How Complex are Real Algebraic Objects?

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Introduction A standard technique to process non-linear curves and surfaces in geometric systems is to approximate them in terms of a piecewise linear object (a simplicial complex). A main goal is to preserve the topological properties of the input objects. Furthermore, geometric properties, such as the position of "critical" points of the object, are often of interest. For algebraic curves and surfaces as inputs, the former problem is usually called *topology computation*, the latter *topological-geometric analysis* of the object. Efficient techniques for curves (e.g, see [4, 5], and references therein) and surfaces [2, 1] have been presented.

In few words, we consider the following question: *How many line segments/triangles are needed to approximate a real algebraic curve/surface of degree n?* For curves, we are able to give sharp bounds: for a topologically correct representation, $\Omega(n^2)$ line segments are needed in the worst case, and we give an algorithm producing $O(n^2)$ line segments for all cases. Although the idea is simple, it seemingly does not appear in the literature yet. For geometric-topological representations, we construct a class of curves such that $\Omega(n^3)$ line segments are necessary. This proves that the *cylindrical algebraic decomposition* [3] ("Find the critical x-coordinates of the curve; compute the fiber at these coordinates and at separating points in between; connect the fiber points by straight-line segments." – compare the pictures on the next page) is asymptotically optimal. This is surprising, because the vertical decomposition strategy seems to introduce much more line segments than actually necessary.

For surfaces, we still have gaps between lower and upper bounds. For the topological approximation, we get a lower bound of $\Omega(n^3)$, and an upper bound of $O(n^5)$ triangles. For the geometric-topological approximation, the bounds are $\Omega(n^4)$ and $O(n^7)$.

A detailed version of this extended abstract will appear in the phd thesis of the first author [6].

Basics A *homeomorphism* between two sets $X,Y \subset \mathbb{R}^d$ is a bijective, continuous map $h: X \to Y$ whose inverse is continuous as well. X and Y are *isotopic*, if they are "connected by homeomorphism", that is, there exists a continuous map $\psi: [0,1] \times X \to \mathbb{R}^d$ such that $\psi(0,x) = x$, $\psi(1,X) = Y$, and $\psi(t_0,x)$ is a homeomorphism for any $t_0 \in [0,1]$. ψ is called the *isotopy* between X and Y. We assume that the reader is familiar with the definition of a simplicial complex. We assume that the complex is embedded into $\mathbb{R}^d$ by fixing its vertices, and we identify the complex and the induced point set.

An *algebraic hypersurface* $\mathscr{O}$ (over $\mathbb{Q}$) in $\mathbb{R}^d$ is the solution set of an equation $f = 0$ with $f \in \mathbb{Q}[x_1,\ldots,x_d]$. Hypersurfaces in dimensions 2 and 3 are called *algebraic curves* and *algebraic surface*, respectively. The *degree* of $\mathscr{O}$ is defined by the total degree of f. An *isolated point* $p \in \mathbb{R}^d$ is a point on $\mathscr{O}$, such that an open neighborhood of p in $\mathbb{R}^d$ does not contain any further point of $\mathscr{O}$. An *isocomplex* of $\mathscr{O}$ is a simplicial complex S that is isotopic to $\mathscr{O}$. A *stable isocomplex* is an isocomplex that is stable at vertices, that is, there exists an isotopy ψ between $\mathscr{O}$ and S such that for each vertex v of S, $\psi(t_0,v) = v$ for any $t \in [0,1]$. Computing the topology of $\mathscr{O}$ means to compute an isocomplex, computing a geometric-topological analysis means to compute a stable isocomplex.

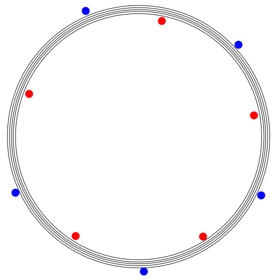
Our main idea for deriving lower bounds is to construct algebraic hypersurfaces with many isolated points. We can even fix the position of each isolated point, up to a ball of arbitrary small radius.

Theorem 1. *For $d,n \in \mathbb{N}$, set $c := \binom{\lfloor n/2 \rfloor + d}{d} - d$. Then, for any $\varepsilon > 0$, and any set of points $p_1,\ldots,p_c \in \mathbb{Q}^d$, there exists a hypersurface $\mathscr{O} \subset \mathbb{R}^d$ of degree n, such that for any p_i, $\mathscr{O}$ contains a (rational) isolated point p'_i with $\|p_i - p'_i\|_2 < \varepsilon$.*

Proof. Choose the points $p'_1,\ldots,p'_c$ in generic position, p'_i in an ε-ball around p_i, such that the hypersurfaces of degree $\lfloor n/2 \rfloor$ through these points define a variety of dimension $d-1$ or larger. Pick d hypersurfaces with equations $f_1,\ldots,f_d$ that form a complete intersection, and define $\mathscr{O}$ by the equation $f_1^2 + \ldots + f_d^2 = 0$. □

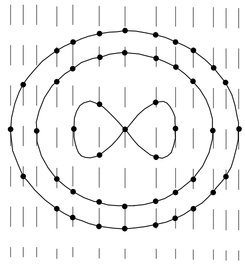
We will consider d as a constant ($d = 2$ or $d = 3$). Then the theorem states that we can choose $\Theta(n^d)$ arbitrary rational points and construct an algebraic curve/surface of degree n with isolated points close to them.

Bounds on stable isocomplexes Best upper bounds we are aware of are given by cylindrical algebraic decomposition: $O(n^3)$ cells for curves, $O(n^7)$ cells for surfaces [2]. We construct lower bounds based on Theorem 1. In $2D$, we consider a set of $\Theta(n)$ distinct circles around the origin, all of radius close to one. We fix $\Theta(n^2)$ many isolated points close to the boundary of the circle, as illustrated in the right figure. The curve is defined by the union of the circles and the isolated points. In the isocomplex, the polyline of the circles have to perform a "slalom" around the vertices in order to ensure topological correctness. Thus, each of the circles must be subdivided in at least $\Omega(n^2)$ many line segments, which proves the lower bound of $\Omega(n^3)$. In $3D$, the idea is similar, considering $\Theta(n)$ many "almost-unit" spheres and $\Theta(n^3)$ isolated points close to the boundary of these spheres. Each sphere is divided in at least $\Omega(n^3)$ triangles. This proves a lower bound $\Omega(n^4)$. Irreducible worst-case curve and surfaces can be obtained by constructing two coprime objects as above and adding their equations.



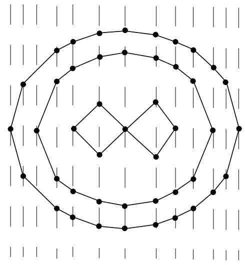
Bounds on general isocomplexes Without stability requirement, the lower bounds of $\Omega(n^d)$ follow from Theorem 1 (or by intersecting n hyperplanes in generic position).

For the upper bound in $2D$, we consider the isocomplex returned by a *cad* (cylindrical algebraic decomposition) algorithm. It returns $O(n^2)$ many fibers of the curves (with respect to some projection direction) and connects the fiber points by straight-line segments. Since any fiber has at most n points, the complexity is $O(n^3)$. We can assume that no segment is vertical and consider the complex as a directed graph from left to right. We re-embed the graph into the plane with the following rules. (1) Each vertex remains the same x-coordinate, and the vertical ordering of the vertices at the same x-coordinate remains unchanged. (2) Each edge from a vertex of in-degree 1 to another vertex of in-degree 1 must be horizontal.



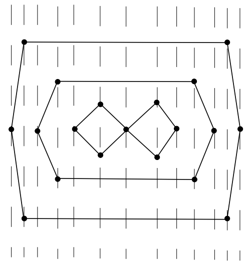
cad of the curve

Properties (1) ensures that the result is isotopic to the original complex. A complex with properties (1) and (2) can be computed by a simple plane sweep algorithm. Vertices adjacent to exactly two horizontal edges are removed afterwards, and the edges are merged. Let $\mathscr{C}_h$ denote this new complex. By construction, any maximal smooth x-monotone segment of the curve is represented by a polyline in $\mathscr{C}_h$ with two bends, running horizontally between the two bends. The number of edges is thus at most three times the number of segments of the curve that leave a critical point. Their number can be bounded by $O(n^2)$, thus the complexity of $\mathscr{C}_h$ is also $O(n^2)$.



induced isocomplex

For the $3D$ case, we consider a *cad* of the surface, and apply the just described algorithm on the projected silhouette of the surface, which is a planar curve of degree $O(n^2)$. This yields an isocomplex with $O(n^4)$ cells in the projection plane. C_h is then extended to a triangulation of $\mathbb{R}^2$ with $O(n^4)$ triangles, using a trapezoidal map. Each triangle is lifted according to the adjacency relations of the surface, producing at most n lifts per triangle in the plane. This yields a triangulation with $O(n^5)$ cells.



simplification

Open problems We believe that the just described $O(n^5)$ triangulation is not optimal. It might be possible to improve it by a method not based on projection. In general, closing the gaps in $3D$ would be interesting as well as a generalization into higher dimensions. Moreover, the bitsize τ of the defining polynomials should be taken into account in the bounds. Regarding algorithmic complexity, it is shown in [6] that the therein presented algorithm to compute a (stable) isocomplex of a planar curve C of degree n requires $\tilde{O}(n^{10}(n+\tau)^2)$ bit operations ($\tilde{O}$ means that we omit logarithmic factors). For a surface $S \subset \mathbb{R}^3$ of degree n we conjecture that $\tilde{O}(n^{22}(n+\tau)^2)$ bit operations are sufficient.

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