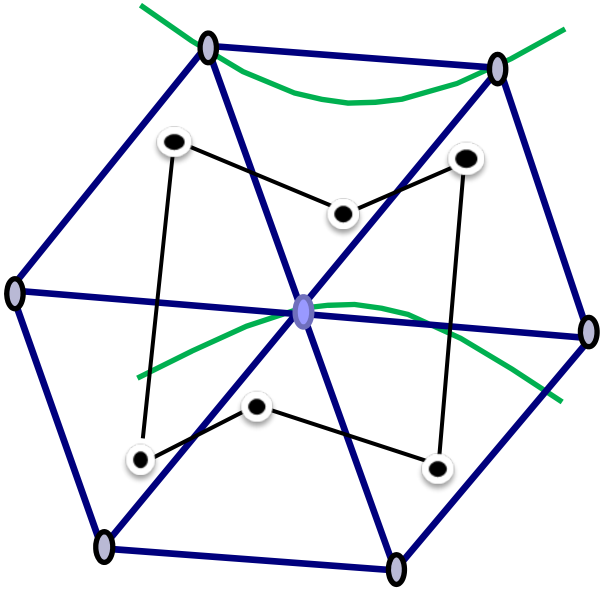
Dupin dual of *T* is the hex mesh formed by connecting Dupin centers of all adjacent triangles.

Consider the assembly of 6 triangles incident to vertex *v*.

Theorem (Dupin Duality): *The hex formed by Dupin centers of the 6 triangles is inscribed in Dupin conic.*

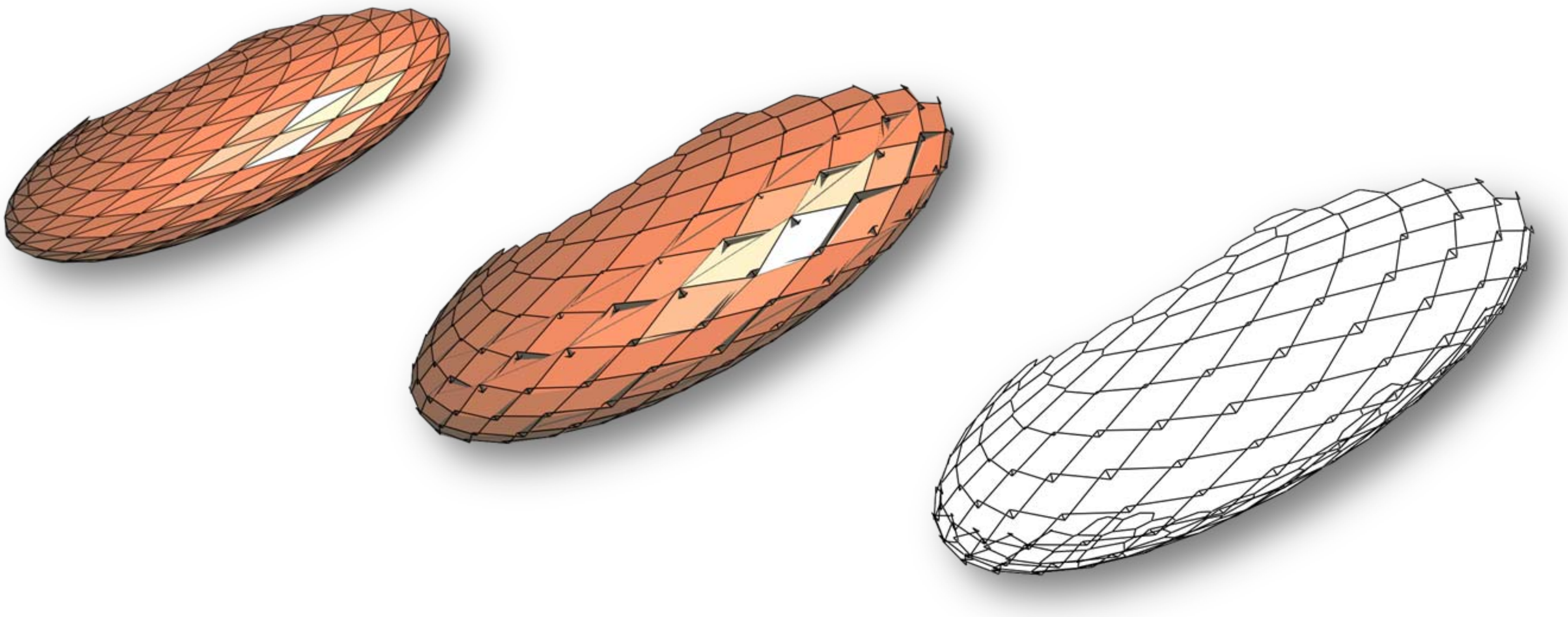


Non-convex P-Hex ---- Hyperbolic Case



What triangulation produces good P-Hex mesh?

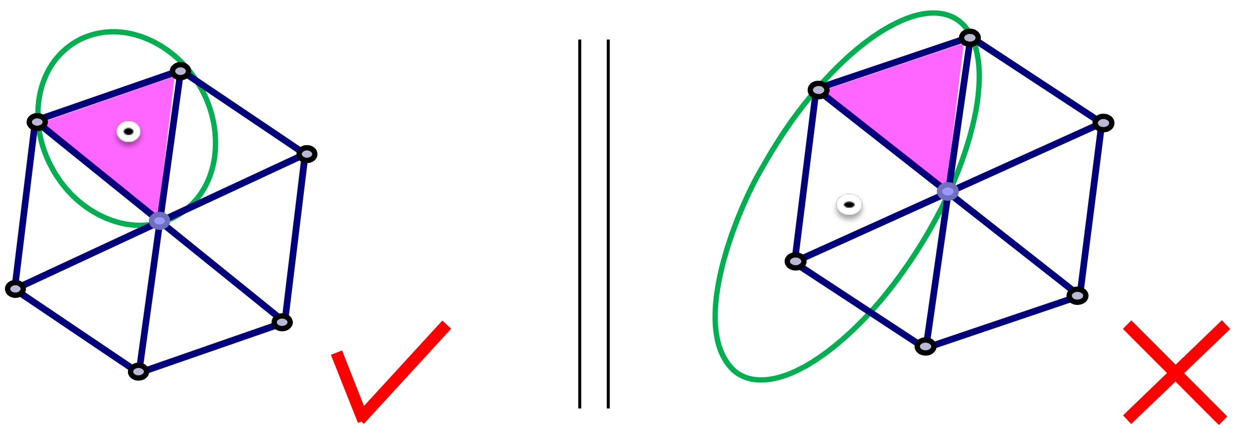
For this regular triangular mesh of ellipsoid, its Dupin dual contains self-intersecting P-Hex faces

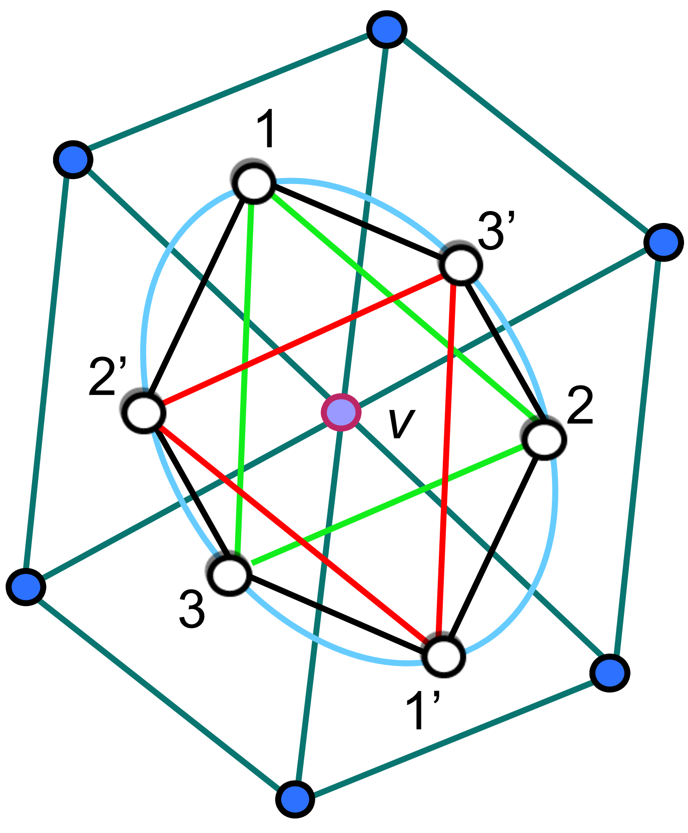


Conditions on P-Hex Free of Self-intersection

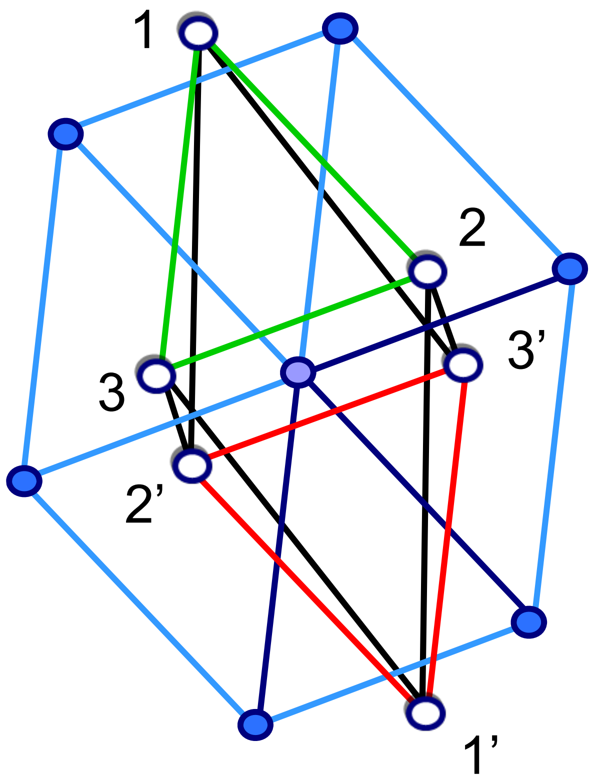
Theorem: *P-Hex mesh is free of self-intersecting faces if and only if locally everywhere the Dupin center of fundamental triangle* t *is contained in* t.

Or, equivalently, *t* is an acute triangle with respect to inner product induced by Dupin conic.

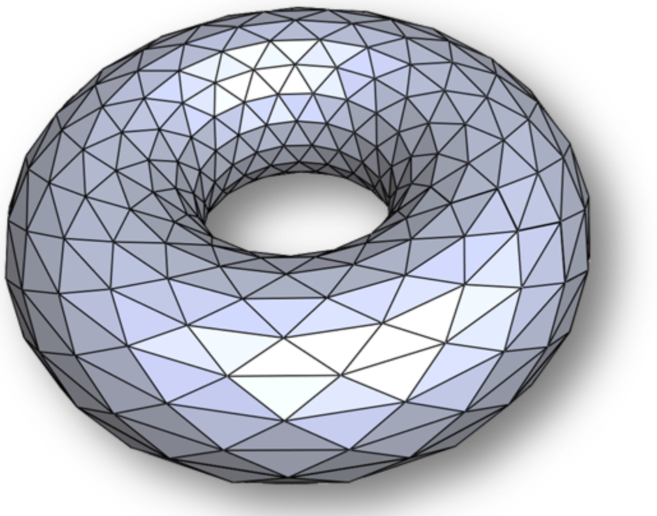




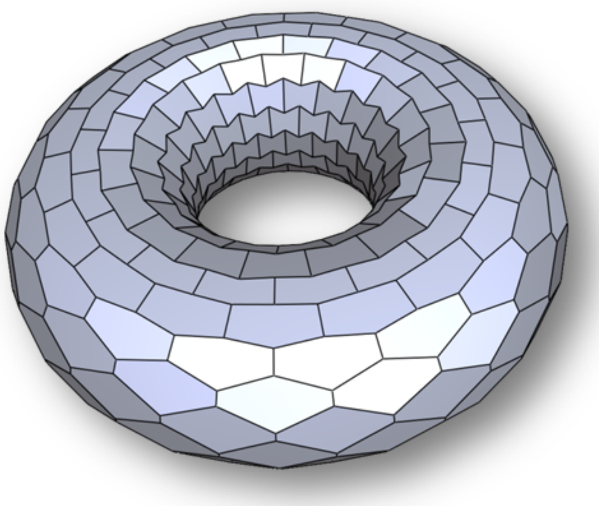
Traversal 1 > 3’ > 2 > 1’ > 3 > 2’ > 1 gives the P-Hex face



Traversal of 1 > 3’ > 2 > 1’ > 3 > 2’ > 1 gives
self-intersecting P-Hex face



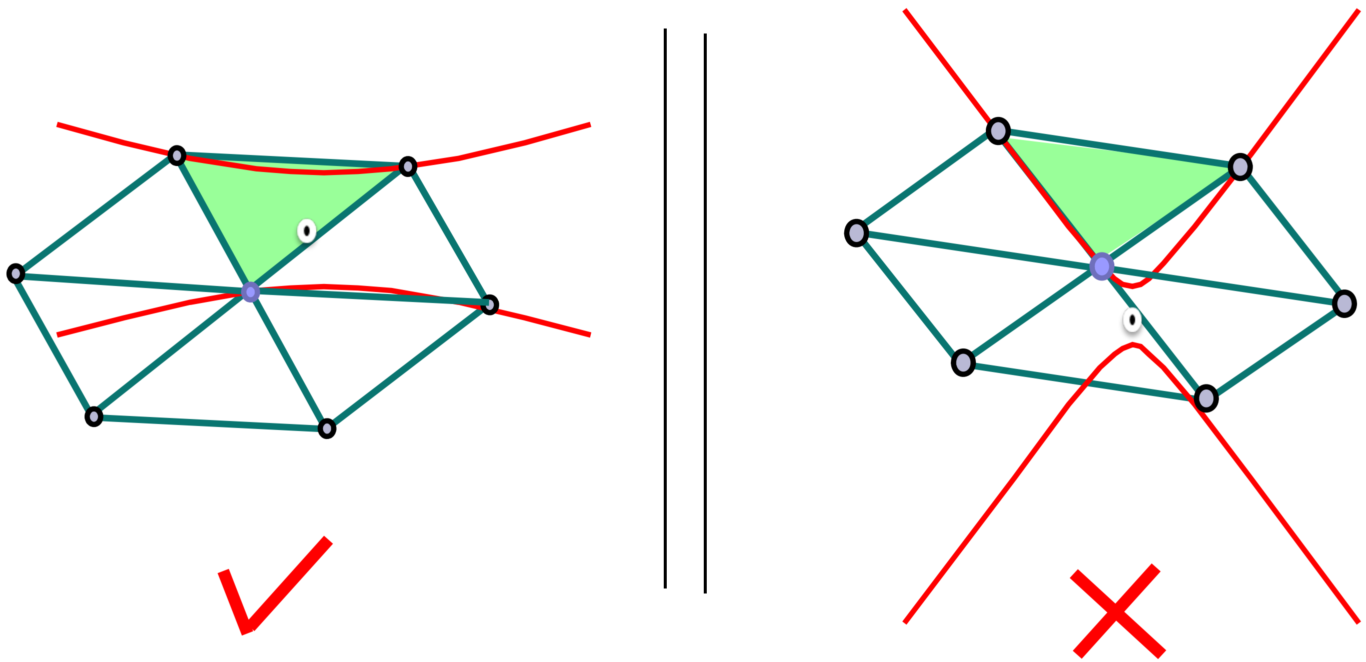
Good triangular mesh of torus



Dupin dual as nearly P-Hex mesh

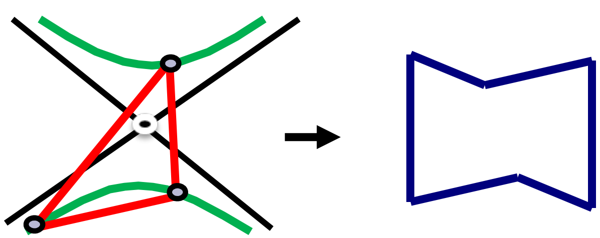
Hyperbolic case – avoidance of self-intersection

Theorem: *A P-Hex face is free of self-intersection if and only if three vertices of fundamental triangle* t *lie on different branches of Dupin hyperbola.*

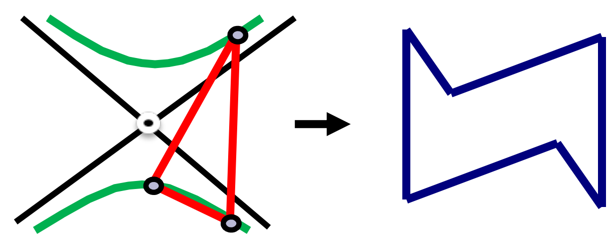


Hyperbolic case -- star-shaped non-convex P-Hex

Theorem: *Suppose that vertices of fundamental triangle* t *are on different branches of Dupin hyperbola. Then P-Hex face is star-shaped if and only if center of Dupin hyperbola is contained in* t*.*



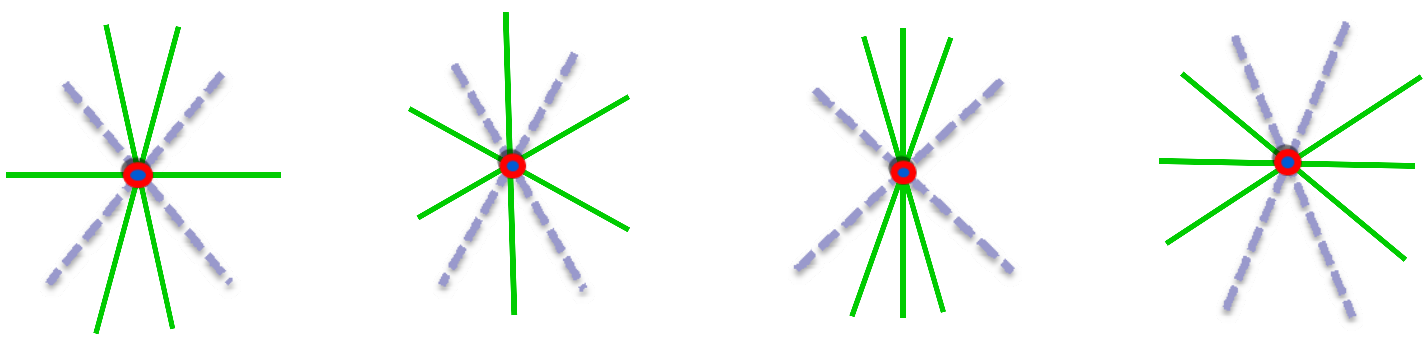
Star-shaped P-Hex



Non-star-shaped P-Hex

Hyperbolic case
– *characterization in terms of asymptotic lines*

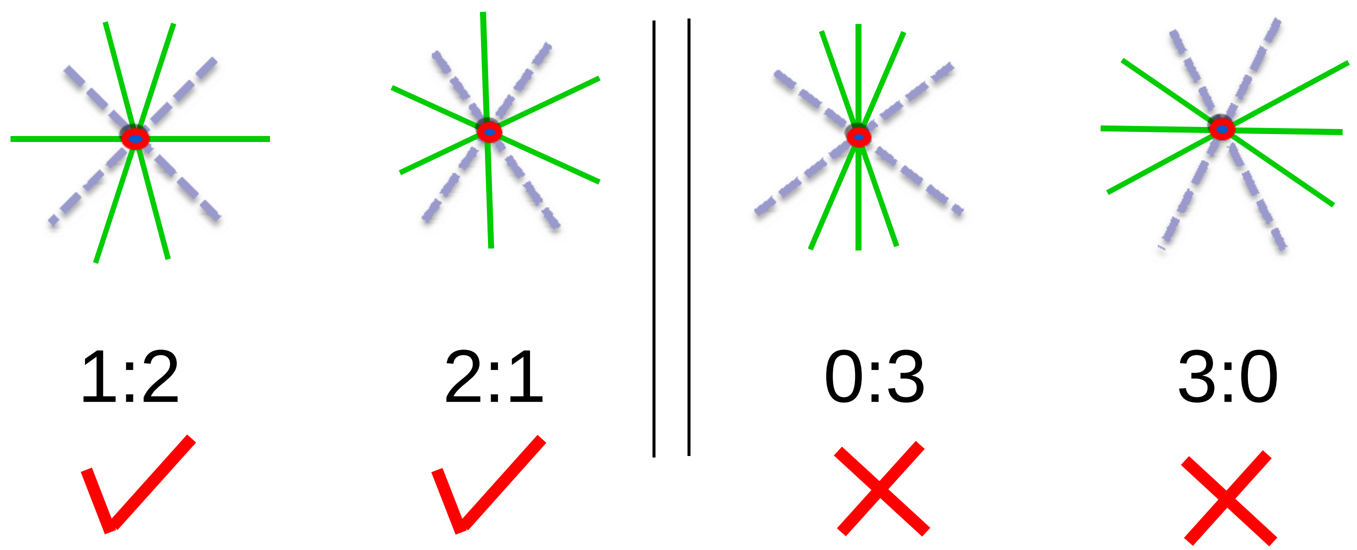
Two asymptotic lines divide 2D direction field originated at surface point *v* into two ranges, with opposite directions being identified.



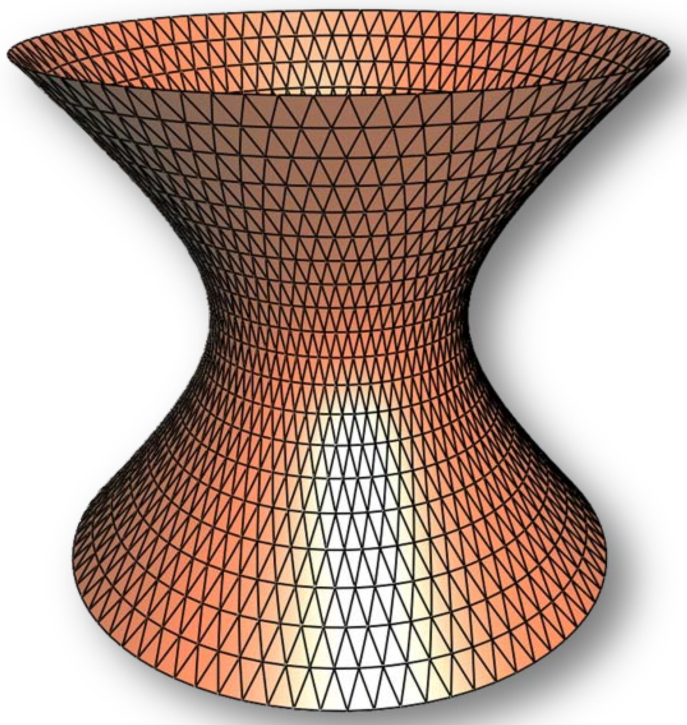
1:2 2:1 0:3 3:0

Condition on non-self-intersection of P-Hex faces

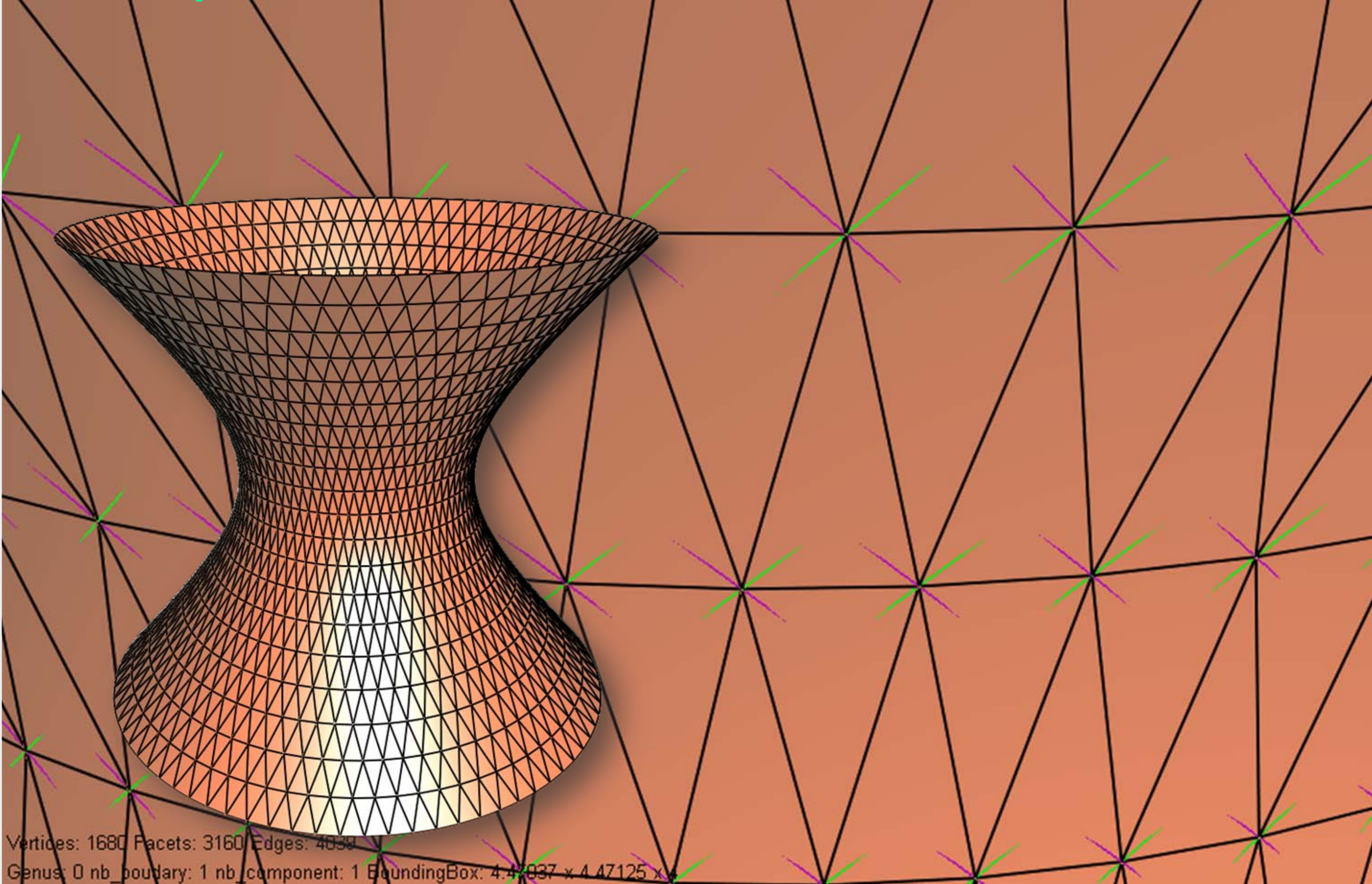
- **Theorem**: *P-Hex mesh is free of self-intersecting faces if only if locally everywhere the three principal line directions of regular triangle mesh are NOT contained in the same range (i.e., 1+2 or 2+1 occurs).*



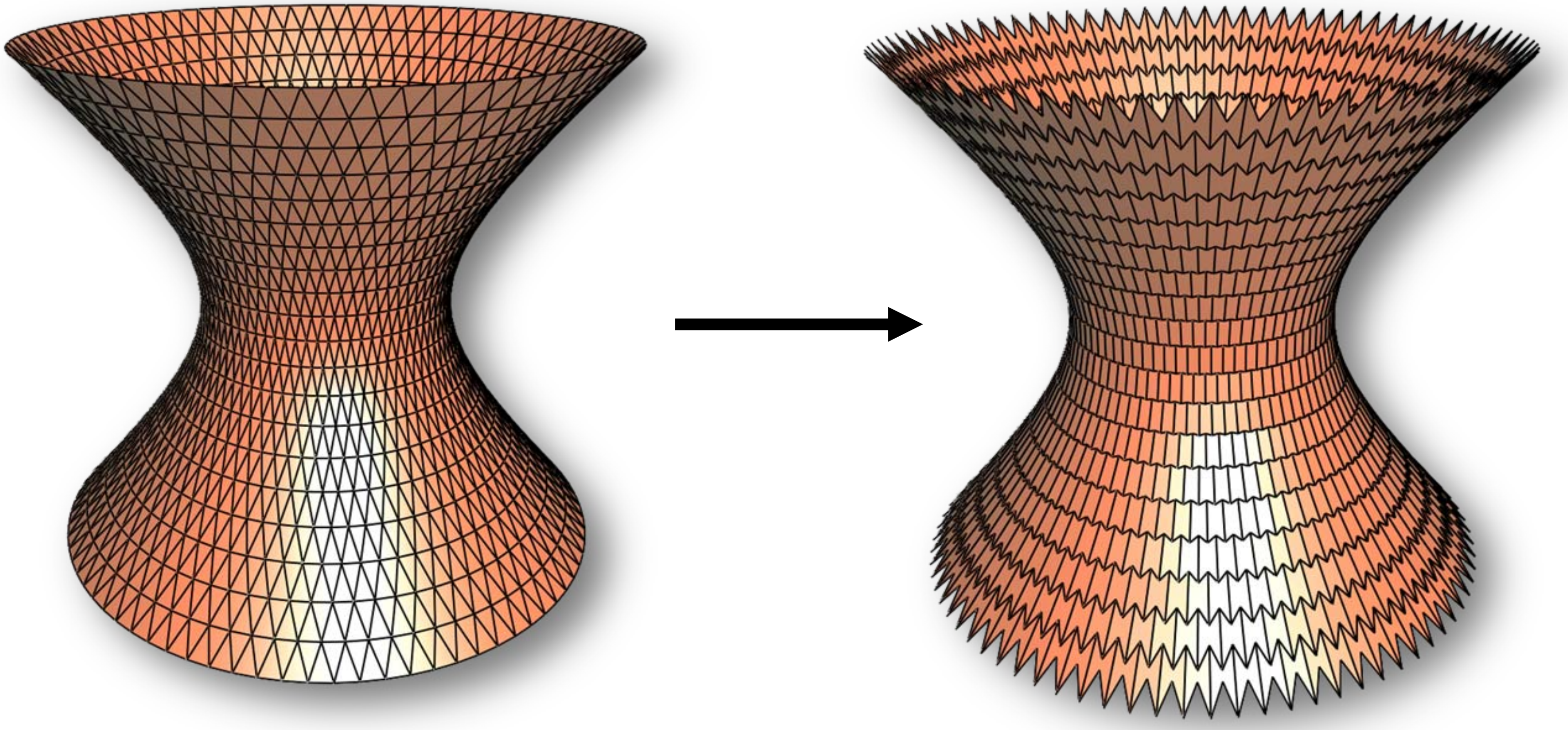
Example 1



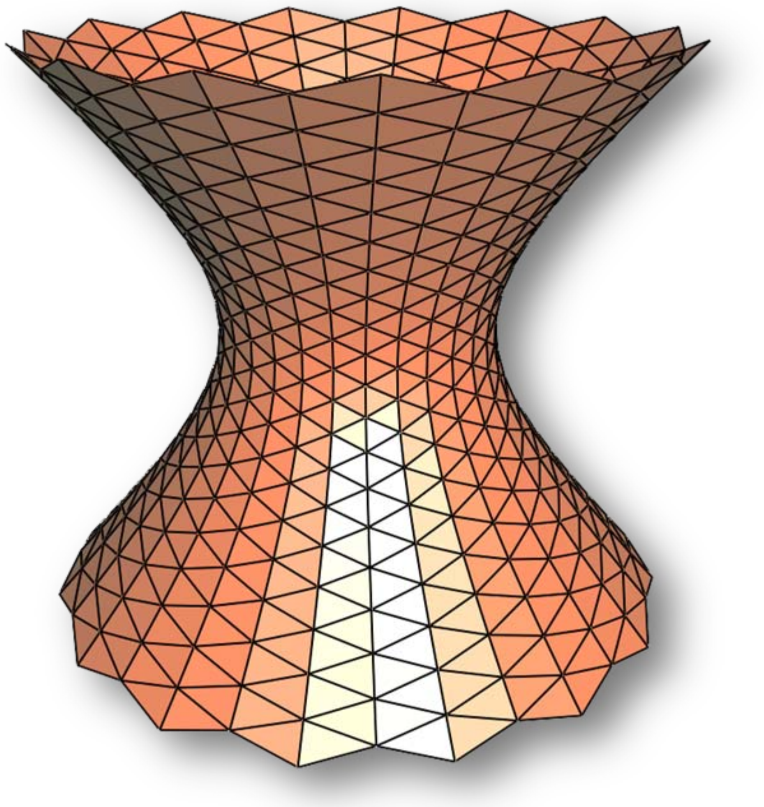
Example 1: Case of 1 + 2



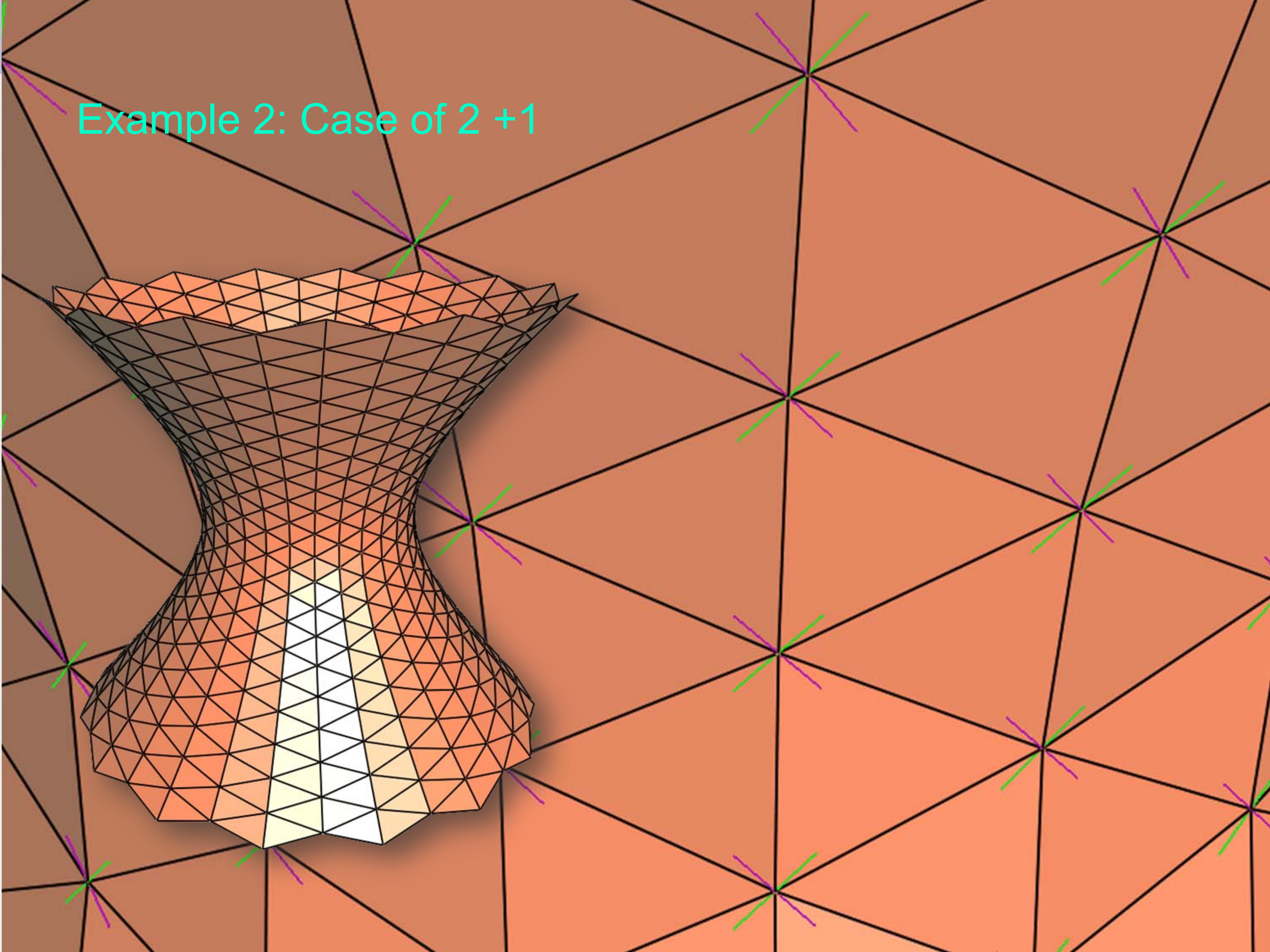
Example 1: Dupin dual (1+2)



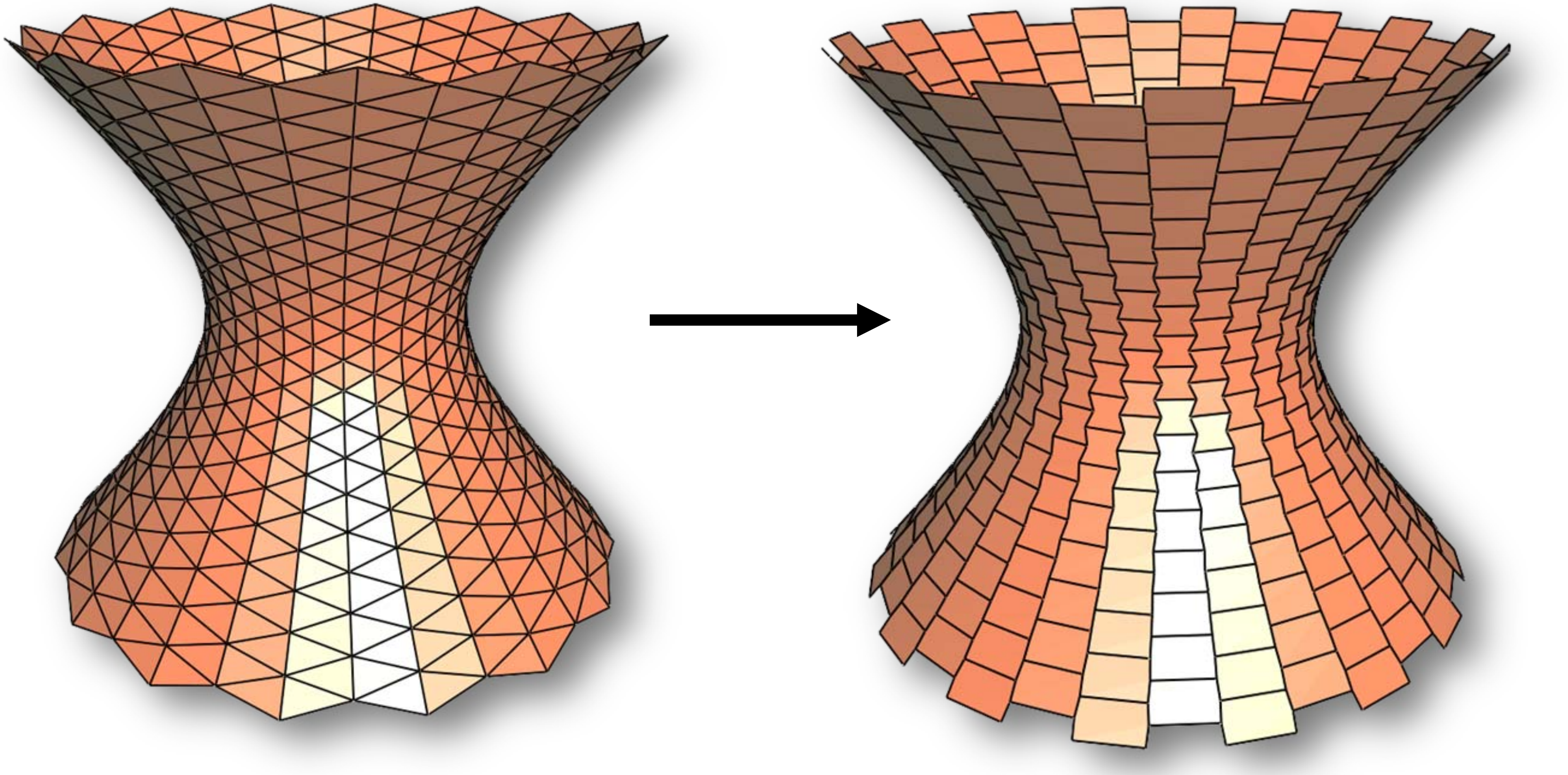
Example 2



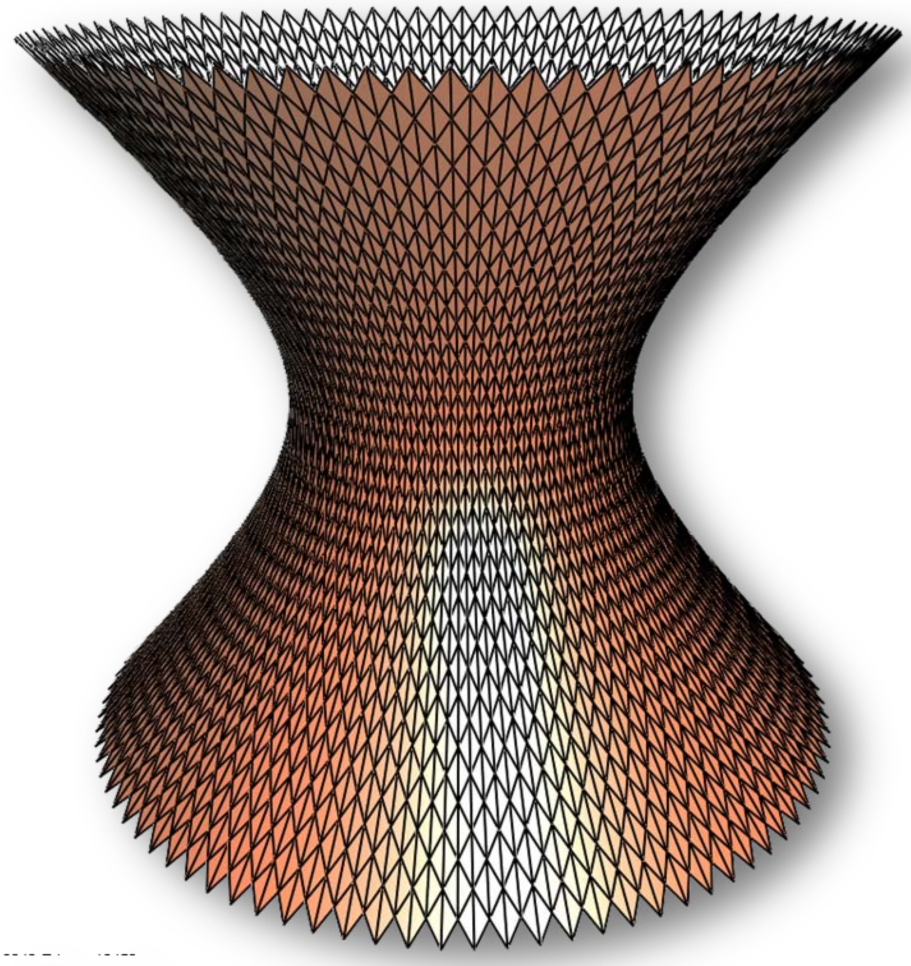
Example 2: Case of 2 +1



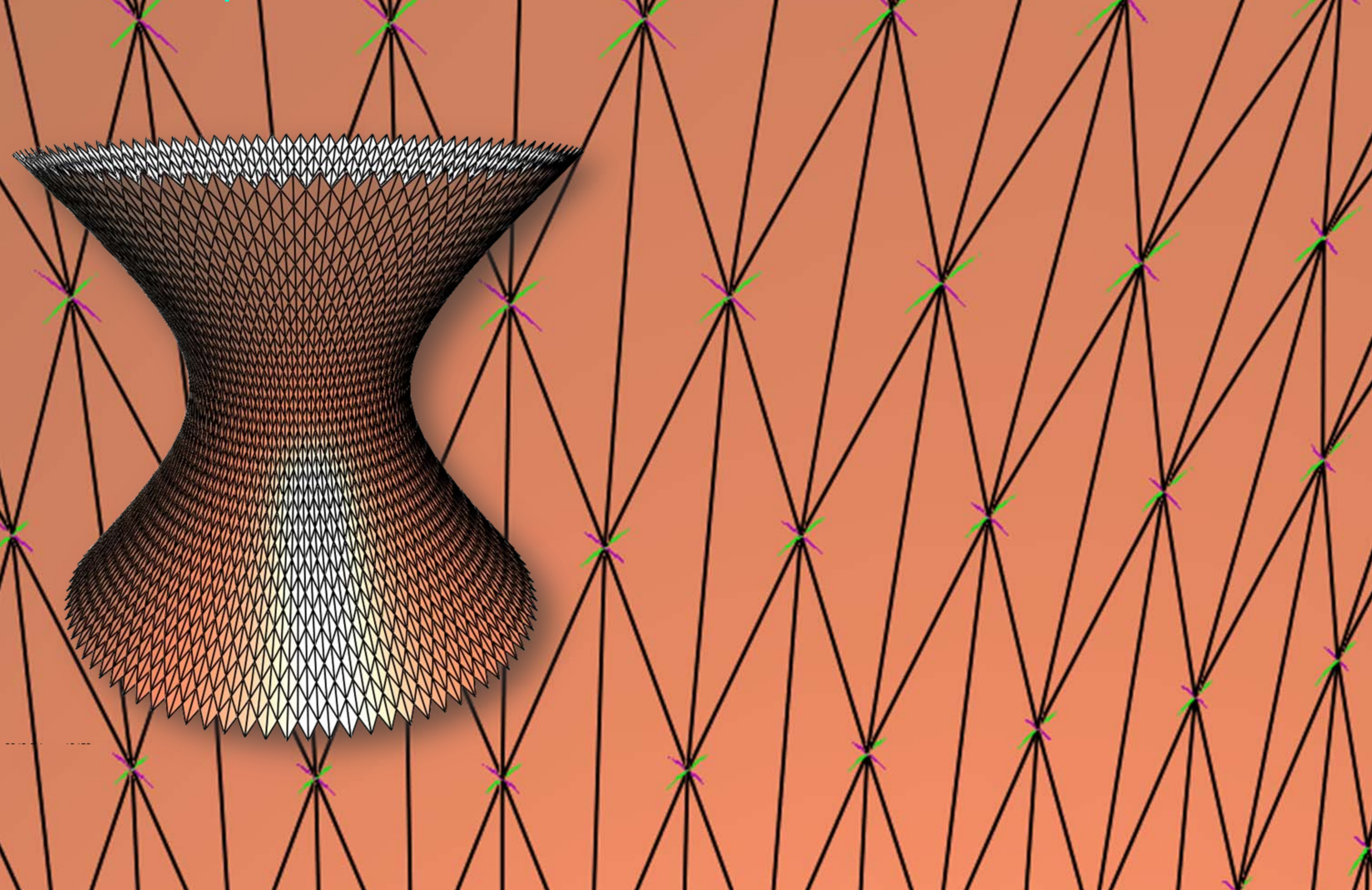
Example 2: Dupin dual (2+1)



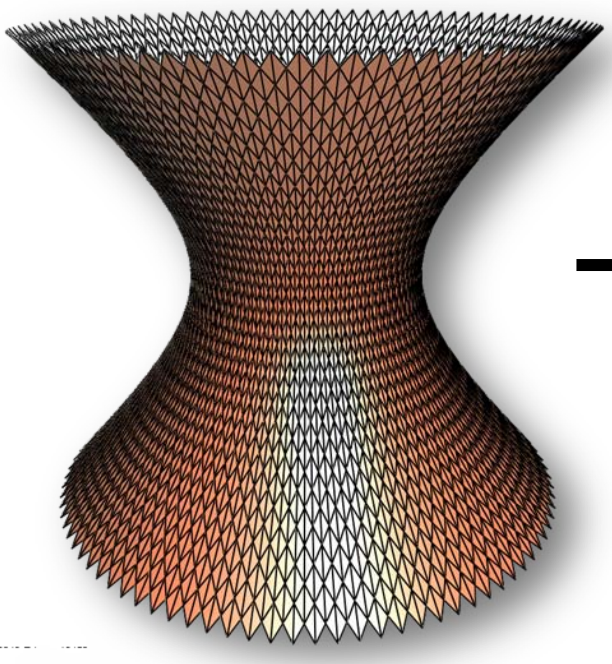
Example 3

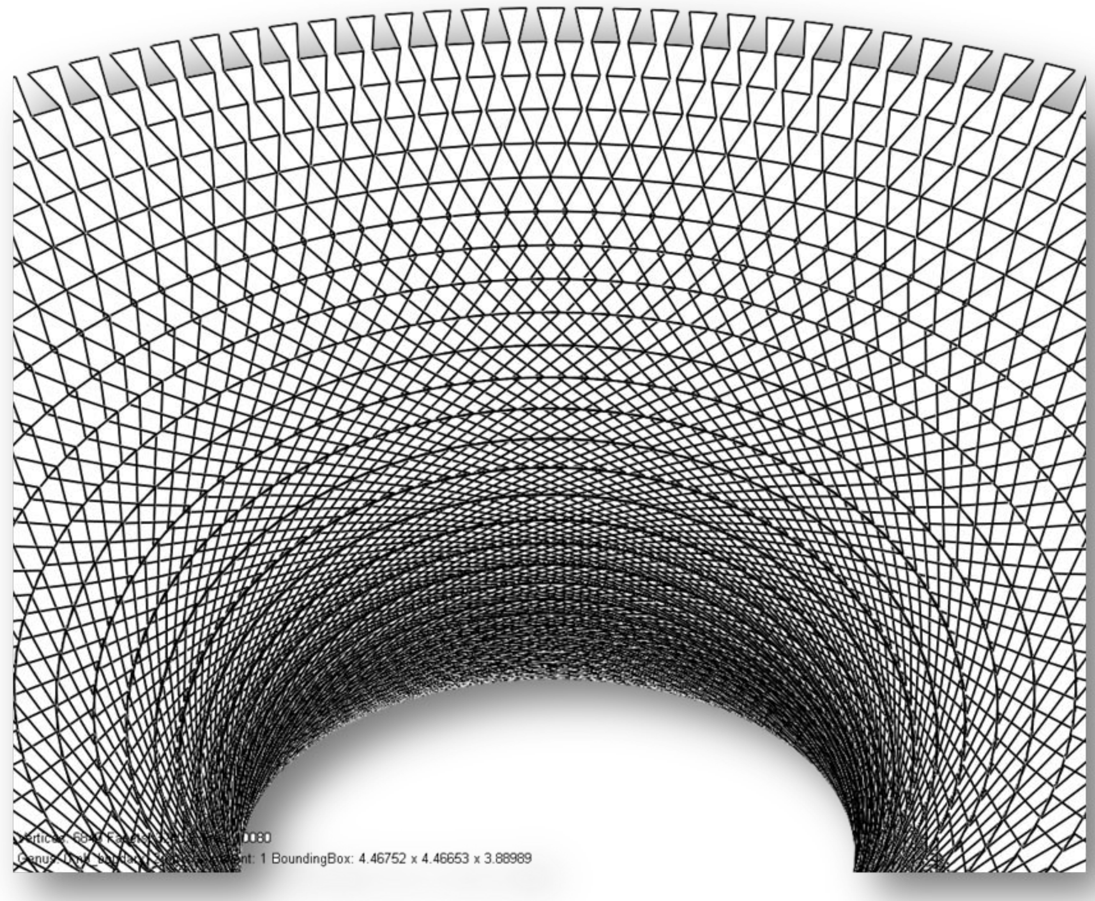


Example 3: Case of 0 + 3

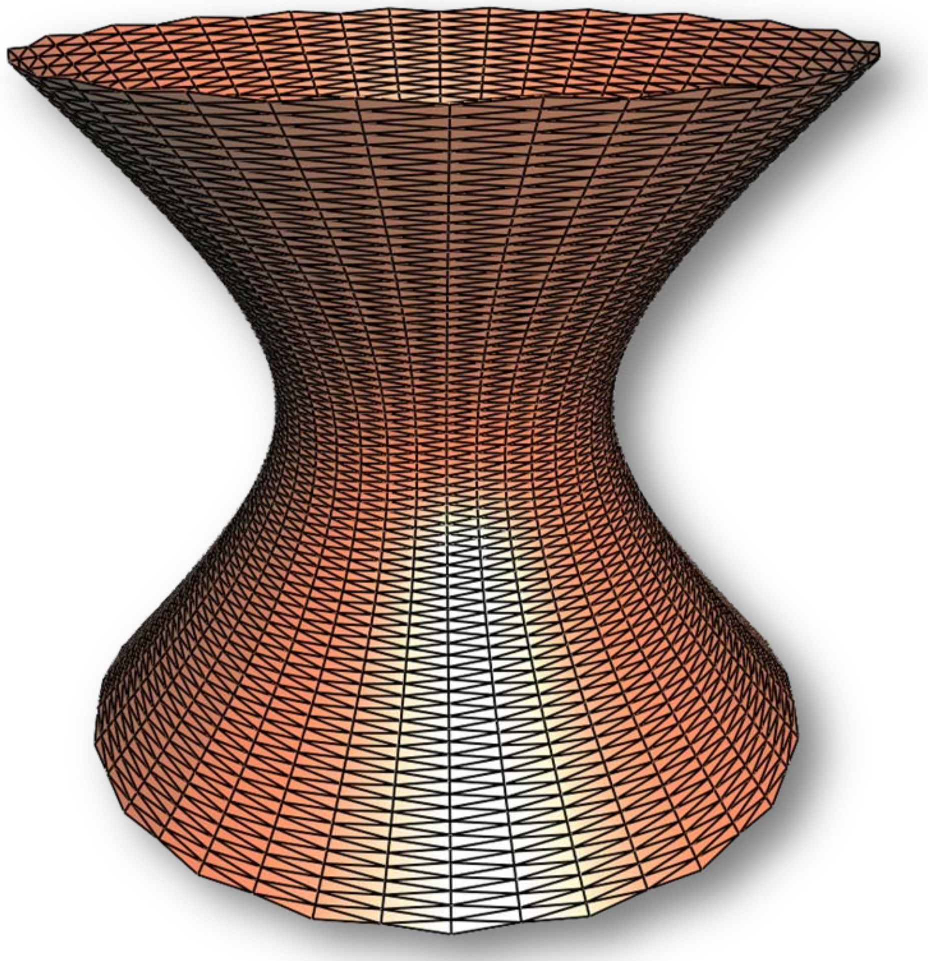


Example 3: Dupin dual (0+3)

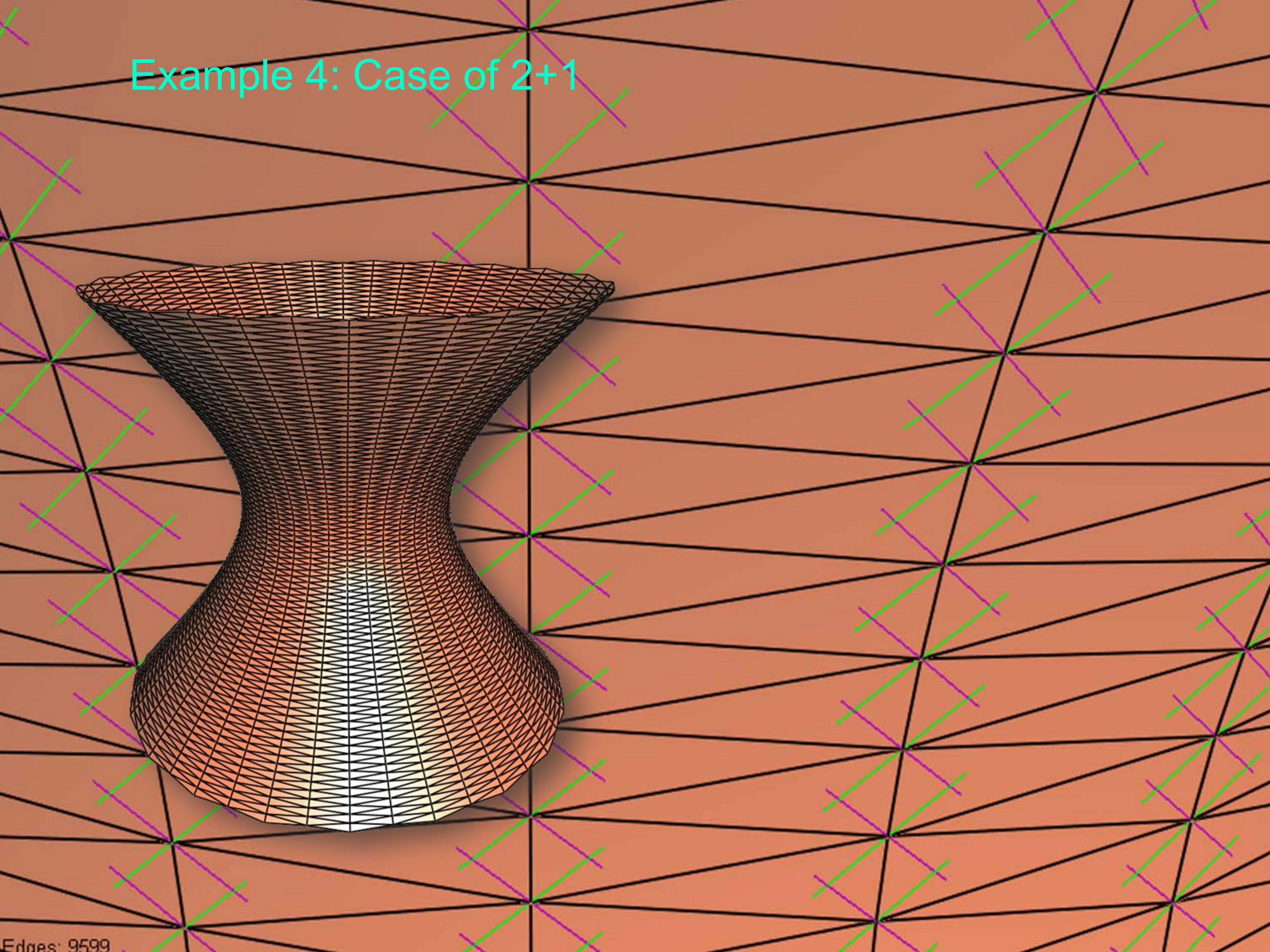




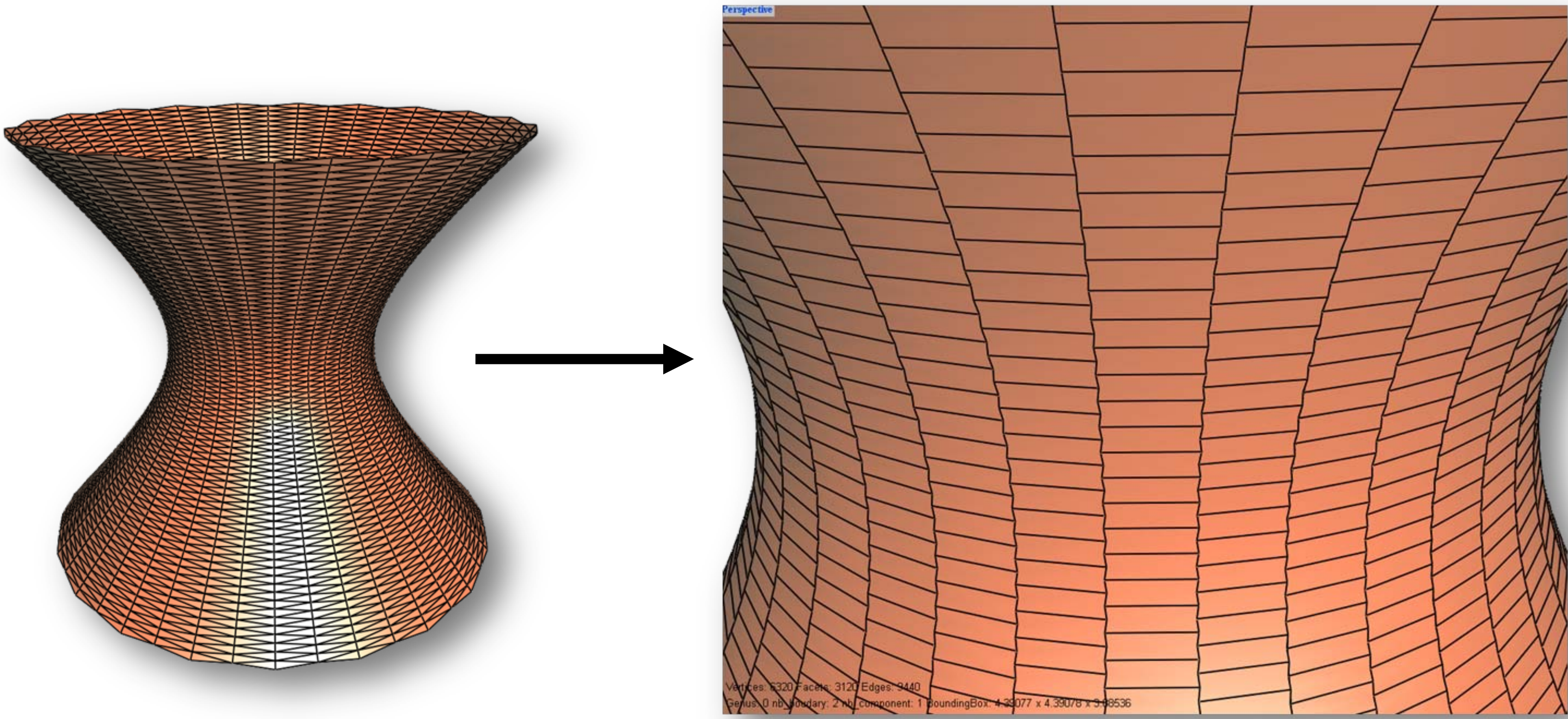
Example 4



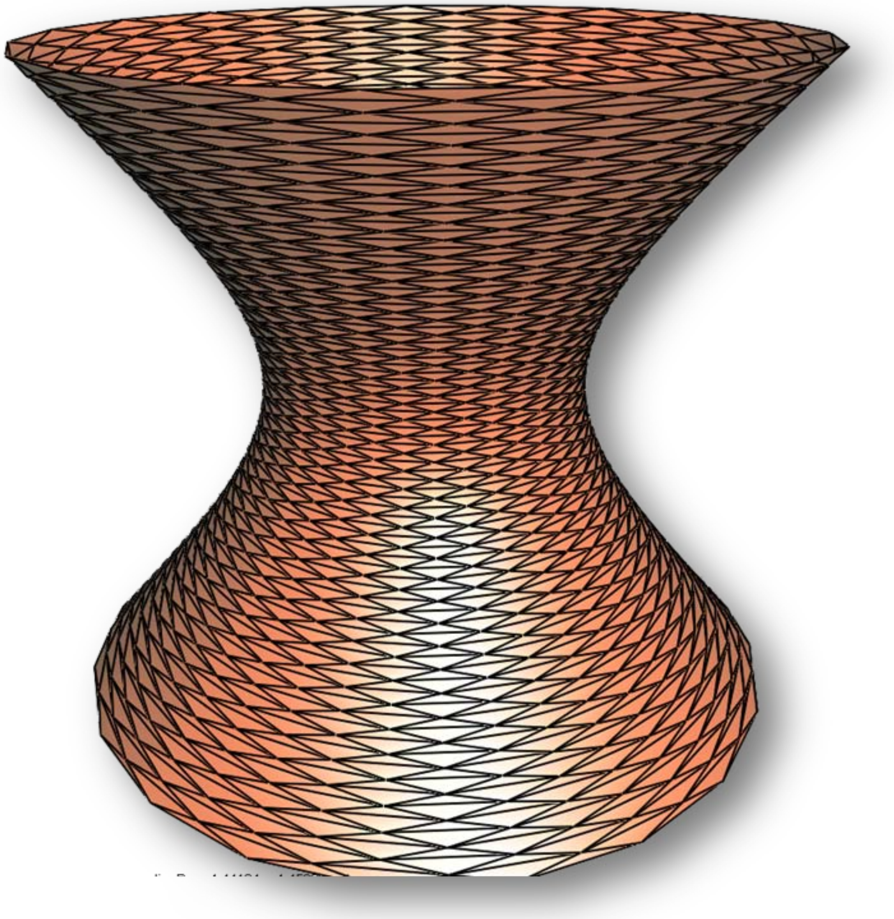
Example 4: Case of 2+1



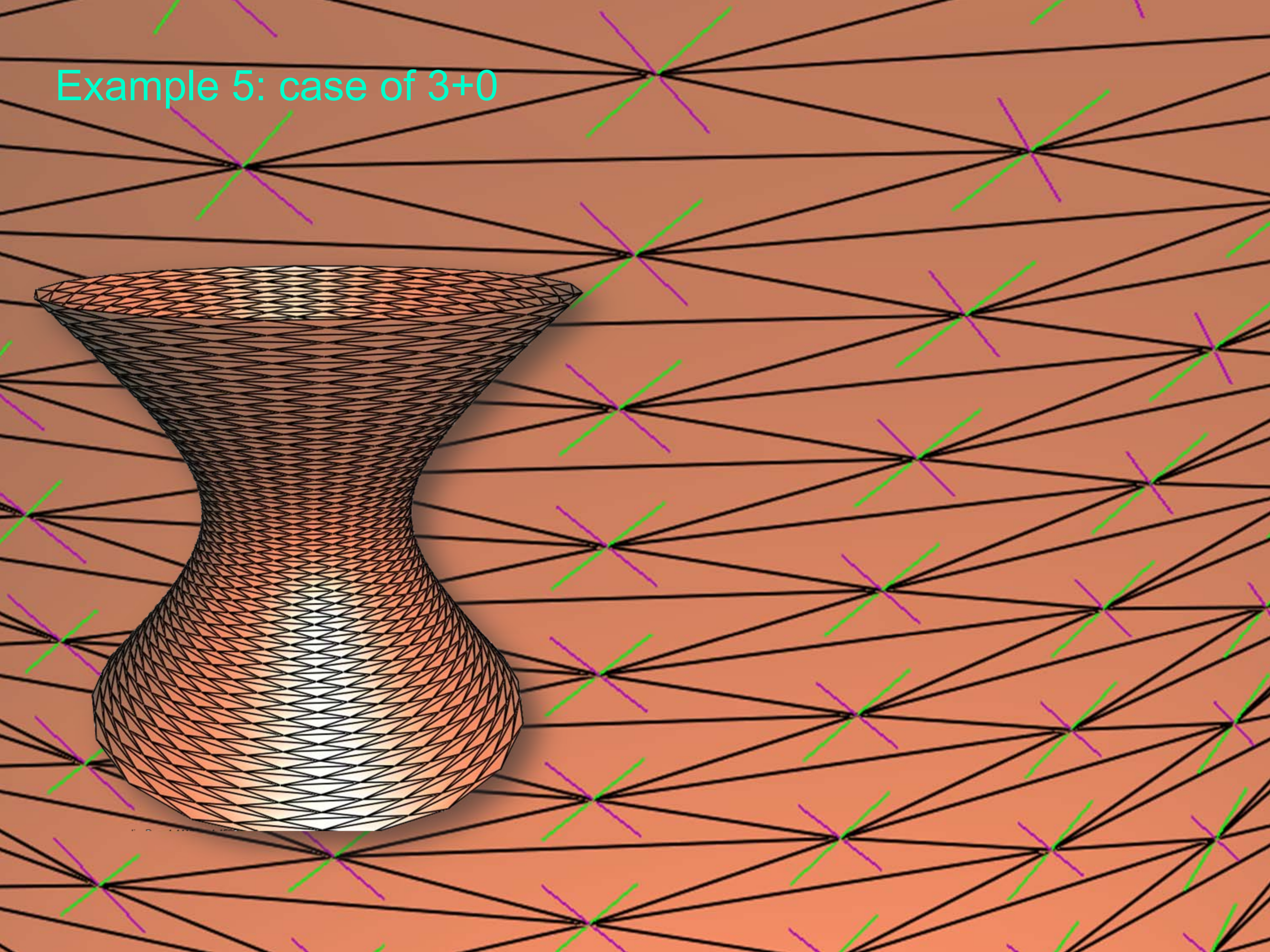
Example 4: Dupin dual (2+1)



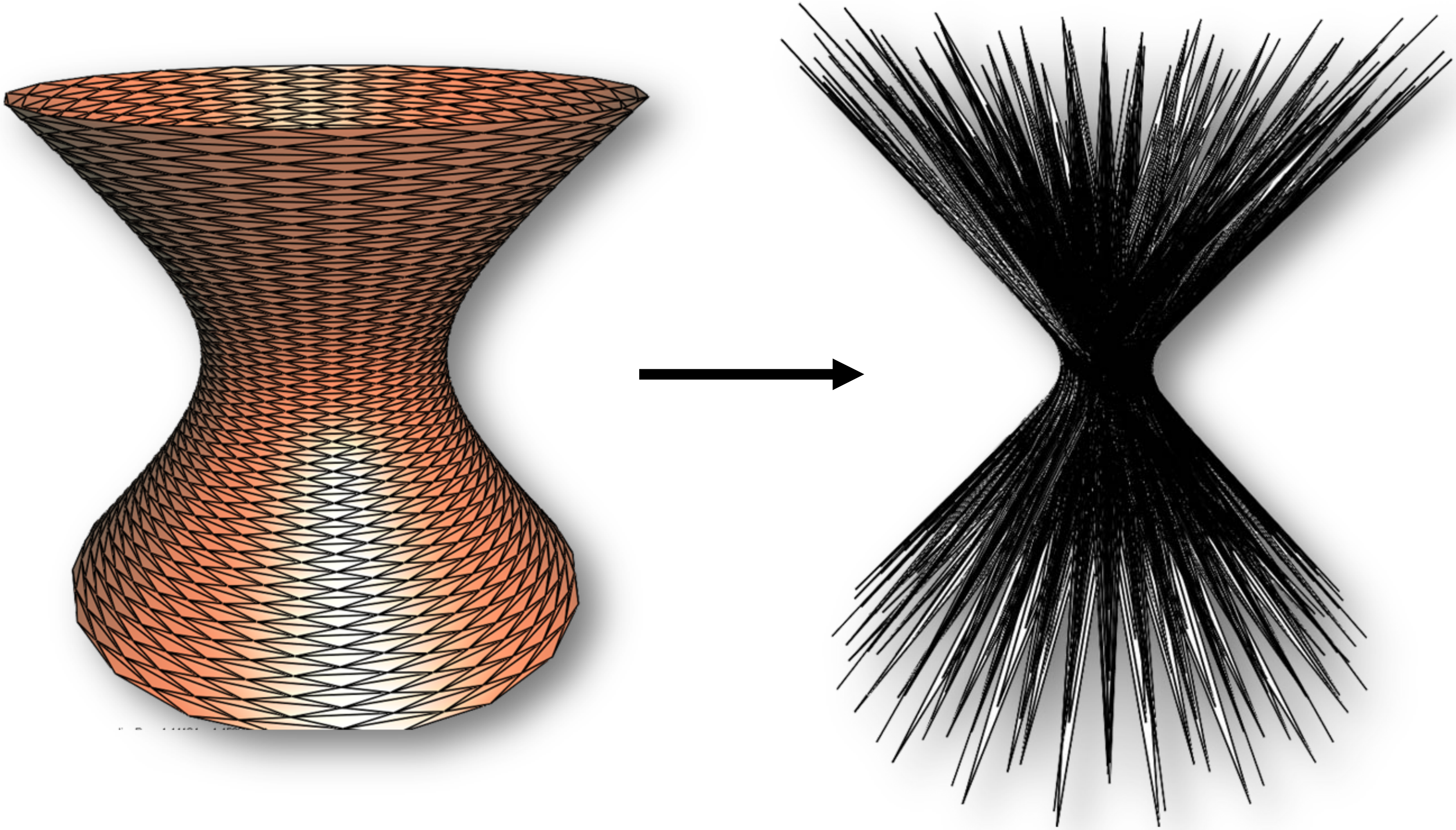
Example 5



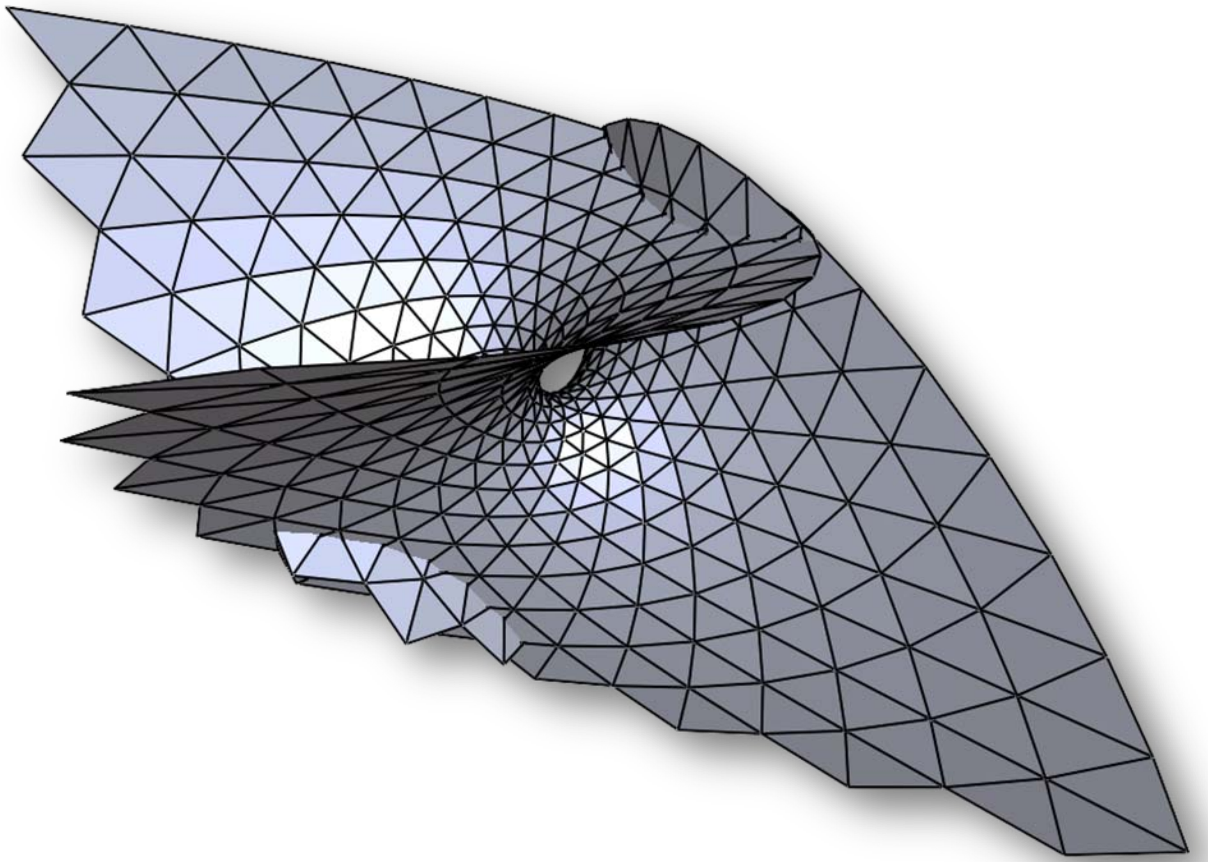
Example 5: case of 3+0



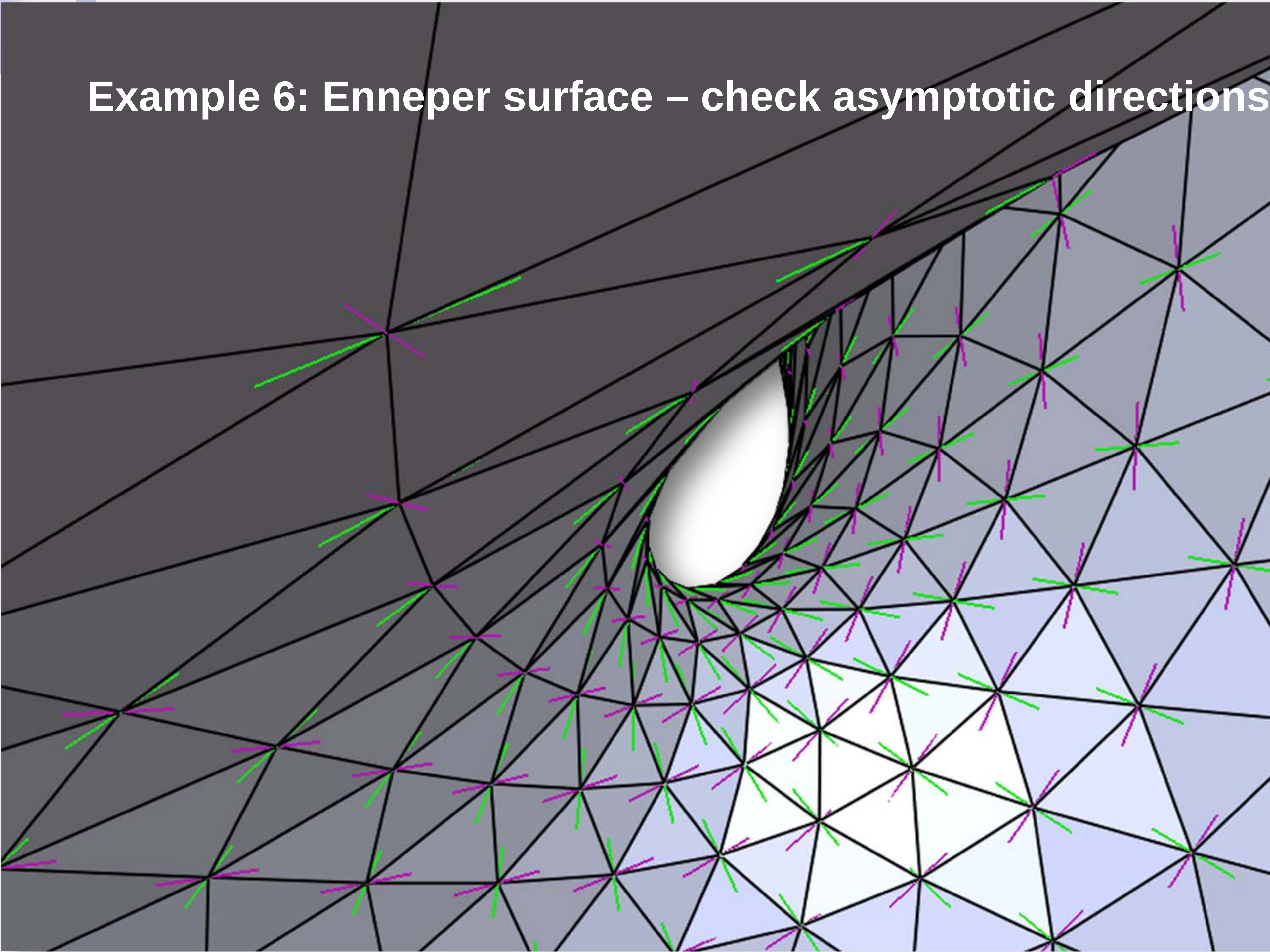
Example 5: Dupin dual (3+0)



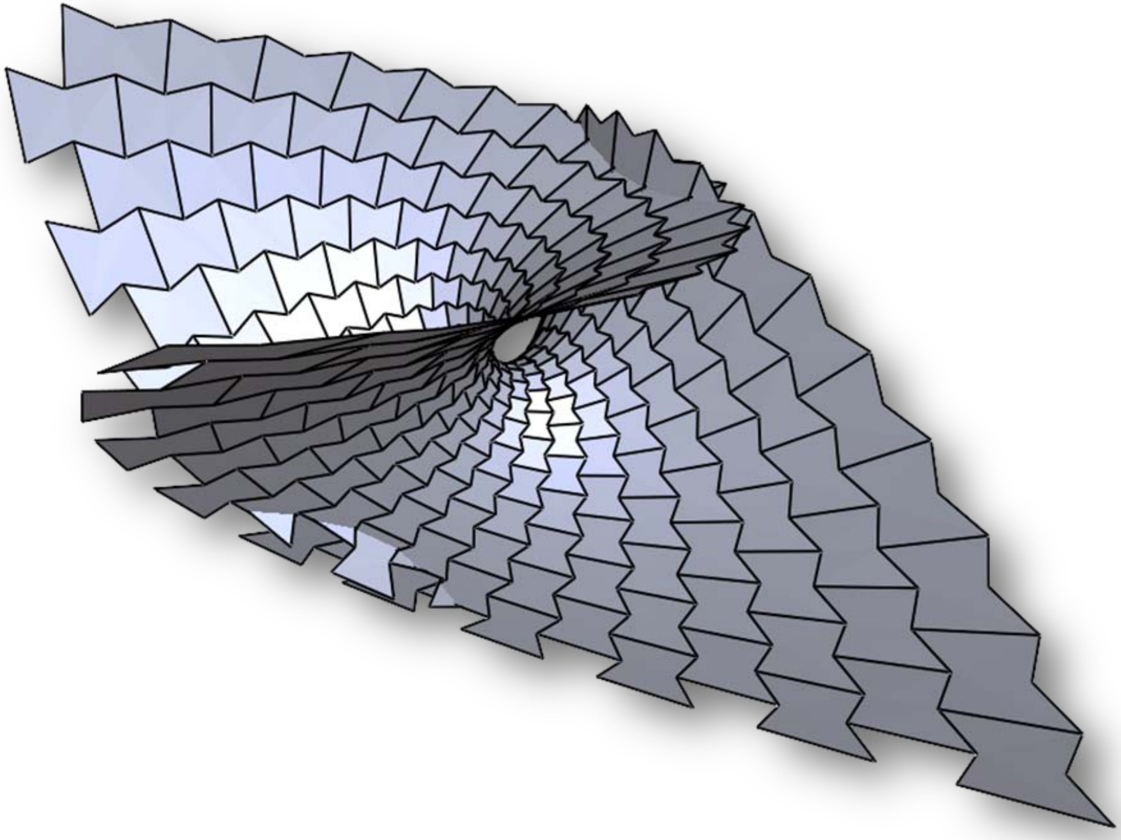
Example 6: Enneper surface



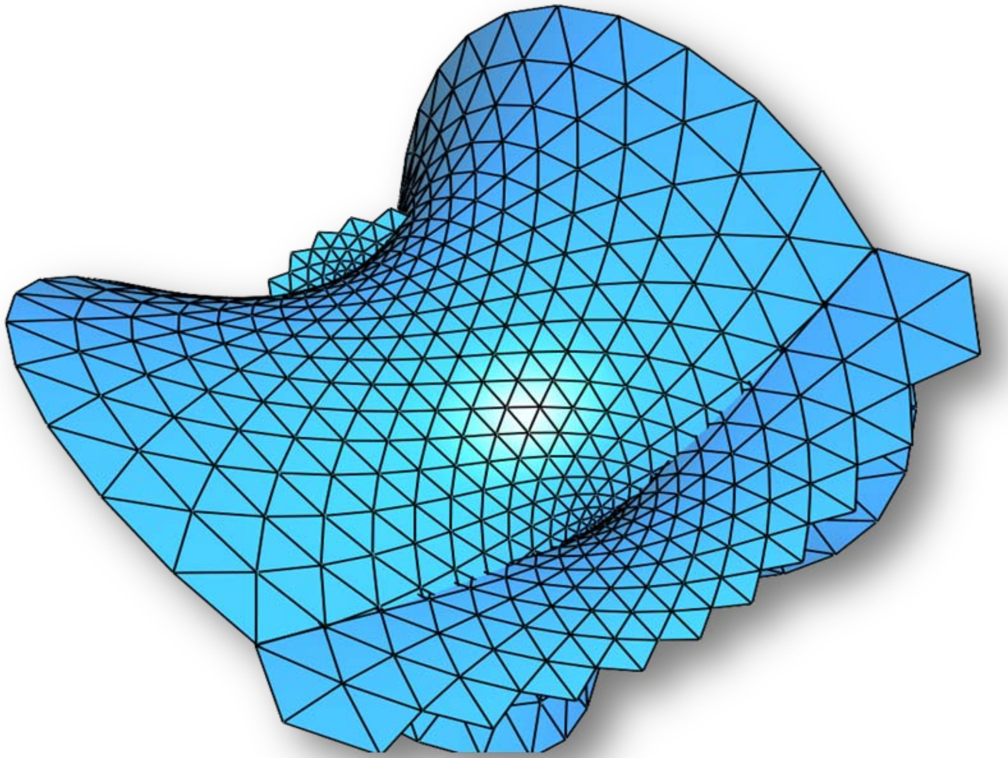
Example 6: Enneper surface – check asymptotic directions



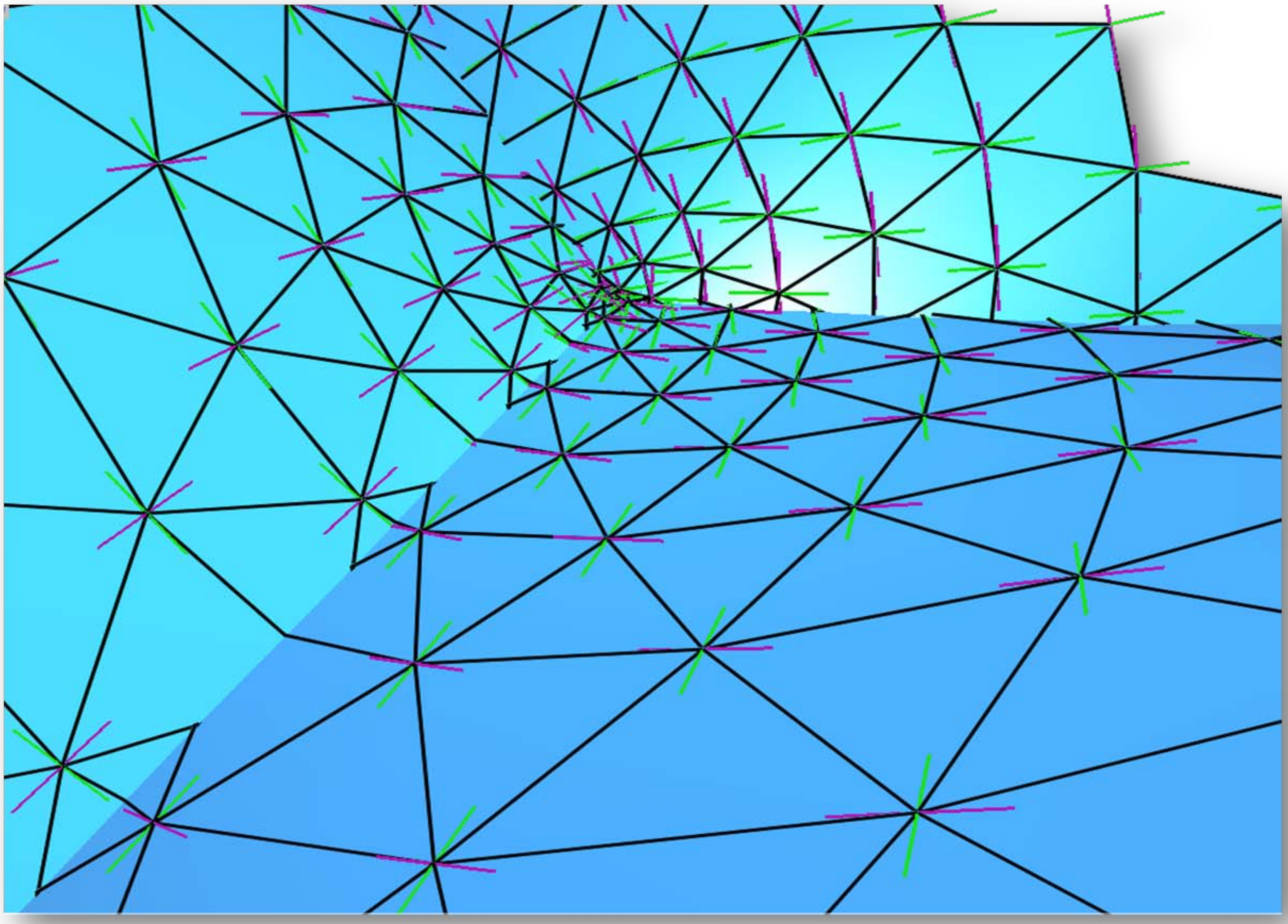
Example 6: Enneper surface – Dupin dual



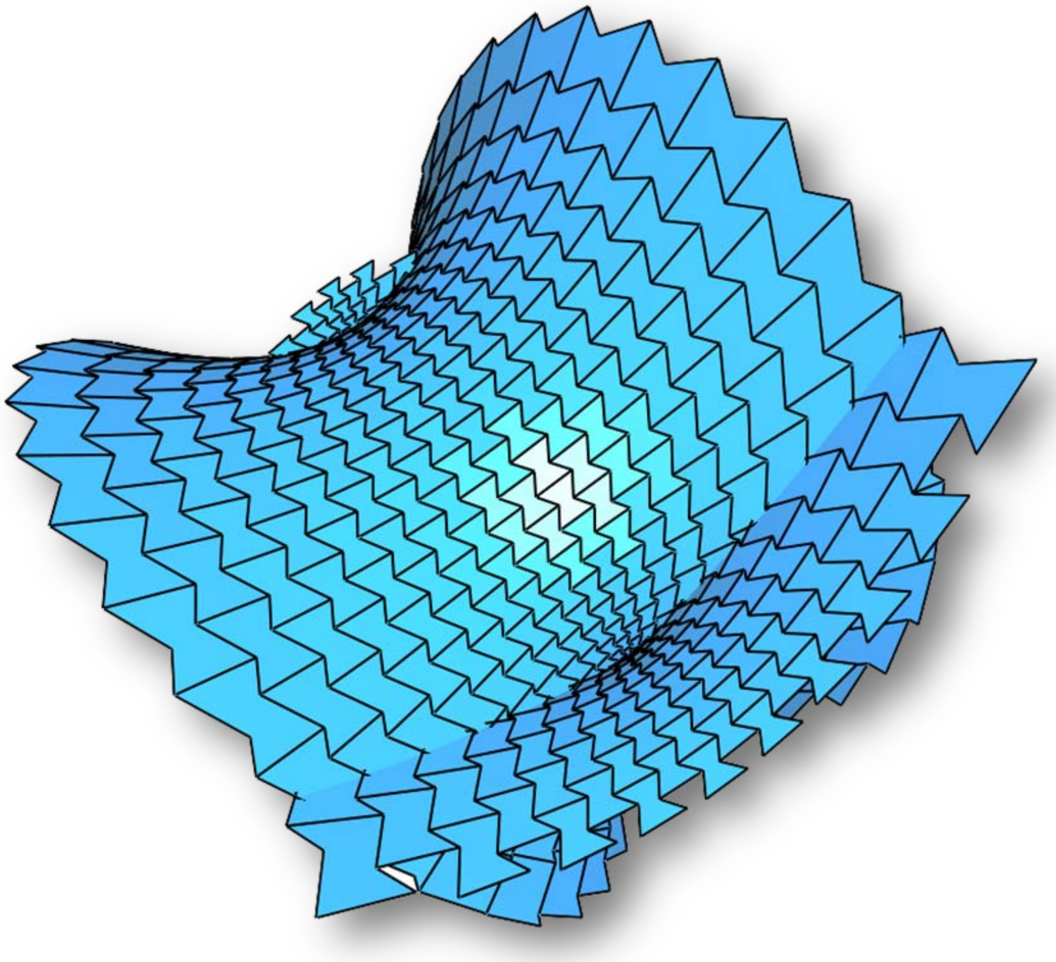
Example 7: Catalan surface – triangulation



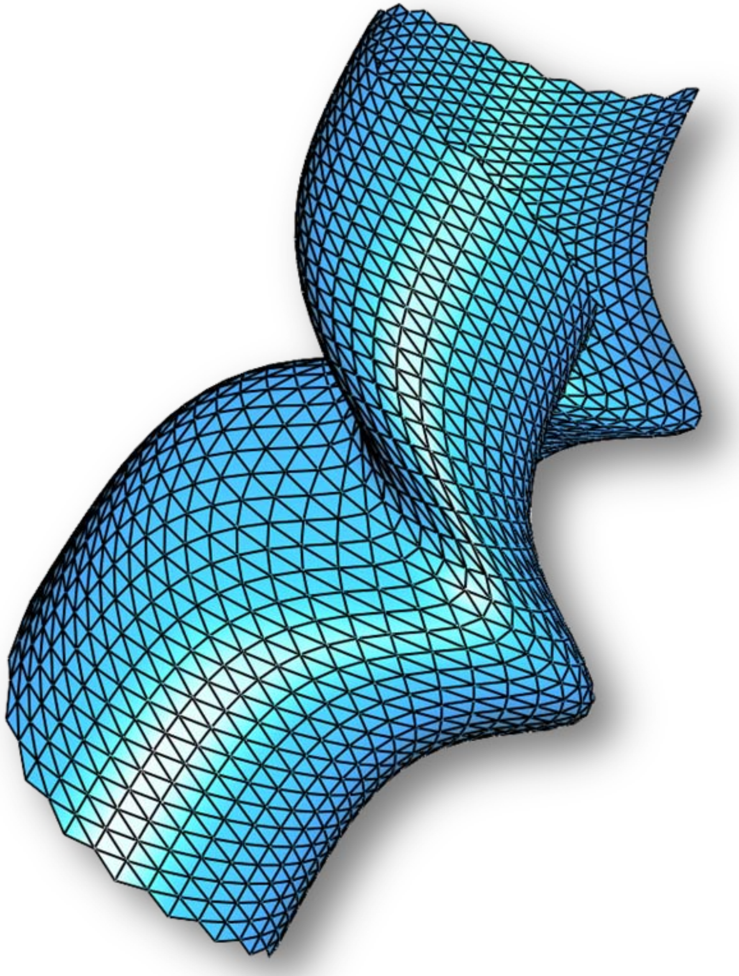
Example 7: Catalan surface – check asymptotic directions

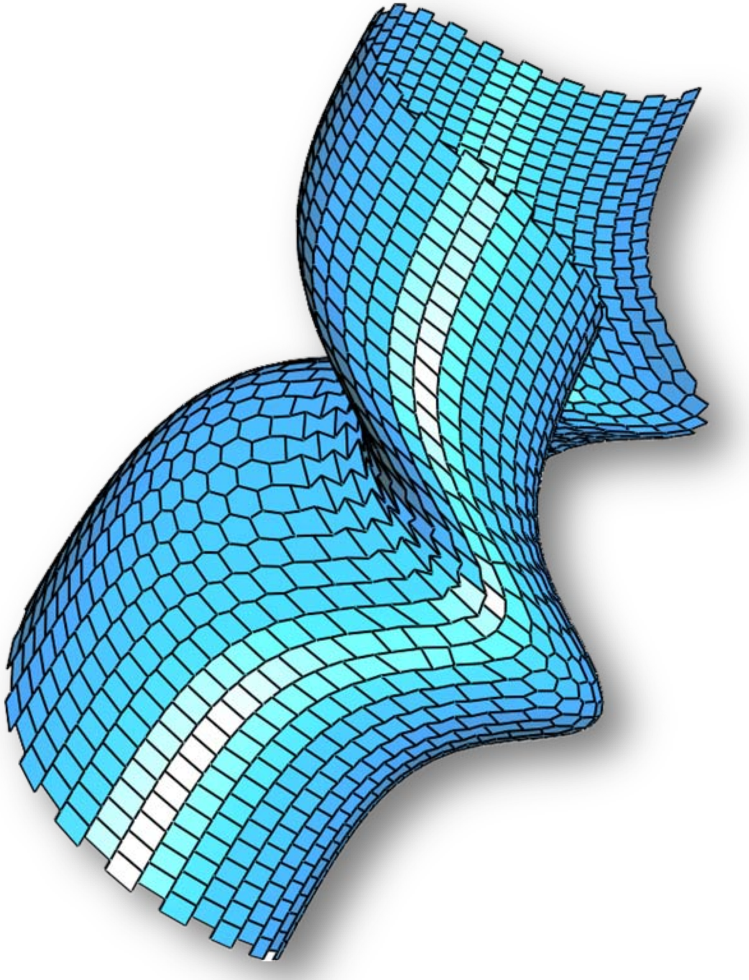


Example 7: Catalan surface – Dupin dual

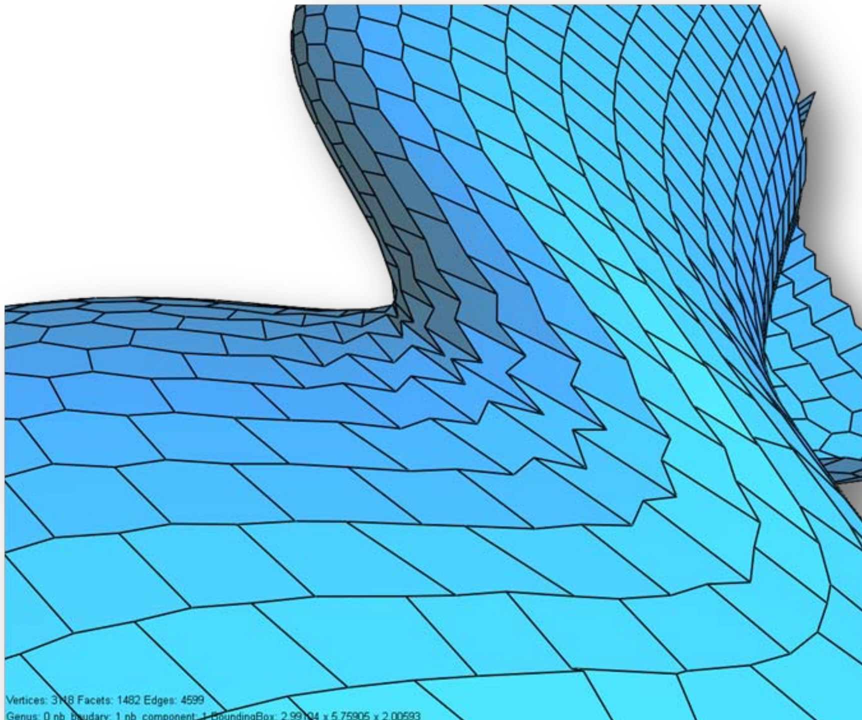


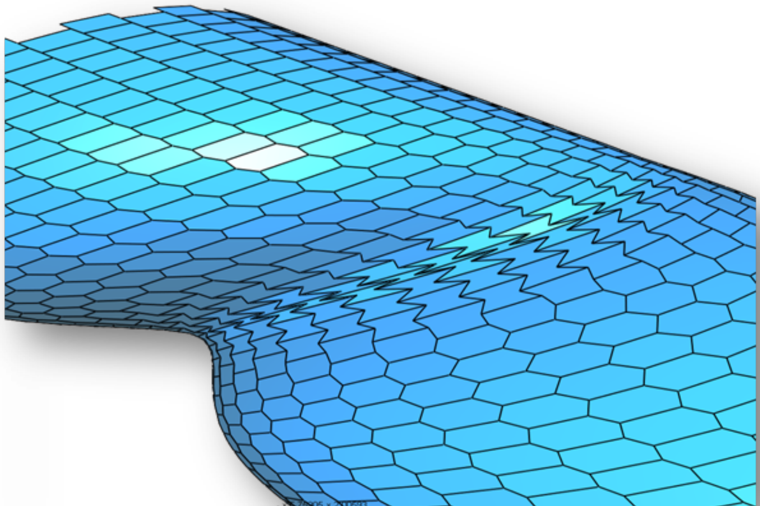
Example 8: Kinky torus – triangulation and Dupin dual





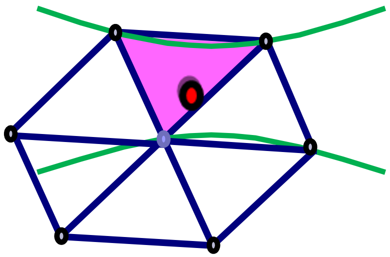
Example 8: Kinky torus – close-up views

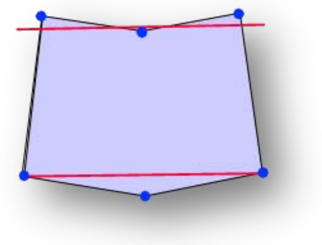




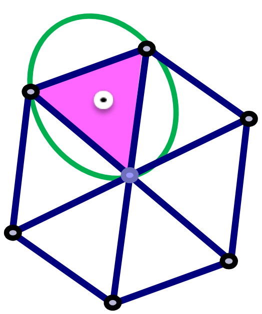
Computational Issues

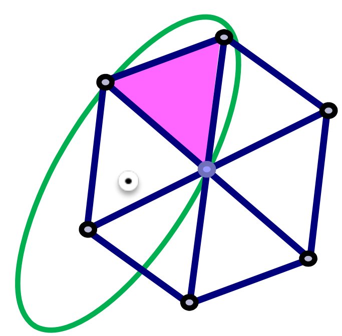
1) Computing Dupin center using curvature information at all three vertices





2) Detecting if Dupin center falls in triangle – done by sign-testing of **inner products**





Summary

We have provided local shape characterization of P-Hex meshes obtained from regular triangle mesh via Dupin duality.

--- Dupin duality allows establishment of simple conditions on existence of valid P-Hex meshes;

--- it also produces good initial hex mesh for effective optimization.

What's next

Develop a complete algorithm for computing P-Hex meshes based on good understandings of properties and constraints.

--- **Design** triangle meshes for computing P-Hex meshes

--- **Control** of shape, size, edge lengths and angles of P-hex faces

--- Compute P-Hex mesh with **special properties**, e.g., with vertex offset or edge offset property

- **Thank you**