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Fundamental issues and new trends in parallel mechanisms and manipulators

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Mapping similarity between parallel and serial architecture kinematics

Paul Zsombor-Murray · Anton Gfrerrer

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Abstract First the principles of mapping spatial *points to surfaces* is introduced in the context of the inverse kinematics of a general six revolute serial wrist partitioned robot. Then the advantage of choosing *ideal frames* is illustrated by showing that in the case of some architectures an image space formulation, though possible, may be an impediment to clear geometric insight and a satisfactory and much simpler solution. After showing how the general point mapping transformation is reduced to classical Blaschke-Grünwald planar mapping a novel three legged planar robot's direct kinematics is solved in image space and then using conventional “distance” constraints. The purpose is to show why the latter approach yields spurious solutions and how the displacement pole rotation performed with kinematic mapping reliably avoids this problem. In conclusion certain other new and/or interesting reduced mobility parallel robots are discussed briefly to point out some advantages and insights gained with an image space approach. Particular effort is made to expose in detail how mapping simplifies and extends the solution of direct kinematics pertaining to Calvel's “Delta” 3D translational robot.

Keywords Kinematic mapping · Parallel · Serial · Manipulator · Inverse · Direct · Kinematics · Distance · Constraints

1 Introduction

We address the engineering reader, with limited background in algebraic geometry, who, thus deterred, finds it difficult to appreciate important results obtained through the application of kinematic mapping. Kinematic mapping has been used to treat inverse and direct kinematics, workspace analysis, singularity and mechanism synthesis, all in a similar fashion. Constraint equations are written, not as loop closure equations in the Euclidean space, but by displacing points from an ideal moving frame or *end effector frame* EE onto surfaces or curves in an ideal base or *fixed frame* FF with a transformation matrix whose elements are expressed in terms of *dual quaternion* coordinates, rather than with conventional translation/rotation matrices. Some recent contributions in this regard are tabulated below.

- Inverse kinematics—Husty et al. [10]
- Direct kinematics—Husty [8]
- Workspace and singularity—Ottaviano et al. [13] and Pernkopf et al. [14]
- Synthesis—Brunnthaler et al. [4] and Schröcker et al. [15]
- Algebraic geometry—Husty [9] and Husty et al. [11]

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Mapping is immediately adaptable to reduced mobility (dof) as found in planar, spherical, translational and Schönflies architectures. This article begins with a 6dof serial formulation and proceeds to easily understood examples with fewer dof. Examine Fig. 1. Shown there is a schematic of six axes of the chain of R-joints (revolute) and their common normals, beginning with the basal R-joint in the fixed frame (FF) and ending on the S-jointed (spherical) wrist, a triad of mutually intersecting R-joints to which the end effector (EE) is attached. The wrist centre and EE are attached to FF via the specification of the EE pose that is given in the case of any inverse kinematics problem. This becomes a “parallel manipulator” direct kinematics problem if one imagines EE, the S-joint and the R-joint on the link attached to it as one leg while the basal R-joint, the next one and the link between is the other. The intermediate assembly shown as the moving frame (MF) becomes the “platform”.

1.1 Overview

Husty et al. [10] did an elegant IK image space solution of a 6R serial robot with six skew axes, drawing heavily on advanced concepts from the classical work of Study [17]. Since the purpose here is to illustrate the point-on-surface notion, formulating the much simpler image space IK problem for a 6R serial wrist-partitioned manipulator is more appropriate and proceeds as follows. There is no loss in generality if one lays out the serial kinematic chain of six points O, P, Q, R, S, W with O the origin in FF on the axis of the first R-joint. Points P and Q are on the axis of the second such that OP is the common normal (CN) between the first axis pair. Points R and S are on the third R-joint axis where QR is the CN between the second and third axes. Finally points S and W define the normal to the third axis where W is the wrist centre that is fixed by the given end effector (EE) pose that defines the problem. Now consider a conveniently chosen, ideal MF where the four points P, Q, R, S are represented homogeneously as $P'\{1:0:0:p'_3\}$, $Q'\{1:0:0:0\}$, $R'\{1:1:0:0\}$, $S'\{1:1:s'_2:s'_3\}$. Note that all length dimensions are scaled to $QR = 1$. Points P', Q', R', S' are mapped to P, Q, R, S by premultiplication by the general point transformation matrix $[a]$. There are six constraining equations $e_1, e_2, e_3, e_4, e_5, e_6$. First, points P and Q are placed on transaxial circles that rule a one-sheet hyperboloid of revolution swept by the second revolute axis on QP revolving freely about the first revolute axis on O in FF when the chain is broken at Q . These circles are on planes specified by their coefficients $p\{0:0:0:1\}$ and $q\{Q_0:0:0:1\}$ as well as the hyperboloid h presented in standard form. Note how opportunity to choose naturally simple poses for the key elements in both MF and FF has been exploited so computations need not resort to specific numerical examples and can be maintained in symbolic form. Similarly the points R and S are on concentric spheres r and s centred on point W , the wrist centre. The spheres represent the surfaces upon which R and S are free to move when the chain is broken at W . The squares of distance $|RW|^2 = R_{rw}$ and $|SW|^2 = R_{sw}$ are given by the known EE pose and shown in the upper sketch in Fig. 1. To summarize the procedure, $MF \rightarrow FF$. The four points in MF are transformed to FF then distributed on six surfaces. Two planes p and q intersect hyperboloid h normal to its axis and $P \in p$ and $P \in h$ as well as $Q \in q$ and $Q \in h$ thus providing four equations. The other two points, $R \in r$ and $S \in s$, go, respectively, on one of the concentric spheres centred on W .

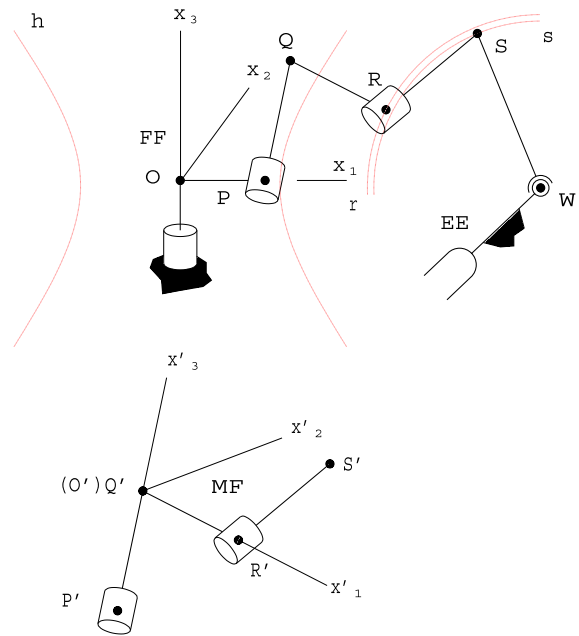


Fig. 1 A six revolute serial chain

lute axis on QP revolving freely about the first revolute axis on O in FF when the chain is broken at Q . These circles are on planes specified by their coefficients $p\{0:0:0:1\}$ and $q\{Q_0:0:0:1\}$ as well as the hyperboloid h presented in standard form. Note how opportunity to choose naturally simple poses for the key elements in both MF and FF has been exploited so computations need not resort to specific numerical examples and can be maintained in symbolic form. Similarly the points R and S are on concentric spheres r and s centred on point W , the wrist centre. The spheres represent the surfaces upon which R and S are free to move when the chain is broken at W . The squares of distance $|RW|^2 = R_{rw}$ and $|SW|^2 = R_{sw}$ are given by the known EE pose and shown in the upper sketch in Fig. 1. To summarize the procedure, $MF \rightarrow FF$. The four points in MF are transformed to FF then distributed on six surfaces. Two planes p and q intersect hyperboloid h normal to its axis and $P \in p$ and $P \in h$ as well as $Q \in q$ and $Q \in h$ thus providing four equations. The other two points, $R \in r$ and $S \in s$, go, respectively, on one of the concentric spheres centred on W .

1.2 Given

The homogeneous general point transformation matrix $[a]$, whose elements will be expanded upon later, the

homogeneous point vectors in MF, P' , Q' , R' , S' , and the five surface, p , q , h , r , s , scalar equations are written as follows.

$$[a] : \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad P' : \begin{bmatrix} 1 \\ 0 \\ 0 \\ p'_3 \end{bmatrix} \quad (1)$$

$$Q' : \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad R' : \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad S' : \begin{bmatrix} 1 \\ 1 \\ s'_2 \\ s'_3 \end{bmatrix}$$

$$p : p_3 = 0, \quad q : Q_0 + q_3 = 0$$

$$h : -H_1 H_2 H_3 + H_2 H_3 h_1^2 + H_3 H_1 h_2^2 - H_1 H_2 h_3^2 = 0$$

$$r : (r_1 - w_1)^2 + (r_2 - w_2)^2 + (r_3 - w_3)^2 - R_{rw} = 0$$

$$s : (s_1 - w_1)^2 + (s_2 - w_2)^2 + (s_3 - w_3)^2 - R_{sw} = 0 \quad (2)$$

Dummy point coordinate variables p_i, q_i, h_i, r_i, s_i will be replaced by those produced by the multiplications

$$[a]P' = P, \quad [a]Q' = Q, \quad [a]R' = R,$$

$$[a]S' = S$$

written below.

$$P : \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ a_{10} + a_{13}p'_3 \\ a_{20} + a_{23}p'_3 \\ a_{30} + a_{33}p'_3 \end{bmatrix}$$

$$Q : \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 \\ a_{10} \\ a_{20} \\ a_{30} \end{bmatrix} \quad (3)$$

$$R : \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1 \\ a_{10} + a_{11} \\ a_{20} + a_{21} \\ a_{30} + a_{31} \end{bmatrix}$$

$$S : \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1 \\ a_{10} + a_{11} + a_{12}s'_2 + a_{13}s'_3 \\ a_{20} + a_{21} + a_{22}s'_2 + a_{23}s'_3 \\ a_{30} + a_{31} + a_{32}s'_2 + a_{33}s'_3 \end{bmatrix}$$

1.3 Constraint equations

After making the substitutions referred to above, the six constraint equations (4) can be written in terms of the twelve matrix elements a_{ij} .

$$e_1 : [P' \rightarrow P \in p] a_{30} + a_{33}p'_3 = 0$$

$$e_2 : [P' \rightarrow P \in h] - H_1 H_2 H_3 + H_2 H_3 (a_{10} + a_{13}p'_3)^2$$

$$+ H_3 H_1 (a_{20} + a_{23}p'_3)^2$$

$$- H_1 H_2 (a_{30} + a_{33}p'_3)^2 = 0$$

$$e_3 : [Q' \rightarrow Q \in q] Q_0 + a_{30} = 0$$

$$e_4 : [Q' \rightarrow Q \in h] - H_1 H_2 H_3 + H_2 H_3 a_{10}^2$$

$$+ H_3 H_1 a_{20}^2 - H_1 H_2 a_{30}^2 = 0 \quad (4)$$

$$e_5 : [R' \rightarrow R \in r] (a_{10} + a_{11} - w_1)^2$$

$$+ (a_{20} + a_{21} - w_2)^2 + (a_{30} + a_{31} - w_3)^2$$

$$- R_{rw} = 0$$

$$e_6 : [S' \rightarrow S \in s] (a_{10} + a_{11} + a_{12}s'_2 + a_{13}s'_3 - w_1)^2$$

$$+ (a_{20} + a_{21} + a_{22}s'_2 + a_{23}s'_3 - w_2)^2$$

$$- R_{sw} = 0$$

1.4 Euler parameters and a point position vector

The twelve elements a_{ij} are defined as follows.

$$a_{10} = c_0 d_1 - c_1 d_0 + c_2 d_3 - c_3 d_2$$

$$a_{11} = c_0^2 + c_1^2 - c_2^2 - c_3^2$$

$$a_{12} = 2(c_1 c_2 - c_0 c_3), \quad a_{13} = 2(c_1 c_3 + c_0 c_2)$$

$$a_{20} = c_0 d_2 - c_2 d_0 + c_3 d_1 - c_1 d_3$$

$$a_{21} = 2(c_2 c_1 + c_0 c_3) \quad (5)$$

$$a_{22} = c_0^2 - c_1^2 + c_2^2 - c_3^2, \quad a_{23} = 2(c_2 c_3 - c_0 c_1)$$

$$a_{30} = c_0 d_3 - c_3 d_0 + c_1 d_1 - c_2 d_1$$

$$a_{31} = 2(c_3 c_1 - c_0 c_2)$$

$$a_{32} = 2(c_3 c_2 + c_0 c_1), \quad a_{33} = c_0^2 - c_1^2 - c_2^2 + c_3^2$$

The c_i are Euler parameters that determine the rotation that takes place in the transformation MF→FF while the d_i are sums of products of the c_i and the displacement vector $\mathbf{a}$ from the origin, $O' \equiv Q'$, in MF to the

position of Q in FF.

$$Q' \rightarrow Q \equiv \mathbf{a} : \begin{bmatrix} 0 \\ a_1 \\ a_2 \\ a_2 \end{bmatrix}$$

The following product completes the definition of d_i .

$$\begin{aligned} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \otimes \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} -(a_1 c_1 + a_2 c_2 + a_3 c_3) \\ a_1 c_0 + a_2 c_3 - a_3 c_2 \\ a_2 c_0 + a_3 c_1 - a_1 c_3 \\ a_3 c_0 + a_1 c_2 - a_2 c_1 \end{bmatrix} \end{aligned} \quad (6)$$

Notice that the system has been reduced to six unknown parameters in spite of the apparent eight represented by c_i, d_i . This fact may be argued in two ways.

- The d_i are defined in terms of only three a_i and the four c_i . However the latter are homogeneous coordinates hence any one -one that does not vanish- may be set to unity.
- The first argument is formally stated by the so-called “Study condition”, i.e., the inner product

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} \cdot \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = 0$$

This can be easily verified by carrying out this operation, making use of the definition provided in (6). Furthermore the more usual way to dehomogenize, rather than arbitrarily setting some $c_i = 1$, is to normalize as specified by the following non-zero condition.

$$c_0^2 + c_1^2 + c_2^2 + c_3^2 = 1$$

This is already implicit in matrix $[a]$. The left-uppermost element, a_{00} if you will, has been set to unity. This is also reflected by the properties of the normed quaternion or Euler parameter vector below.

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \cos \frac{\phi}{2} \\ \cos \alpha \sin \frac{\phi}{2} \\ \cos \beta \sin \frac{\phi}{2} \\ \cos \gamma \sin \frac{\phi}{2} \end{bmatrix}$$



Fig. 2 A very simple 6R robot

Cosines of α, β, γ are direction numbers that define the axis of rotation implied by the quaternion while ϕ is the rotation angle, in a right hand screw sense with respect to the axis direction numbers, about that axis. The *dual quaternion* vectors $\mathbf{c}$ and $\mathbf{d}$, defined in (6) and (5), and related algebraic operations are dealt with by Blaschke [2].

2 A simple wrist partitioned robot

Shown in Fig. 2 is a small, popular 6R industrial robot, [6]. It fits nicely into the six surface paradigm outlined for the general wrist partitioned case. Moreover one might easily solve the inverse problem, using kinematic mapping. However by choosing *naturally simple pose and an ideal frame* for FF and EE it becomes obvious that the solution reduces to that of a 2R serial chain.

To see this, note that in Fig. 3 identical labeling as that in Fig. 1 has been used. The moving frame MF is still depicted as a pictorial. The origin O' has been moved to P' and the chain $P'Q'R'S'$ is a bit contrived, i.e., unnecessary offsets $P'Q'$ and $R'S'$ were introduced so as to be able to formulate the six equations required to solve a spatial problem. These six point-on-surface conditions are stated succinctly in the following (7).

$$\begin{aligned} P \in z, \quad P \in p, \quad Q \in z, \quad Q \in q, \\ R \in r, \quad S \in s \end{aligned} \quad (7)$$

- z is the horizontal plane on the origin, $x_3 = 0$.
- p and q are concentric, vertical cylinders -the hyperboloid h has been replaced by two surfaces- on the x_3 -axis.

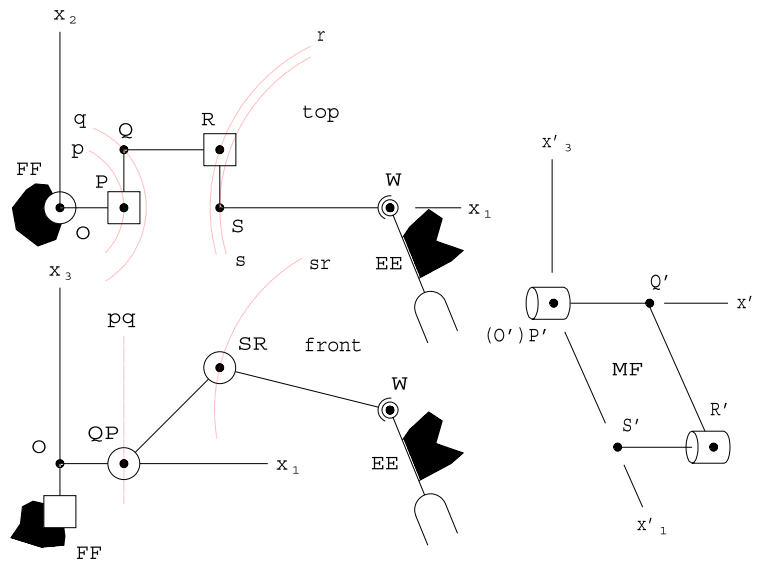


Fig. 3 Points and surfaces

- r and s are, like before, concentric spheres centred on W . Now it becomes evident what those offsets were for, here and in the case of surfaces p and q , above.

Reduction to the solution of a 2R serial chain occurs because the principal plane $x_2 = 0$ was chosen on W , the wrist centre, as well as implicitly on the x_3 -axis. The position of W determines the angle of the first joint rotation about this axis. The front (lower) view in Fig. 3 is the projection plane $x_2 = 0$ so the other two joint angles are available by inspection. In this view the single line through P and Q is the coincident view of two cylinder generators while the concentric, circular cross sections of these cylinders are visible in the top view, a projection on principal plane $x_3 = 0$. The concentric circles, labeled r and s in both top and front views, are not sphere limbs but sections of the spheres r and s on four planes

$$x_3 = r_3, \quad x_3 = s_3, \quad x_2 = 0, \quad x_2 s_2,$$

$$R(r_1, 0, r_3), \quad S(s_1, s_2, s_3)$$

respectively, where due to the dummy offsets and choice of MF $r_3 = s_3$.

3 Spatial—vs—planar point mapping

Having made a very superficial introduction into the essentials of spatial mapping, simpler versions—subgroups, actually—will be used to demonstrate application procedures in more detail. The first will entail planar displacements and a simple parallel manipulator. Afterward an example of pure spatial translation will be included. Finally some parallel Schönflies architectures will be mentioned. Before proceeding to analyze the direct kinematics (DKP) of a new and interesting planar three legged, three degree of freedom (dof) robot and to show the benefit of an image space approach, the reduction of the general rigid body transformation matrix to the abbreviated planar form will be examined. Then the equivalence between this and the mapping convention found in Bottema and Roth's text [3], the form that will be used, is described. The spatial transformation given by [a], (5) is expanded as (8), below.

$$\begin{bmatrix} c_0^2 + c_1^2 + c_2^2 + c_3^2 & 0 \\ c_0 d_1 - c_1 d_0 + c_2 d_3 - c_3 d_0 & c_0^2 + c_1^2 - c_2^2 - c_3^2 \\ c_0 d_2 - c_2 d_0 + c_3 d_1 - c_3 d_1 & 2(c_2 c_1 + c_0 c_3) \\ c_0 d_3 - c_3 d_0 + c_1 d_2 - c_2 d_1 & 2(c_3 c_1 - c_0 c_2) \\ 0 & 0 \\ 2(c_1 c_2 - c_0 c_3) & 2(c_1 c_3 + c_0 c_2) \\ c_0^2 - c_1^2 + c_2^2 - c_3^2 & 2(c_2 c_3 - c_0 c_1) \\ 2(c_3 c_2 + c_0 c_1) & c_0^2 - c_1^2 - c_2^2 + c_3^2 \end{bmatrix} \quad (8)$$

Examine (6) and note the significance of vector **a** above it. Since there are only two translational dof and one rotational one, obviously $a_3 = c_1 = c_2 = d_0 = d_3 = 0$ so (8) becomes (9).

$$\begin{bmatrix} c_0^2 + c_3^2 & 0 & 0 & 0 \\ c_0 d_1 - c_3 d_2 & c_0^2 - c_3^2 & -2c_0 c_3 & 0 \\ c_0 d_2 + c_3 d_1 & 2c_0 c_3 & c_0^2 - c_3^2 & 0 \\ 0 & 0 & 0 & c_0^2 + c_3^2 \end{bmatrix} \quad (9)$$

Note that only the upper left 3×3 block is required to transform homogeneous planar point vectors and $c_0^2 + c_3^2 = 1$. To complete the picture

$$\begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ a_1 c_0 + a_2 c_3 \\ a_2 c_0 - a_1 c_3 \\ 0 \end{bmatrix}$$

This transformation is illustrated in Fig. 4. Note the vector **a**, the angle of rotation ϕ about invariant pole $P(p_1, p_2)$ and the frame and point transformations

$$O'x'y' \rightarrow Oxy, \quad O' \rightarrow O, \quad A' \rightarrow A$$

In what follows (9) will assume the conventional form (10)

$$\begin{bmatrix} 1 & 0 & 0 \\ 2(X_0 X_2 + X_1 X_3) & X_0^2 - X_3^2 & -2X_0 X_3 \\ -2(X_0 X_1 - X_2 X_3) & 2X_0 X_3 & X_0^2 - X_3^2 \end{bmatrix} \quad (10)$$

where

$$\begin{aligned} \{X_0 : X_1 : X_2 : X_3\} &= \{2c_0 : -d_2 : d_1 : 2c_3\} \\ &= \left\{ 2 \cos \frac{\phi}{2} : a \sin \frac{\phi}{2} - b \cos \frac{\phi}{2} \right. \\ &\quad \left. : a \cos \frac{\phi}{2} + b \sin \frac{\phi}{2} : 2 \sin \frac{\phi}{2} \right\} \end{aligned} \quad (11)$$

and the dual quaternion parameters get switched about and

$$a = a_1/2, \quad b = a_2/2$$

as shown below. Pole coordinates can be conveniently calculated by dividing first by X_0 then by X_3 .

Confusion may arise in attempting to obtain P with ϕ and position vector **a**(a_1, a_2) so proceeding with the

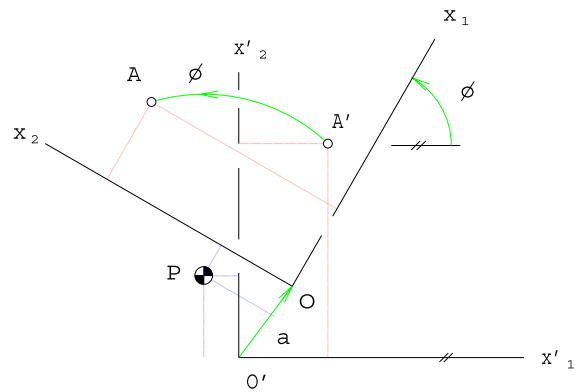


Fig. 4 Planar displacement via rotation about the pole

right hand side of the first line in (11) one sees that it becomes

$$\{2c_0 : a_1 c_3 - a_2 c_0 : a_1 c_0 + a_2 c_3 : 2c_3\}$$

then

$$\left\{ 1 : \frac{a_1}{2} \tan \frac{\phi}{2} - \frac{a_2}{2} : \frac{a_1}{2} + \frac{a_2}{2} \tan \frac{\phi}{2} : \tan \frac{\phi}{2} \right\}$$

and finally

$$P \left(\frac{a_1}{2} - \frac{a_2}{2} \cot \frac{\phi}{2}, \frac{a_1}{2} \cot \frac{\phi}{2} + \frac{a_2}{2} \right)$$

Digression into this little bit of detail was undertaken because the author has not seen this done before and believes that the lack of such seemingly trivial explanations probably deters many from trying to apply methods of kinematic mapping. Furthermore this will be useful in what follows.

4 A fully rotational planar parallel manipulator

Seoul National University is a hotbed of imaginative parallel robot design. Recently In et al. [16] revealed the one shown in Fig. 5.

Although actuated “trucks”, that move on a circular track, are quaintly called “circular prismatic” joints one may conceptually replace them with three centrally coaxial, actuated R-joints supporting three passive P-joints that in turn support the passive, equilaterally distributed R-joints on EE, the central platform.

4.1 Direct kinematics

The spatial DK model, introduced *a-priori*, becomes, in the plane a problem to map three ideally configured points F', G', H' on $MF \equiv EE$ on lines f, g, h in FF , the surfaces in space are replaced by lines, or, if you will, planes normal to $x_3 = 0, x'_3 = 0$. This two solution DK mapping is shown in Fig. 6. The polar rotations depicted are executed about the red and green spots, labeled P in Fig. 4. In the following numerical example the problem will be to locate these and to determine the angles ϕ .

Since all three lines, with their normal direction numbers annotated, are on the origin in FF and the equilateral triangle platform is placed initially with corner F' on the coincident frame MF origin and point H' on the $x'_2 = 0$ axis the equations of f, g, h and the

coordinates of F', G', H' can be written by inspection. No loss in generality is incurred by setting unit platform edge lengths, e.g., $F'H' = 1$.

$$\begin{aligned} f : f_2 &= 0, & g : -3g_1 + 4g_2 &= 0, \\ h : h_1 + h_2 &= 0 \\ F'\{1 : 0 : 0 : 0\}, & & G'\{1 : 1/2 : \sqrt{3}/2 : 0\} \\ H'\{1 : 1 : 0 : 0\} \end{aligned} \quad (12)$$

After multiplying vectors F', G', H' by (9), where $c_0^2 + c_3^2 = 1$, these become

$$\begin{aligned} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} &= \begin{bmatrix} 1 \\ c_0d_1 - c_3d_2 \\ c_0d_2 + c_3d_1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix} &= \begin{bmatrix} 1 \\ (c_0d_1 - c_3d_2) + (c_0^2 - c_3^2)/2 - \sqrt{3}c_0c_3 \\ (c_0d_2 + c_3d_1) + c_0c_3 + \sqrt{3}(c_0^2 - c_3^2)/2 \\ 0 \end{bmatrix} \\ \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix} &= \begin{bmatrix} 1 \\ (c_0d_1 - c_3d_2) + (c_0^2 + c_3^2) \\ (c_0d_1 + c_3d_1) + 2c_0c_3 \\ 0 \end{bmatrix} \end{aligned} \quad (13)$$

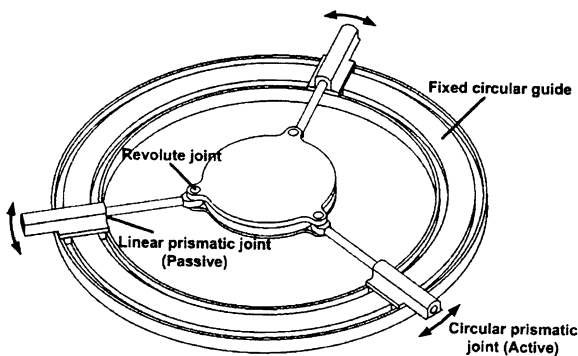


Fig. 5 Three legged fully rotational design

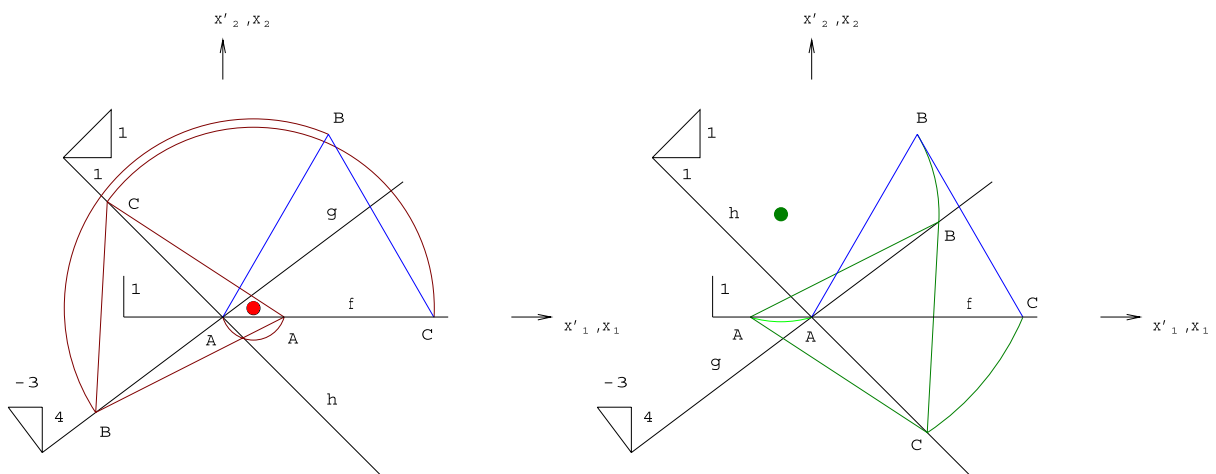


Fig. 6 Three points on three lines

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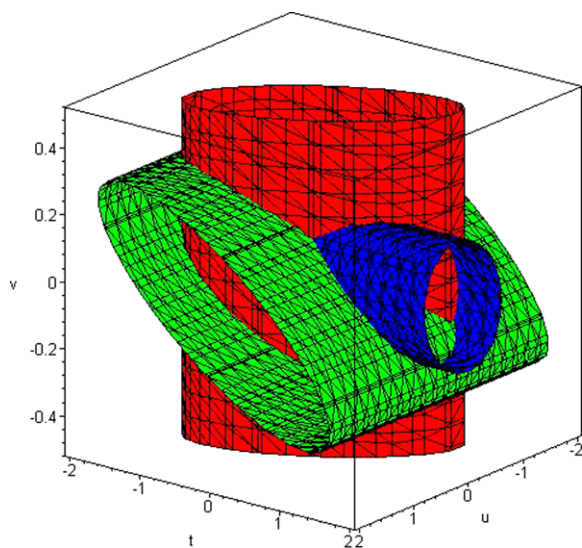


Fig. 8 Three cylinders in t, u, v space

turned over from “heads” to “tails” to assume two of the four assemblies. The coloured images correspond to the two obtained previously. Polar rotations implied by kinematic mapping avoid the black ones. Furthermore the solutions were implemented with a little geometric “cheating”. With four values of t , circles of unit radius were constructed on centres at $(\pm 1.407156, 0)$ and $(\pm 0.290214, 0)$ and appropriate intersections with the lines were selected. Algorithmically combing out the spurious roots would be time-consuming and unreliable; not a good foundation on which to base a motion controller.

4.2.1 Geometric algebra

To appreciate problems created by distance constraints, when used through misguided impression that their intuitive simplicity persists in the implied geometry, consider Fig. 8. It shows the three cylinders in parametric t, u, v space.

Four real solutions are represented by common intersections among the three surfaces. The red one is the first of (20), blue, the second, while the third has the conic right section in t, v . Two solutions are clearly visible and the others, behind the red and green cylinders, are easy to visualize. With a somewhat different disposition of the given lines, i.e., different actuator coordinates, the blue cylinder might turn out flatter and wider and eight real solutions would emerge. This cannot occur with the hyperbolic paraboloids that represent the three image space constraint equations.

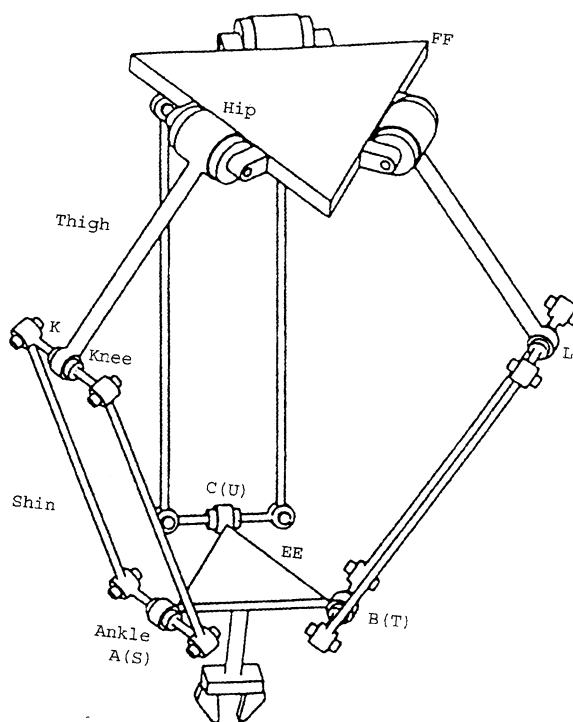


Fig. 9 The “Delta” robot

5 Conclusions

Rather than recap points and issues that are adequately summed up previously in the abstract and in closing statements of various sections and subsections let us instead consider some reduced mobility, familiar and not-so-familiar, parallel manipulators and briefly mention, without specific details, how image space formulation applies here as well.

5.1 Translational manipulators

Many spatial translational parallel manipulators have been proposed, analyzed, built and tested. Clavel’s “Delta” robot, [5], Fig. 9 is typical. It and others in the same category can be DKP modeled with a three-points-on-three-spheres paradigm. Mapping makes it easier to analyze completely asymmetric designs. The IKP is handled in an almost identical fashion [18].

5.1.1 Procedure

To establish the positions of the three anchor points (“ankles”), initially placed ideally on end effector

(EE) at points $A(0, 0, 0)$, $B(b_1, 0, 0)$, $C(c_1, c_2, 0)$, these are moved onto the three spheres, σ, τ, ν , in fixed frame (FF). They are swept when the three actuators at the “hips” put the sphere centres on points $K(k_1, k_2, k_3)$, $L(l_1, l_2, l_3)$, $M(m_1, m_2, m_3)$ on the “knees” and the anchor points on the other end of the “shins” trace the spheres of known, respective radii, $\sqrt{s}$, $\sqrt{t}$, $\sqrt{u}$. The necessary translation is accomplished by a simple point *translation* matrix

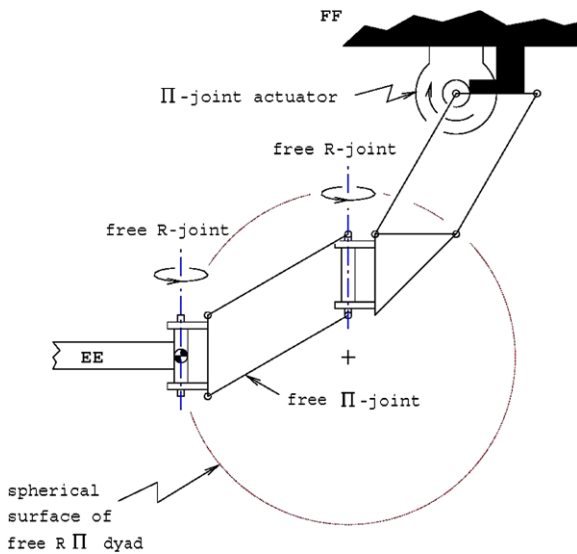


Fig. 10 PIRPIR Schönflies leg architectures

[**T**]. The transformation puts $A \rightarrow S$, $B \rightarrow T$, $C \rightarrow U$, where $S(s_1, s_2, s_3)$, $T(t_1, t_2, t_3)$, $U(u_1, u_2, u_3)$ are points on the respective spheres.

- Three sphere equations are

$$\sigma : (s_1 - k_1)^2 + (s_2 - k_2)^2 + (s_3 - k_3)^2 - s = 0 \quad (22)$$

$$\tau : (t_1 - l_1)^2 + (t_2 - l_2)^2 + (t_3 - l_3)^2 - t = 0 \quad (23)$$

$$\nu : (u_1 - m_1)^2 + (u_2 - m_2)^2 + (u_3 - m_3)^2 - u = 0 \quad (24)$$

- Subtracting (23) and (24) from the (22) produces the following two linear equations, (25) and (26), that may then be solved simultaneously with, say (22).

$$\begin{aligned} \sigma - \tau : & 2(l_1 t_1 - k_1 s_1) + 2(l_2 t_2 - k_2 s_2) \\ & + 2(l_3 t_3 - k_3 s_3) + k_1^2 - l_1^2 + k_2^2 - l_2^2 \\ & + k_3^2 - l_3^2 + t - s = 0 \end{aligned} \quad (25)$$

$$\begin{aligned} \sigma - \nu : & 2(m_1 u_1 - k_1 s_1) + 2(m_2 u_2 - k_2 s_2) \\ & + 2(m_3 u_3 - k_3 s_3) + k_1^2 - m_1^2 + k_2^2 \\ & - m_2^2 + k_3^2 - m_3^2 + u - s = 0 \end{aligned} \quad (26)$$

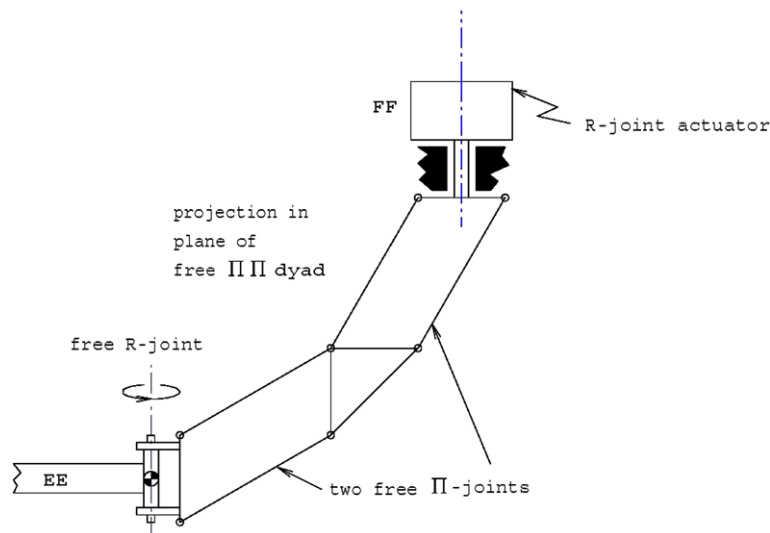


Fig. 11 RPIIR Schönflies leg architectures

- Move $A \rightarrow S$, $B \rightarrow T$, $C \rightarrow U$ via the translation vector $\mathbf{x}$.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad [\mathbf{T}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x_1 & 1 & 0 & 0 \\ x_2 & 0 & 1 & 0 \\ x_3 & 0 & 0 & 1 \end{bmatrix} \quad (27)$$

$$[\mathbf{T}] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{s} \end{bmatrix} \quad (28)$$

$$[\mathbf{T}] \begin{bmatrix} 1 \\ b_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ x_1 + b_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{t} \end{bmatrix} \quad (29)$$

$$[\mathbf{T}] \begin{bmatrix} 1 \\ c_1 \\ c_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ x_1 + c_1 \\ x_2 + c_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{u} \end{bmatrix} \quad (30)$$

- Substitute the expressions for vectors $\mathbf{s}$, $\mathbf{t}$, $\mathbf{u}$, i.e., the last three elements of the vectors in (28), (29) and (30), in (22), (25) and (26) and solve simultaneously for x_1, x_2, x_3 then compute the actual positions of S, T, U with (28), (29) and (30). Equations (22), (25) and (26) become, after substitution, (31), (32) and (33).

$$\sigma : (x_1 - k_1)^2 + (x_2 - k_2)^2 + (x_3 - k_3)^2 - s = 0 \quad (31)$$

$$\begin{aligned} \sigma - \tau : & 2(l_1 - k_1 - b_1)x_1 + 2(l_2 - k_2)x_2 \\ & + 2(l_3 - k_3)x_3 + k_1^2 + k_2^2 + k_3^2 - l_2^2 \\ & - l_3^2 - (b_1 - l_1)^2 - s + t = 0 \end{aligned} \quad (32)$$

$$\begin{aligned} \sigma - v : & 2(m_1 - k_1 - c_1)x_1 + 2(m_2 - k_2 - c_2)x_2 \\ & + 2(m_3 - k_3)x_3 + k_1^2 + k_2^2 + k_3^2 \\ & - (c_1 - m_1)^2 - (c_2 - m_2)^3 \\ & - m_3^2 - s + u = 0 \end{aligned} \quad (33)$$

- The advantage of this method is that it does not assume a symmetric manipulator. Upper and lower platform triangles and hip and shin lengths can be chosen arbitrarily. However the design must be configured so that all “hip”, “knee” and “ankle” R-joint axes are horizontal, i.e., normal to vertical direction x_3 .



Fig. 12 A very special two leg architecture

5.2 Schönflies robots

Aside from the serial “SCARA” robot, a few parallel architectures have been produced or conceived. Examples of two with four singly base-actuated legs and one with two doubly base-actuated legs will be mentioned. Schönflies mapping is produced by transformation of points with (8) when $c_1 = c_2 = 0$. Parallel architectures are usually implemented with parallelogram or Π -joints like those seen in Figs. 9, 11 and 12. Extensive, unified treatment of parallel Schönflies robots is available in [19].

5.2.1 A four legged robot

Two variants, not unlike robots introduced by Nabat et al. [12], are shown in Figs. 10 and 11. It is easy to see that the four required constraints are furnished by placing four EE points on four surfaces in FF; four tori if the legs have an actuated basal Π -joint and four planes in the case of a robot with four legs of the kind with a basally actuated R-joint.

5.2.2 A two legged robot

The photo in Fig. 12 shows a Schönflies robot with two legs. It was constructed and is undergoing tests at McGill in the lab of J. Angeles and is described in an article [1], and in a thesis [7]. Each leg is doubly base-actuated via a planetary differential gearbox that simultaneously but independently moves the first R- and second Π -joints in the chain. The mapping paradigm is the placement of two EE points on two circles in FF, each represented by an plane/sphere intersection.

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