Optimization Techniques for Geometry Processing

-Part II-

Justin Solomon
Princeton University

David Bommes
RWTH Aachen University

PRINCETON UNIVERSITY

RWTH AACHEN UNIVERSITY

Visual Computing Institute

Constrained Optimization

Constrained Optimization

- general form

objective function
$f : \mathbb{R}^n \to \mathbb{R}$

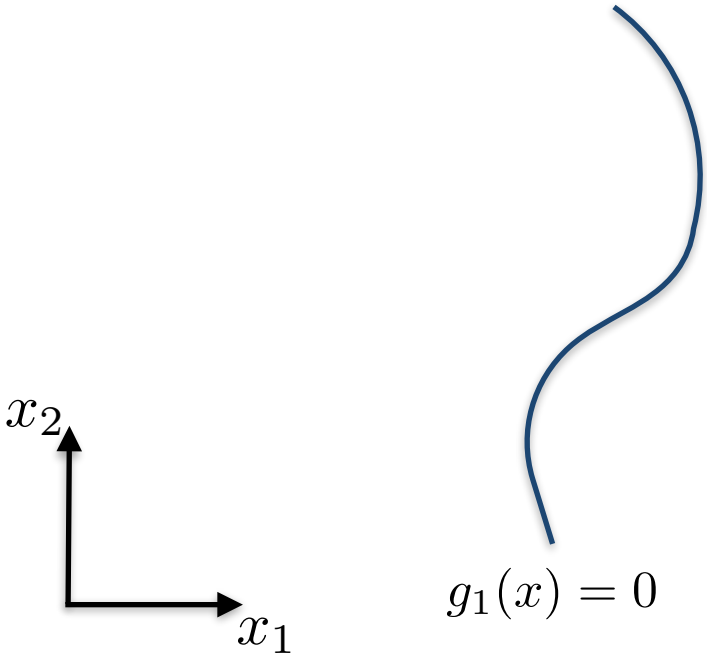
$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0 \qquad i = 1 \dots m \end{aligned}$$

constraint functions
$g_i : \mathbb{R}^n \to \mathbb{R}$

Constrained Optimization

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0 \qquad i = 1 \dots m \end{aligned}$$

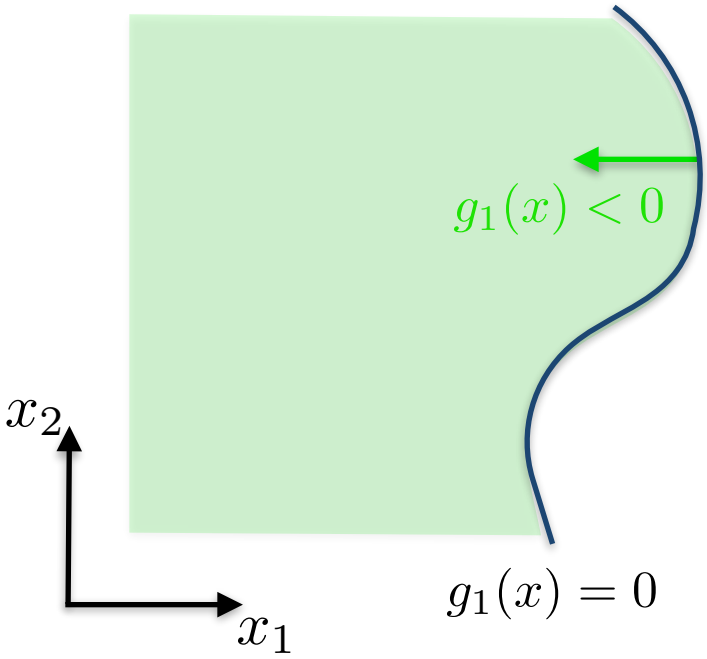
- geometric interpretation



Constrained Optimization

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0 \qquad i = 1 \dots m \end{aligned}$$

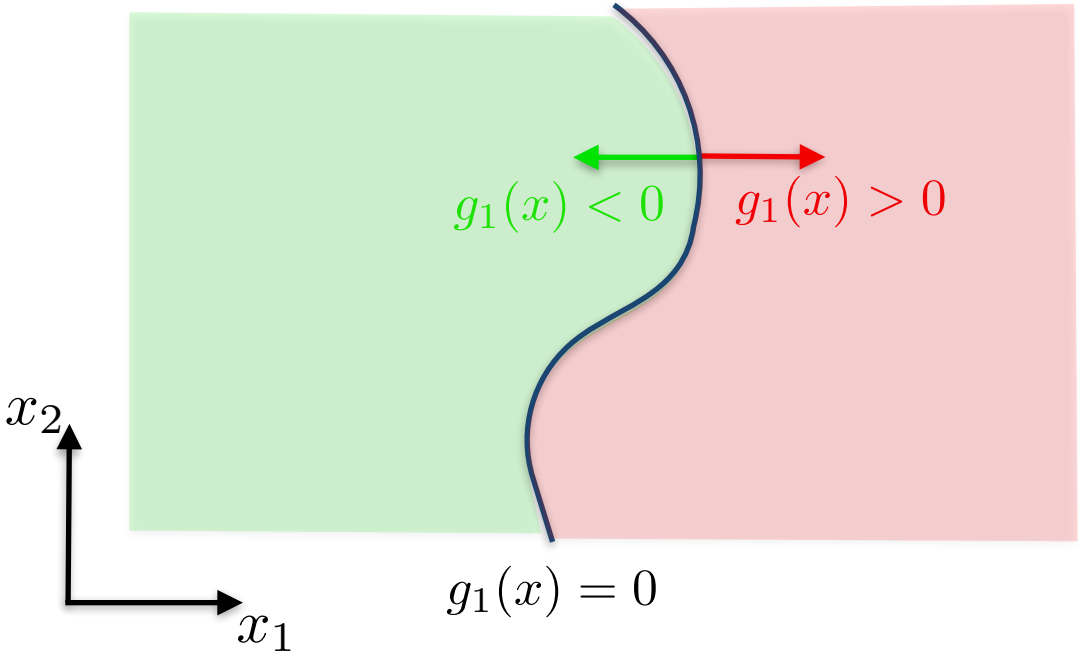
- geometric interpretation



Constrained Optimization

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0 \qquad i = 1 \dots m \end{aligned}$$

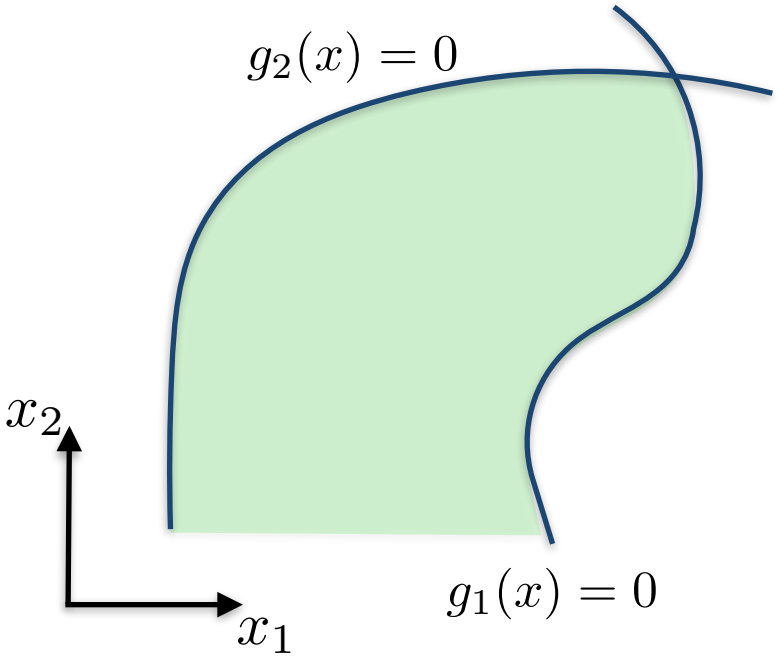
- geometric interpretation



Constrained Optimization

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0 \qquad i = 1 \dots m \end{aligned}$$

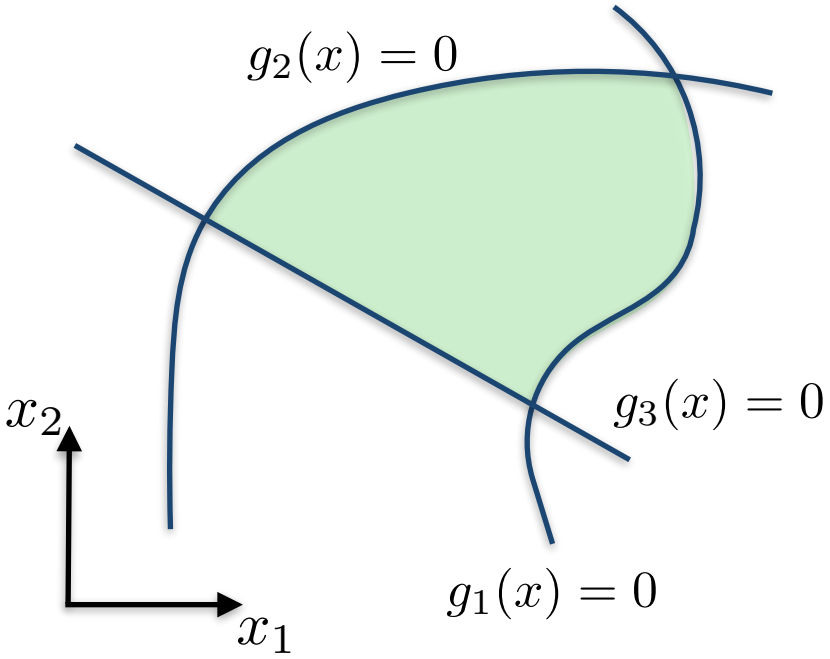
- geometric interpretation



Constrained Optimization

$$\begin{array}{lll} \text{minimize} & f(x) & \\ \text{subject to} & g_i(x) & \leq 0 \qquad i = 1 \dots m \end{array}$$

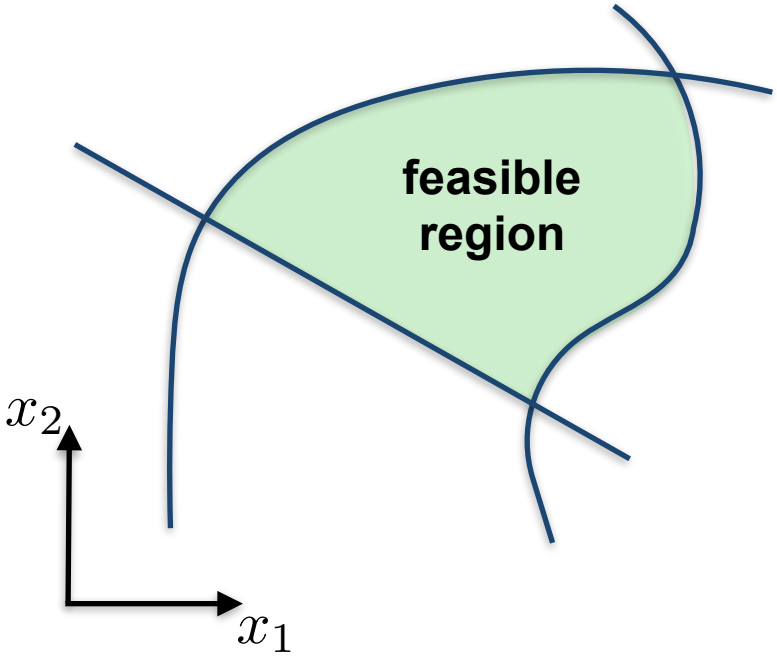
- geometric interpretation



Constrained Optimization

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0 \qquad i = 1 \dots m \end{aligned}$$

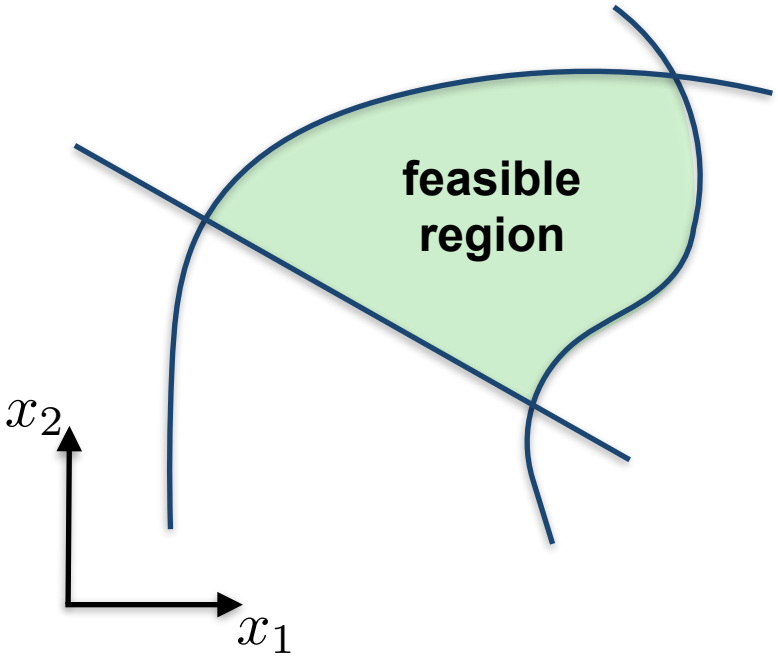
- geometric interpretation

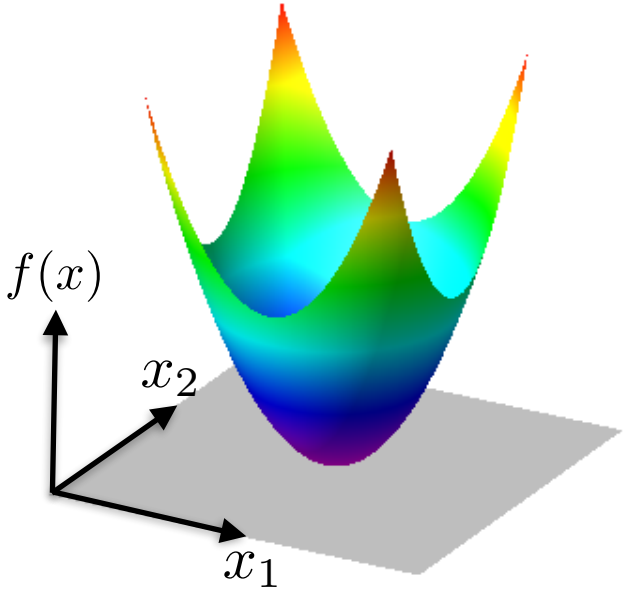


Constrained Optimization

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0 \qquad i = 1 \dots m \end{aligned}$$

- geometric interpretation

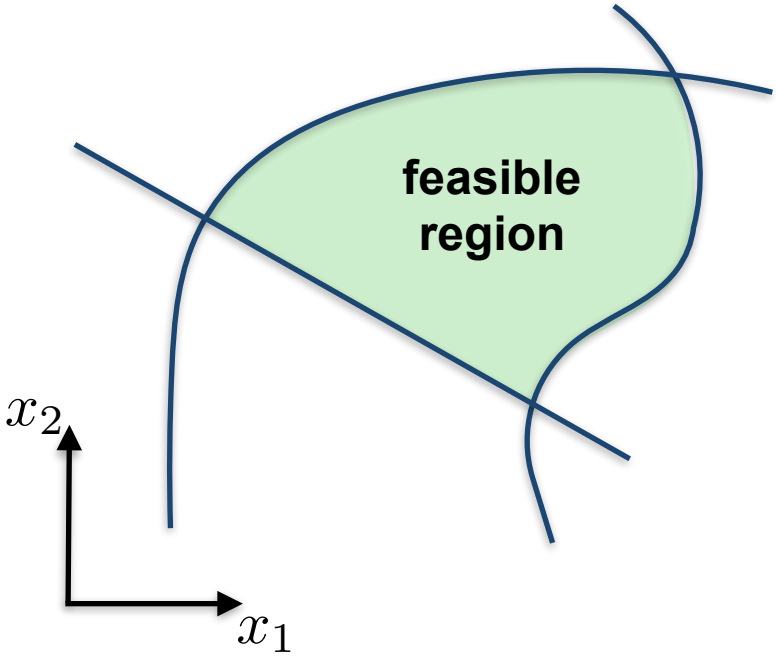


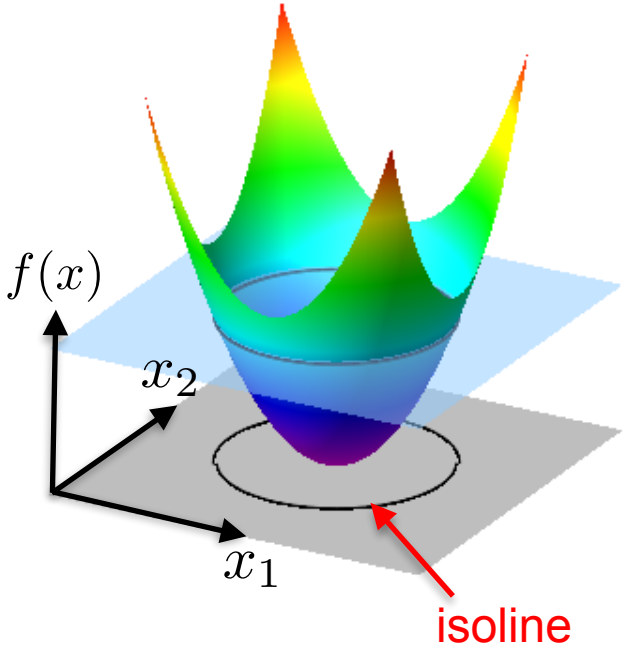


Constrained Optimization

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0 \qquad i = 1 \dots m \end{aligned}$$

- geometric interpretation

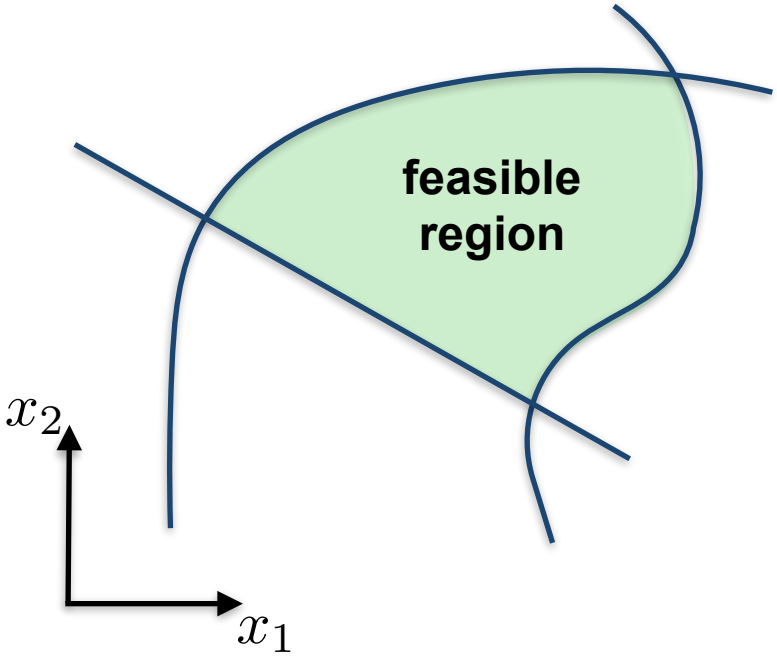


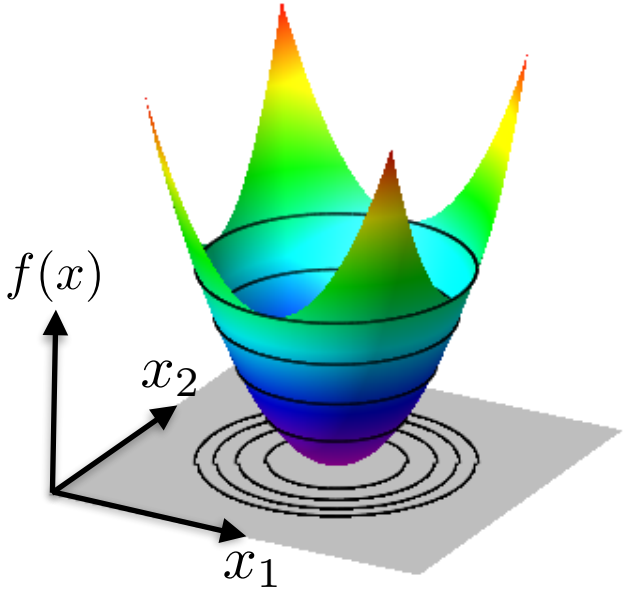


Constrained Optimization

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0 \qquad i = 1 \dots m \end{aligned}$$

- geometric interpretation

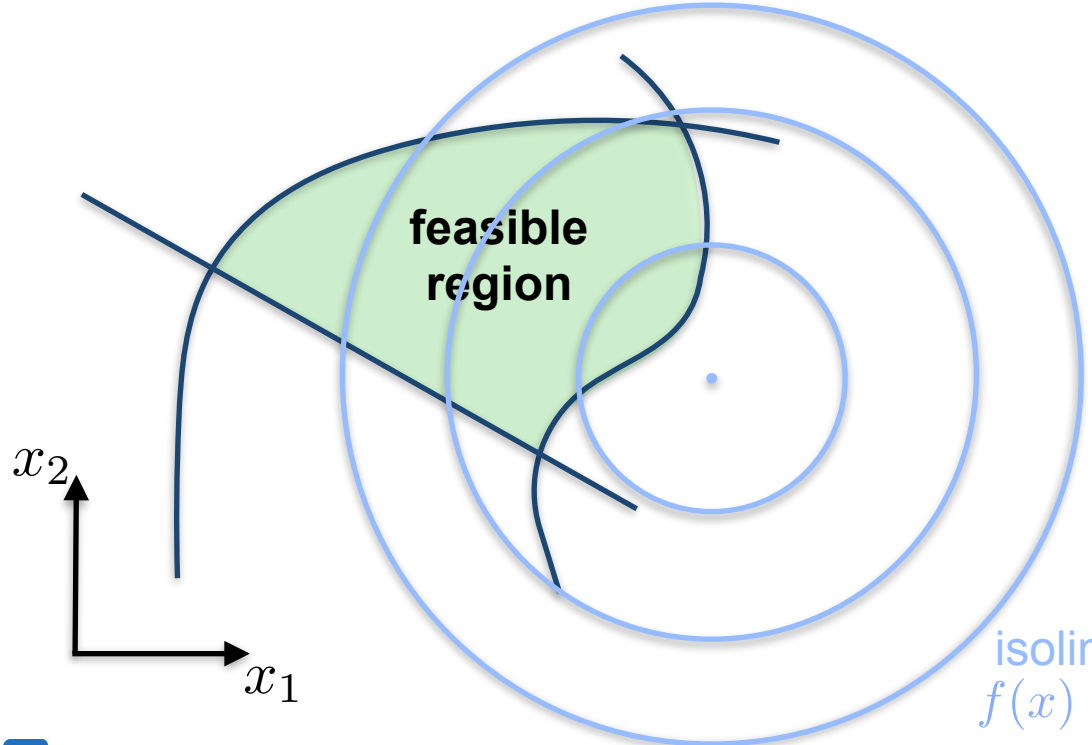


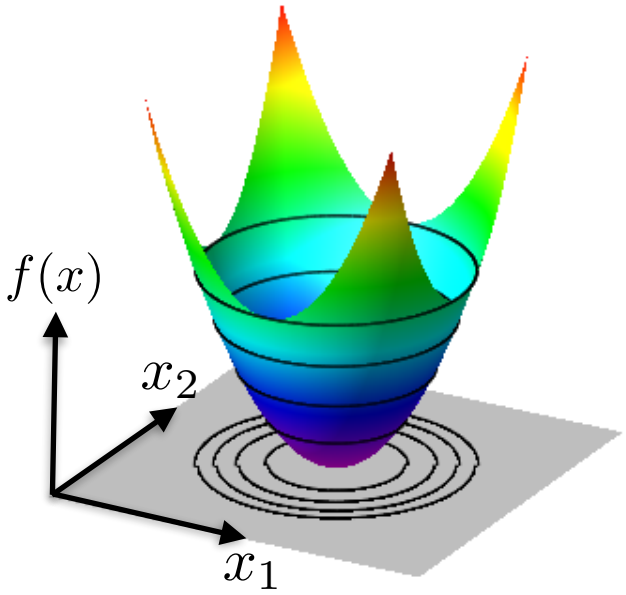


Constrained Optimization

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0 \qquad i = 1 \dots m \end{aligned}$$

- geometric interpretation



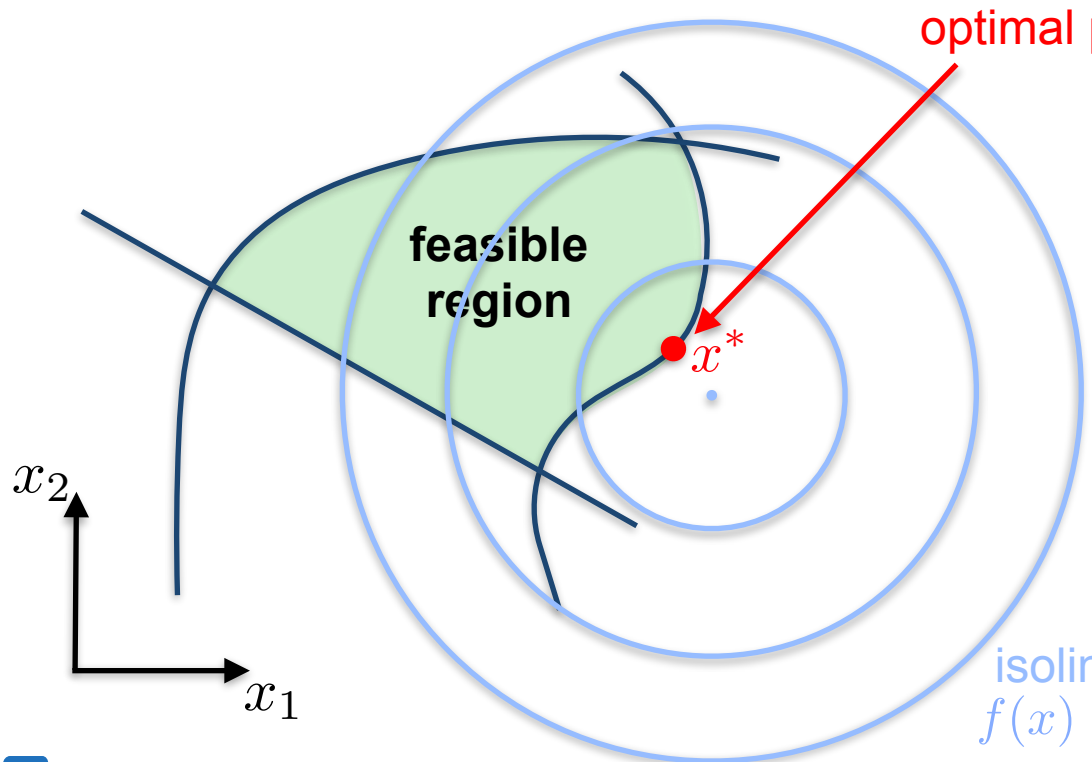


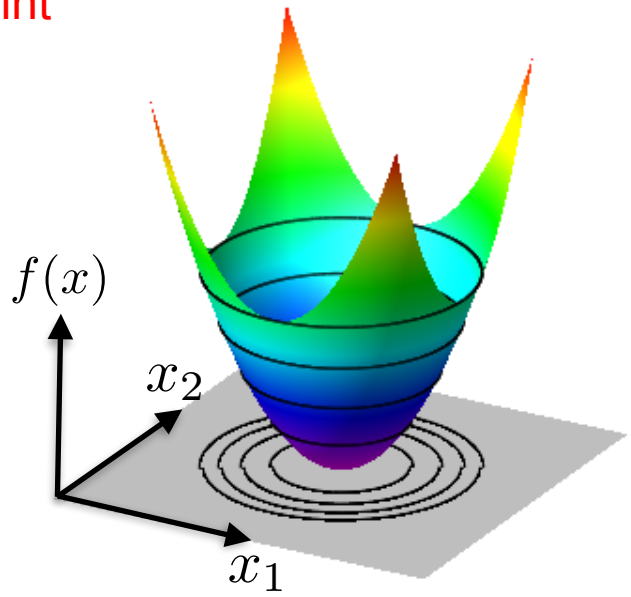
isolines with $f(x) = const$

Constrained Optimization

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0 \qquad i = 1 \dots m \end{aligned}$$

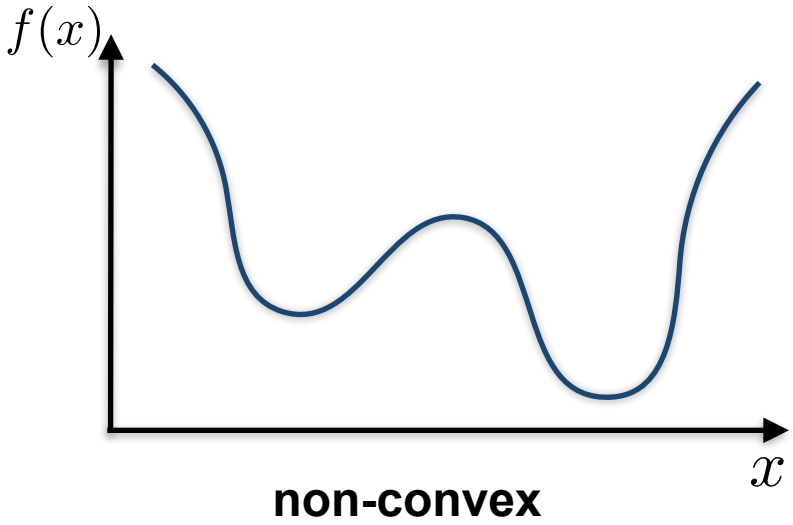
- geometric interpretation





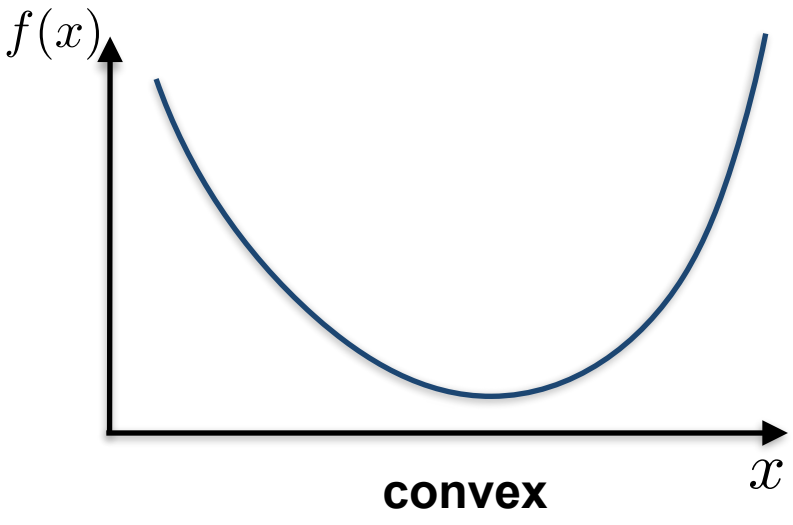
A few words on Convexity

- Searching globally optimal solutions usually requires **convexity**!



non-convex

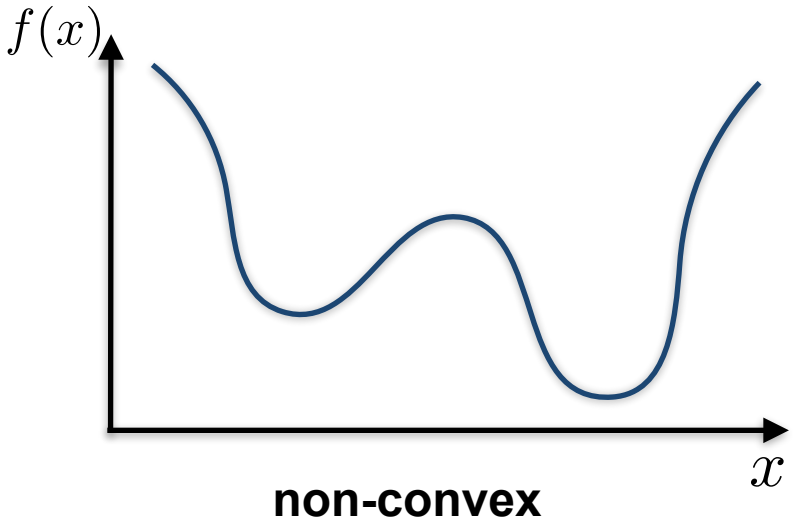
vs.



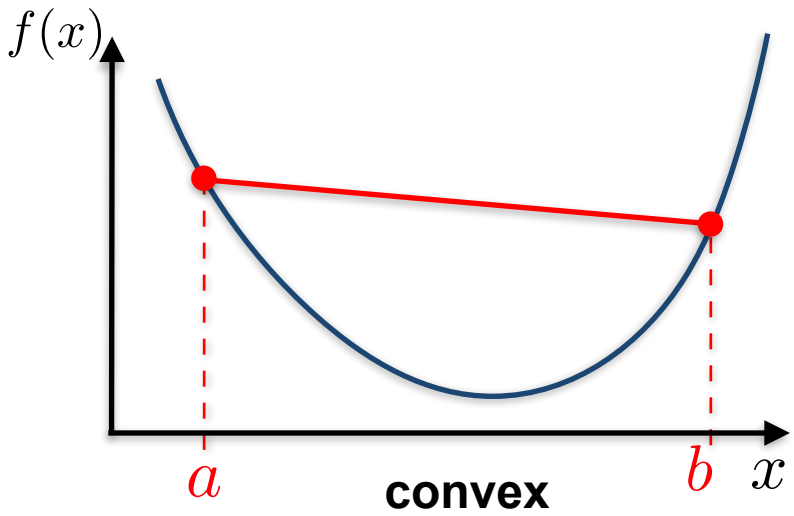
convex

A few words on Convexity

- Searching globally optimal solutions usually requires **convexity**!
- f convex if: $f((1-t)a+tb) \leq (1-t)f(a)+tf(b) \quad t \in [0,1]$

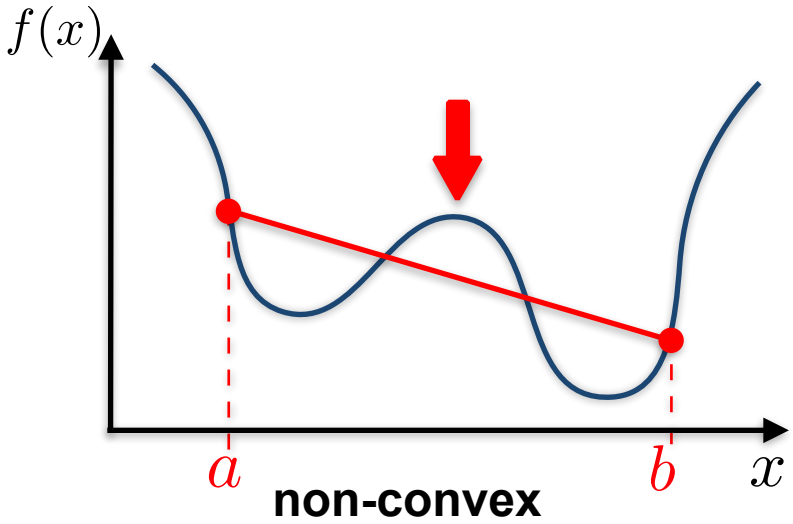


vs.

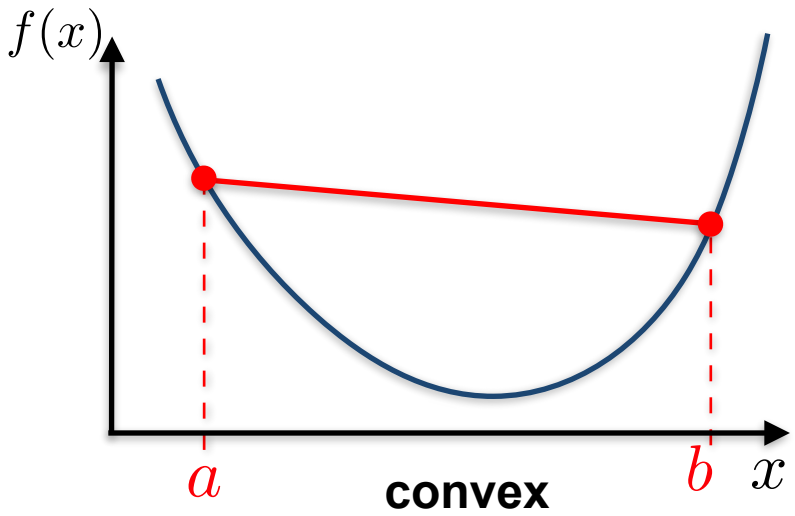


A few words on Convexity

- Searching globally optimal solutions usually requires **convexity**!
- f convex if: $f((1-t)a+tb) \leq (1-t)f(a)+tf(b) \quad t \in [0,1]$



vs.



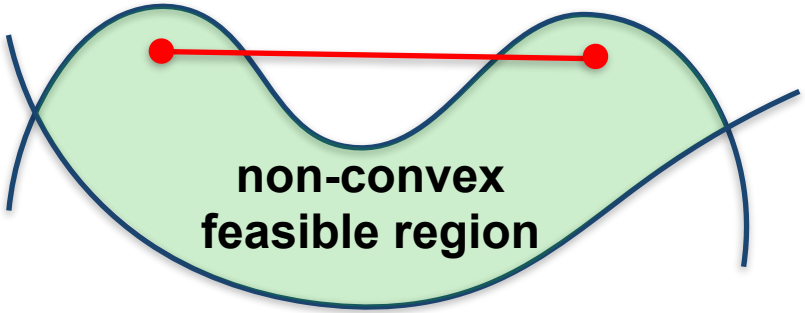
A few words on Convexity

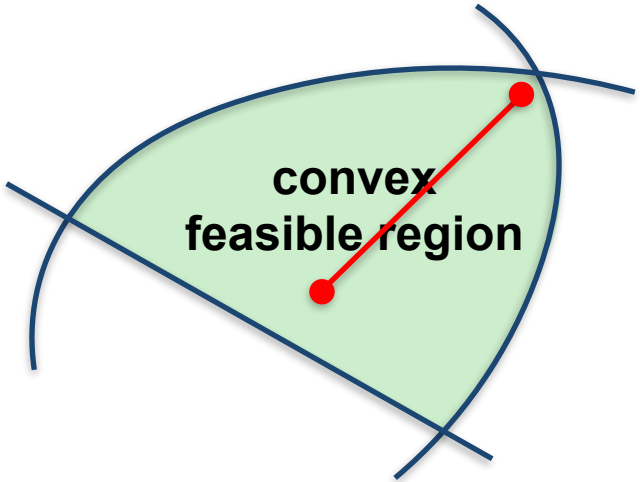
$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0 \qquad i = 1 \dots m \end{aligned}$$

is **convex optimization problem** if $f(x)$ and all $g_i(x)$ are convex functions

consequences

- feasible region is convex set





A few words on Convexity

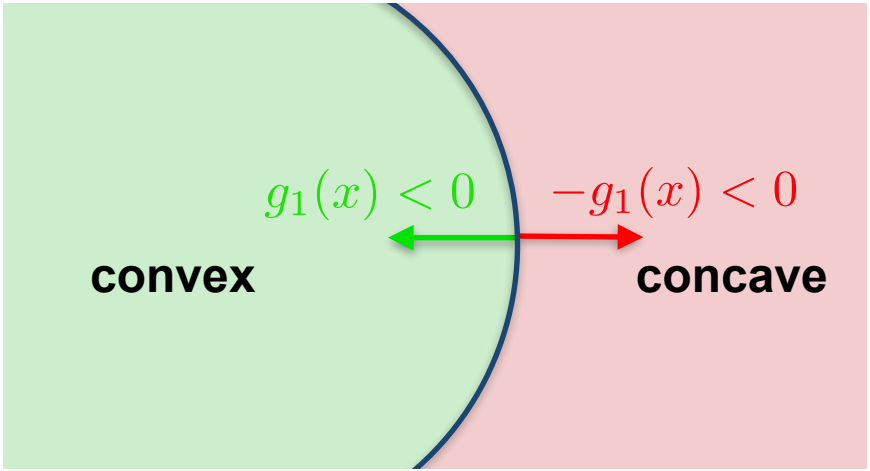
$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0 \qquad i = 1 \dots m \end{aligned}$$

is **convex optimization problem** if $f(x)$ and all $g_i(x)$ are convex functions

consequences

- feasible region is convex set
- equality constraints can only be affine, i.e. $g_i(x) = a^T x + b$ since

$$g_i(x) = 0 \iff \begin{cases} g_i(x) & \leq 0 \\ -g_i(x) & \leq 0 \end{cases}$$



A few words on Convexity

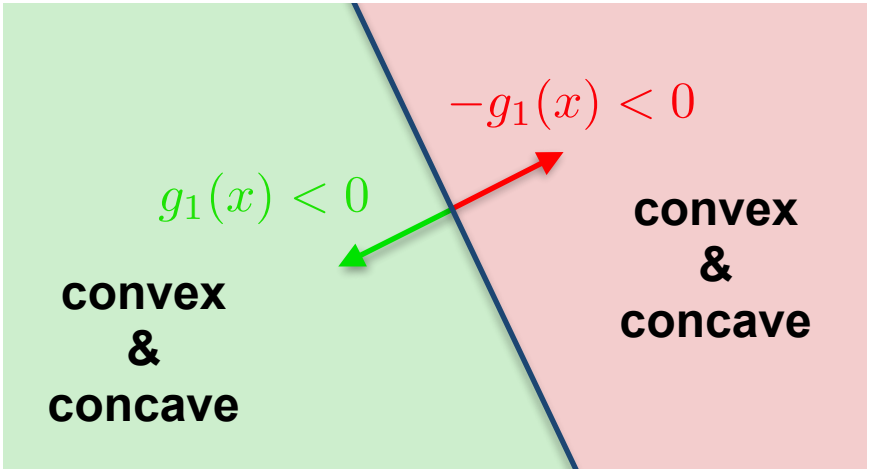
$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0 \qquad i = 1 \dots m \end{aligned}$$

is **convex optimization problem** if $f(x)$ and all $g_i(x)$ are convex functions

consequences

- feasible region is convex set
- equality constraints can only be affine, i.e. $g_i(x) = a^T x + b$ since

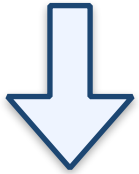
$$g_i(x) = 0 \iff \begin{cases} g_i(x) & \leq 0 \\ -g_i(x) & \leq 0 \end{cases}$$



Convexity

- Important Take Home Message:

Always Search for Convex Formulations



Global Optimum can be Found Efficiently

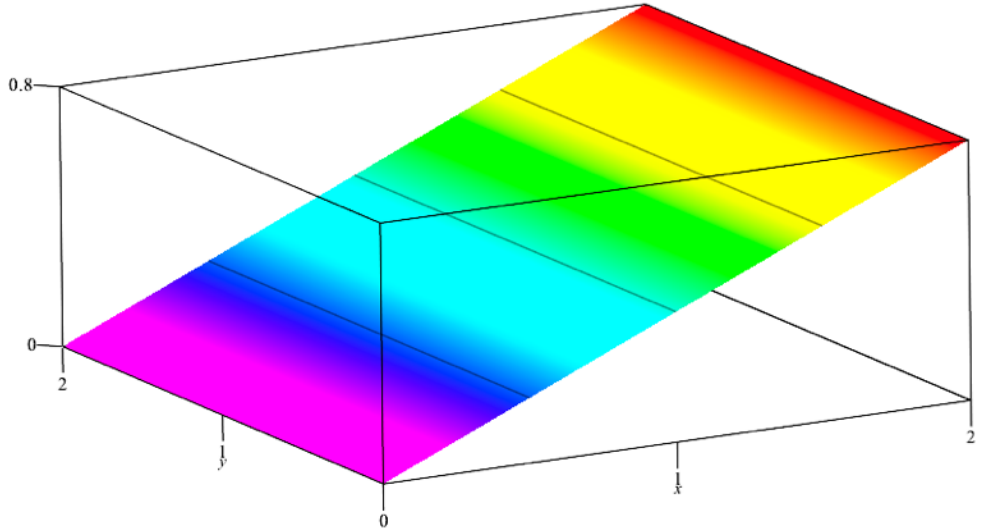
Important (convex) Problem Classes

Linear Program (LP)

- general form

linear objective

$$\begin{aligned} \text{minimize} \quad & a^T x \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$



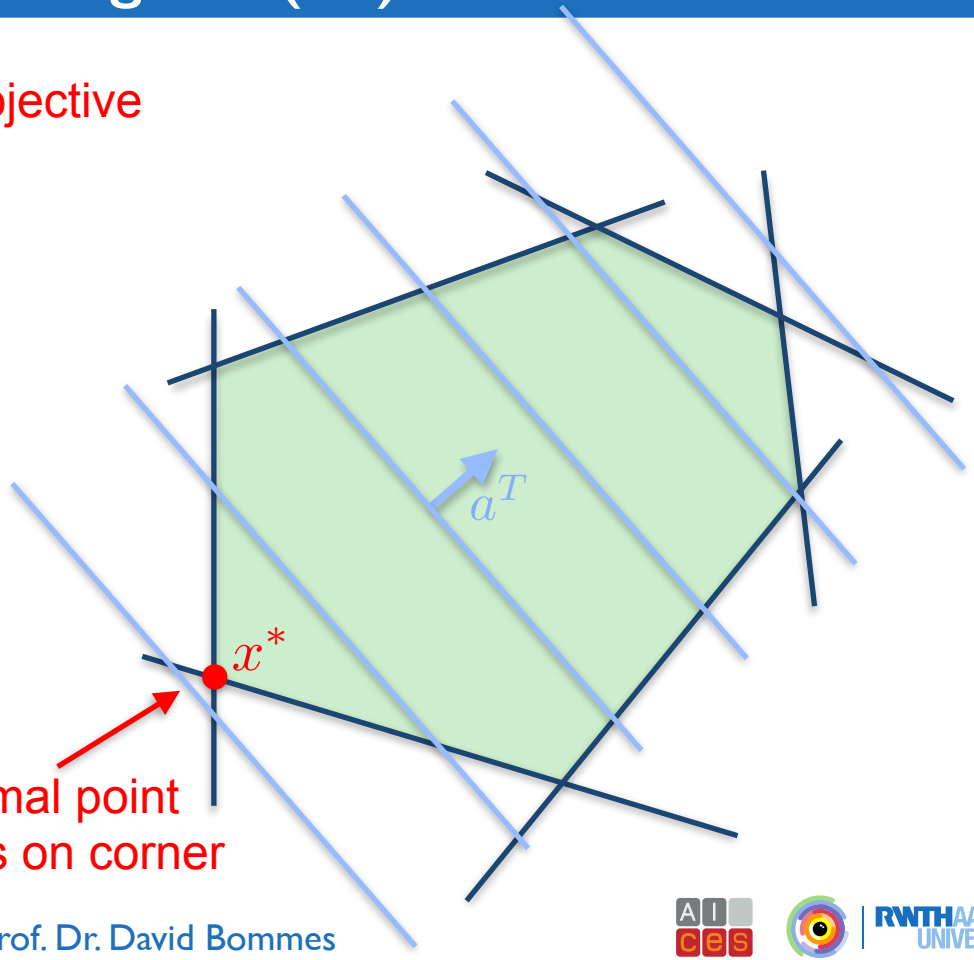
Linear Program (LP)

- general form

linear objective

$$\begin{aligned} \text{minimize} \quad & a^T x \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

linear inequalities
=
intersection of halfspaces
=
polyhedron



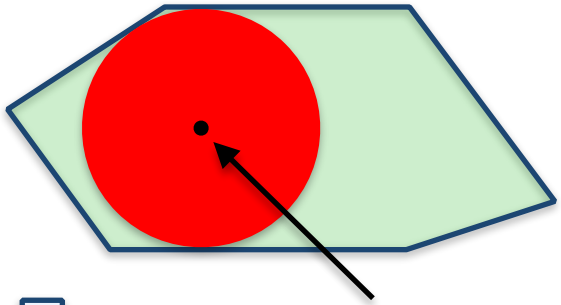
optimal point
always on corner

Example: Chebyshev Center

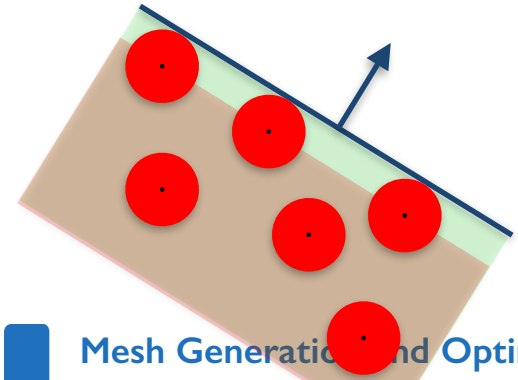
- Find largest ball inside a polyhedron in $\mathbb{R}^n$
- Ball $\mathcal{B} = \{x_c + u \mid ||u||_2 \leq r\}$
- Ball of radius r is inside halfspace

  $a^T x \leq b$ if $a^T x + r||a||_2 \leq b$

  offset



Chebyshev center x_c



$$\begin{array}{lll} \text{minimize} & -r & \\ \text{subject to} & a_i^T x + r||a_i||_2 & \leq b_i \qquad i = 1 \ldots m \end{array}$$

(useful) LP Reformulations

- 1-norm:

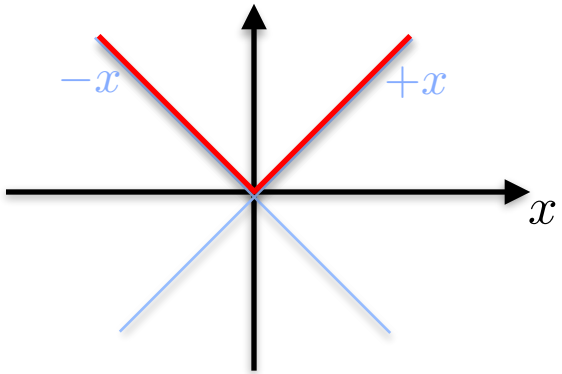
$$\text{minimize} \quad ||Ax - b||_1 = \sum_i |a_i^T x - b_i|$$

LP

$$\begin{aligned} \text{minimize} \quad & \sum_i y_i \\ \text{subject to} \quad & y_i \geq a_i^T x - b_i \qquad \forall i \\ & y_i \geq -(a_i^T x - b_i) \end{aligned}$$

$$y_i \geq \max \pm(a_i^T x - b_i) = |a_i^T x - b_i|$$

$$\max\{x, -x\} = |x|$$



(useful) LP Reformulations

- 1-norm:

$$\text{minimize} \quad ||Ax-b||_1 = \sum_i |a_i^T x - b_i|$$

LP

$$\begin{aligned} \text{minimize} \quad & \sum_i y_i \\ \text{subject to} \quad & y_i \geq a_i^T x - b_i \qquad \forall i \\ & y_i \geq -(a_i^T x - b_i) \end{aligned}$$

- ∞-norm:

$$\text{minimize} \quad ||Ax-b||_\infty = \max_i |a_i^T x - b_i|$$

LP

$$\begin{aligned} \text{minimize} \quad & y \\ \text{subject to} \quad & y \geq a_i^T x - b_i \qquad \forall i \\ & y \geq -(a_i^T x - b_i) \end{aligned}$$

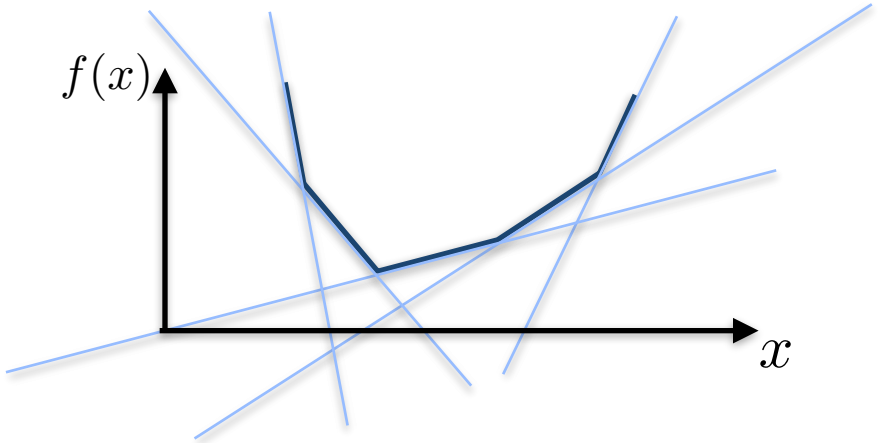
(useful) LP Reformulations

- (convex) piecewise linear function:

$$\text{minimize} \quad f(x) = \begin{cases} f_1(x) & \text{if } x \in [t_1, t_2) \\ f_2(x) & \text{if } x \in [t_2, t_3) \\ \vdots & \vdots \\ f_k(x) & \text{if } x \in [t_k, t_{k+1}) \end{cases}$$

LP

$$\text{minimize} \quad \max_i |a_i^T x - b_i|$$



- approximate arbitrarily complex convex functions by LP!
  (e.g. Outer Approximation)
- works in $\mathbb{R}^d$

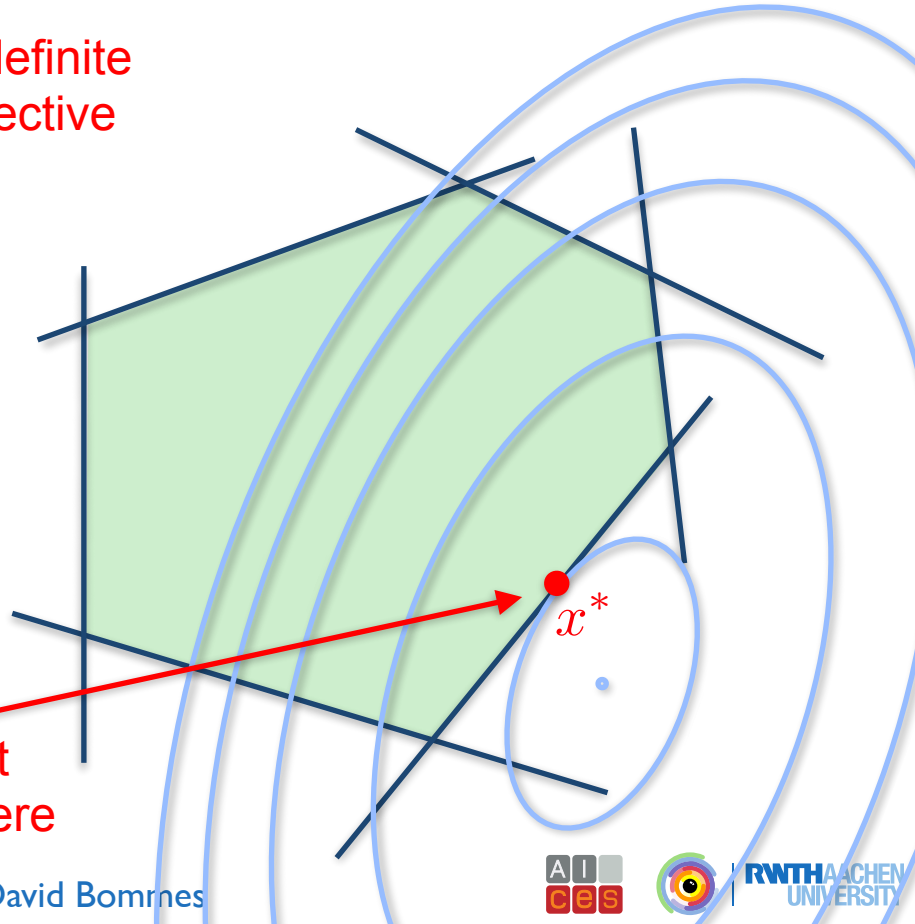
Quadratic Program (QP)

- general form

positive semidefinite quadratic objective

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2} x^T Q x + p^T x + c \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

linear inequalities
=
intersection of halfspaces
=
polyhedron



optimal point
can be anywhere

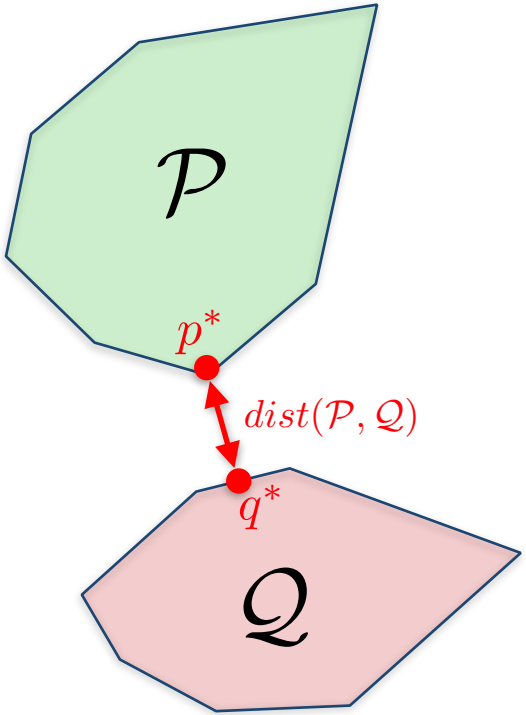
Example QP

- Distance between polyhedra:

$$dist(\mathcal{P}, \mathcal{Q}) = \inf\{||p - q||_2 \mid p \in \mathcal{P},\, q \in \mathcal{Q}\}$$

QP

$$\begin{array}{lll} \text{minimize} & ||p-q||_2^2 & \text{with } x = (p, q) \\ \text{subject to} & A_P p & \leq b_P \\ & A_Q q & \leq b_Q \end{array}$$



Second-Order Cone Program (SOCP)

- general form

$$
\begin{aligned}
&\text{minimize} && a^T x && && \\
&\text{subject to} && ||A_i x + b_i||_2 && \leq c_i^T x + d_i && i = 1 \dots m
\end{aligned}
$$

Second-Order Cone Program (SOCP)

- general form

$$\begin{aligned} \text{minimize} \quad & a^T x \\ \text{subject to} \quad & ||A_i x + b_i||_2 \leq c_i^T x + d_i \qquad i = 1 \dots m \end{aligned}$$

- $\text{LP} \subset \text{QP} \subset \text{QCQP} \subset \text{SOCP}$

Second-Order Cone Program (SOCP)

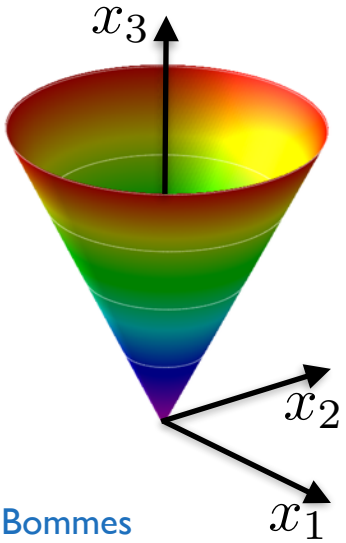
- general form

$$\begin{aligned} \text{minimize} \quad & a^T x \\ \text{subject to} \quad & ||A_i x + b_i||_2 \leq c_i^T x + d_i \qquad i = 1 \dots m \end{aligned}$$

- $\text{LP} \subset \text{QP} \subset \text{QCQP} \subset \text{SOCP}$

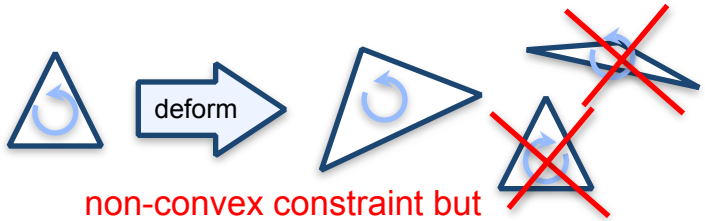
- example constraint:

$$\sqrt{x_1^2 + x_2^2} \leq x_3$$

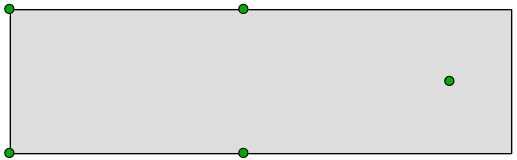


Example SOCP

- Bounded (conformal) distortion mappings in 2D **[Lipman 2012]**:

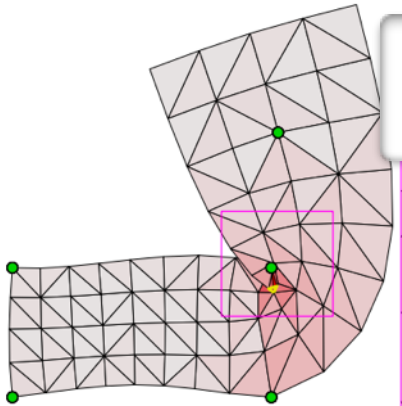


non-convex constraint but approximation by SOCP possible!

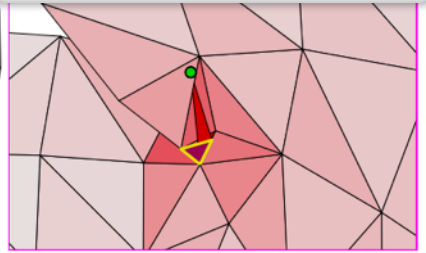


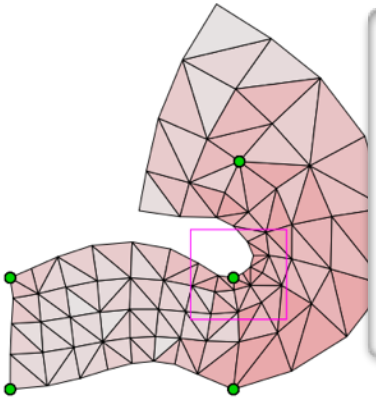
LSCM

BD-LSCM



minimize "*distortion*"





minimize "*distortion*"
subject to "*all triangles have bounded conformal distortion*"

Semidefinite Program (SDP)

- general form

$$\begin{aligned} \text{minimize} \quad & \operatorname{tr}(CX) && = \sum_{i,j} C_{ij} X_{ij} \\ \text{subject to} \quad & \operatorname{tr}(A_i X) && \leq b_i \qquad i = 1 \dots m \\ & X && \succeq 0 \end{aligned}$$

linear function in matrix space

with $X, C, A_i \in S^n$
(symmetric nxn matrices)

X is required to be
positive semidefinite

Finding Minima

First-Order Optimality Conditions

- **Necessary condition for minimum of**

$$\begin{array}{lll} \text{minimize} & f(x) & \\ \text{subject to} & g_i(x) & \leq 0 \qquad i = 1 \dots m \end{array}$$

- **Langrangian**: $L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$

- **Karush-Kuhn-Tucker (KKT)**
  conditions for **minimum** x^*
  1. Stationarity: $\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) = 0$
  2. Primal feasibility: $g_i(x^*) \leq 0$
  3. Dual feasibility: $\lambda_i \geq 0$
  4. Complementary slackness: $\lambda_i g_i(x^*) = 0$

without constraints just

$$\nabla f(x) = 0$$

First-Order Optimality Conditions

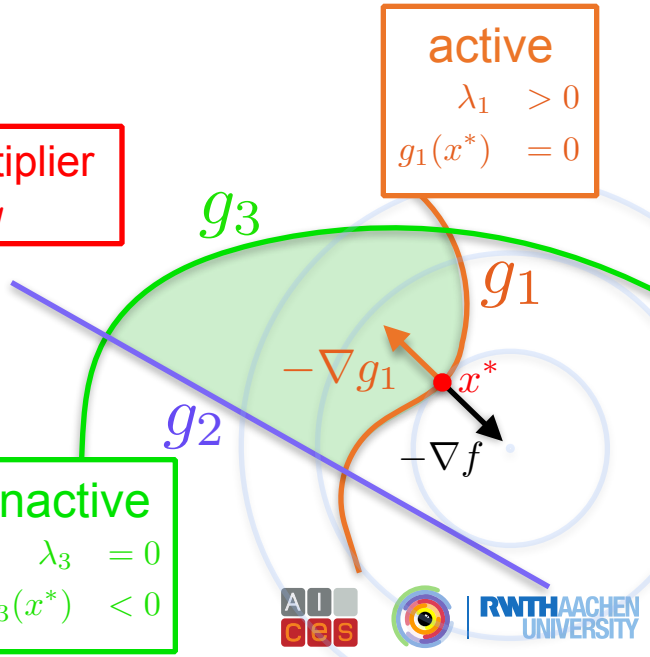
- **Necessary condition for minimum of**

$$\begin{aligned} &\text{minimize} && f(x) && \\ &\text{subject to} && g_i(x) && \leq 0 \qquad i = 1 \dots m \end{aligned}$$

- **Langrangian**: $L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$

- **Karush-Kuhn-Tucker (KKT)** conditions for **minimum** x^*
  1. Stationarity: $\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) = 0$
  2. Primal feasibility: $g_i(x^*) \leq 0$
  3. Dual feasibility: $\lambda_i \geq 0$
  4. Complementary slackness: $\lambda_i g_i(x^*) = 0$

Recall Lagrange Multiplier
$\nabla f = -\lambda \nabla g$



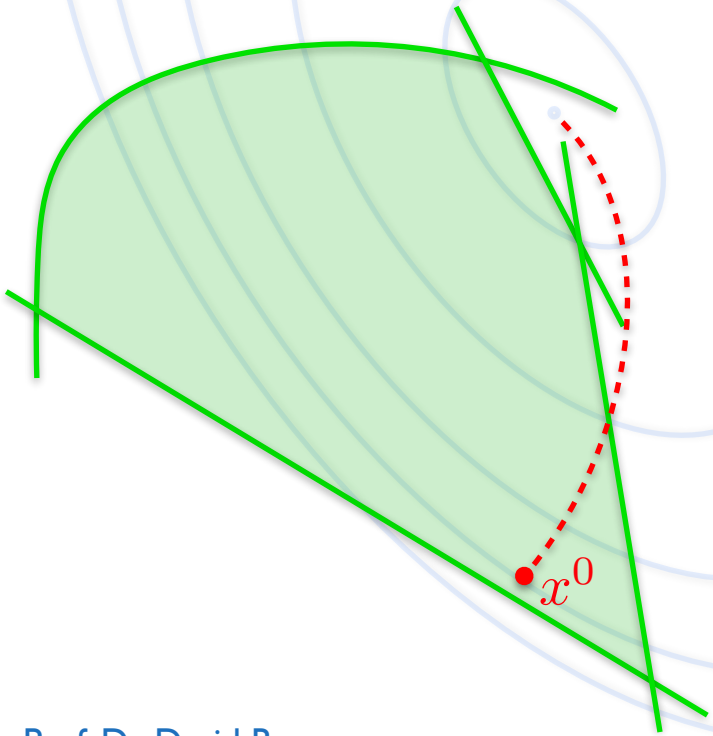
Algorithms

Active Set Method

Active Set Method

- Assume feasible x^0
- Initialize Active Set $A = \{\}$

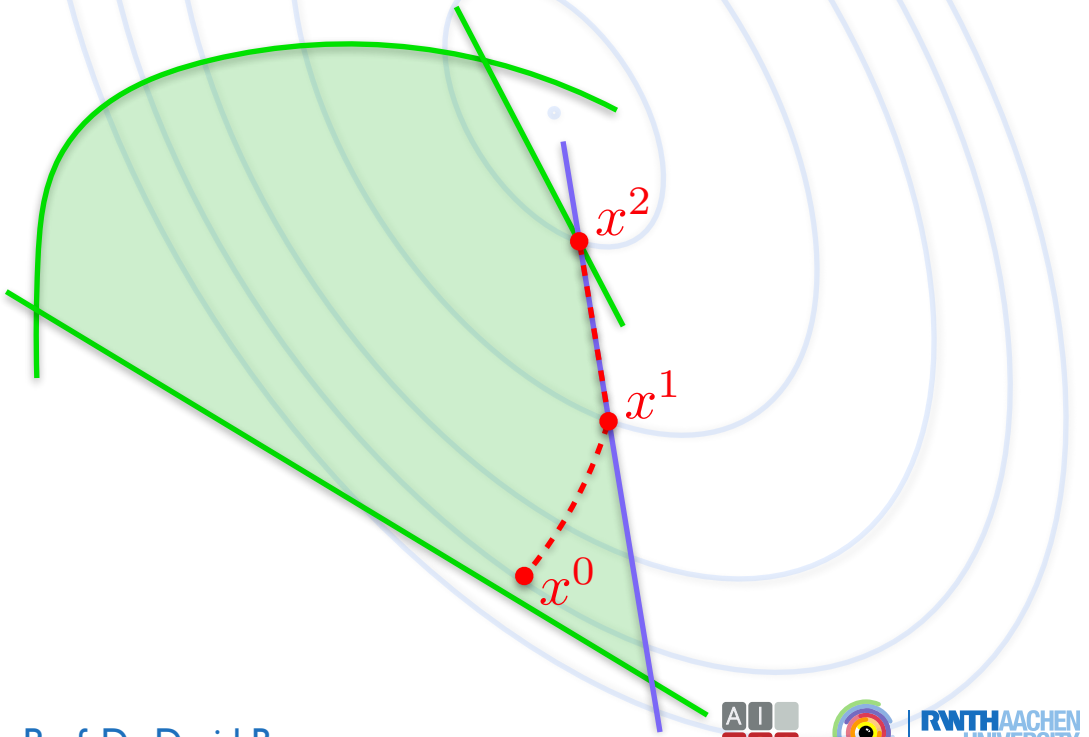
- While KKT not satisfied
  - Optimize with equality constraints of A
  - if constraint gets violated, **activate** it



Active Set Method

- Assume feasible x^0
- Initialize Active Set $A = \{\}$

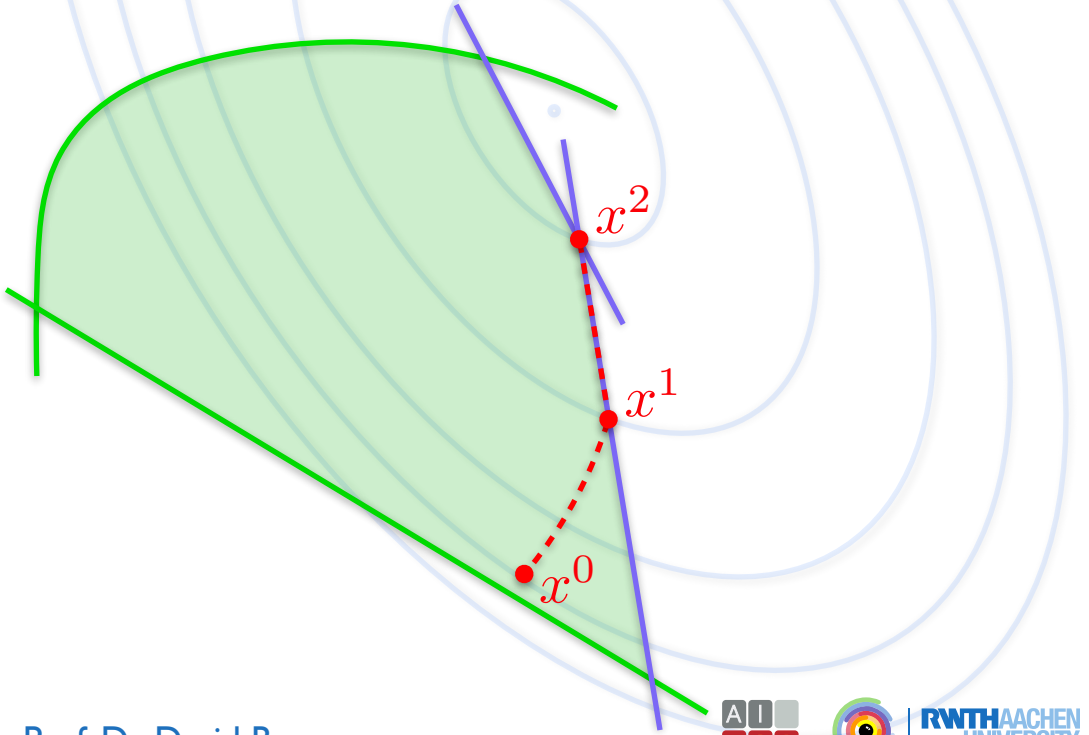
- While KKT not satisfied
  - Optimize with equality constraints of A
  - if constraint gets violated, **activate** it



Active Set Method

- Assume feasible x^0
- Initialize Active Set $A = \{\}$

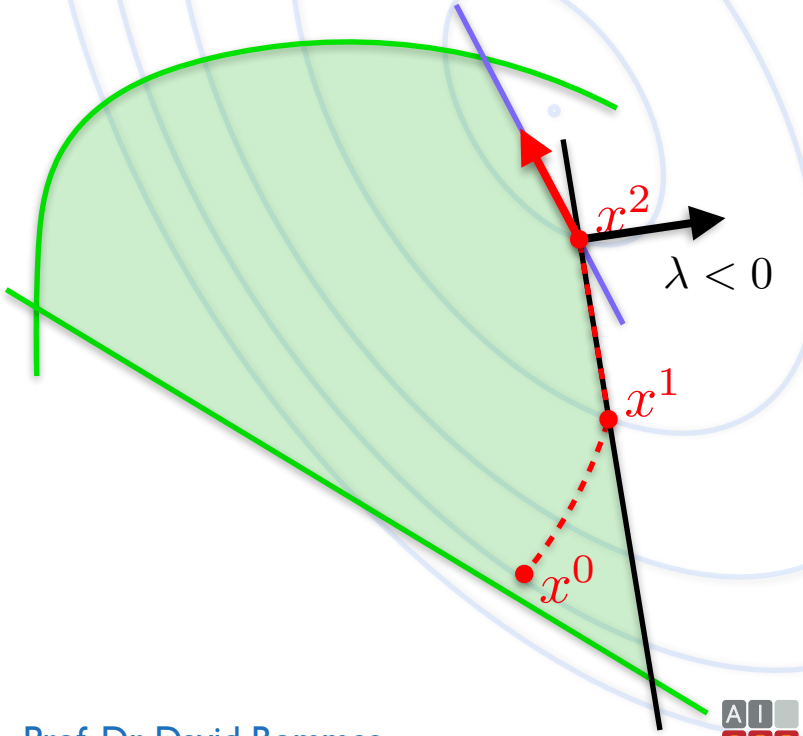
- While KKT not satisfied
  - Optimize with equality constraints of A
  - if constraint gets violated, **activate** it



Active Set Method

- Assume feasible x^0
- Initialize Active Set $A = \{\}$

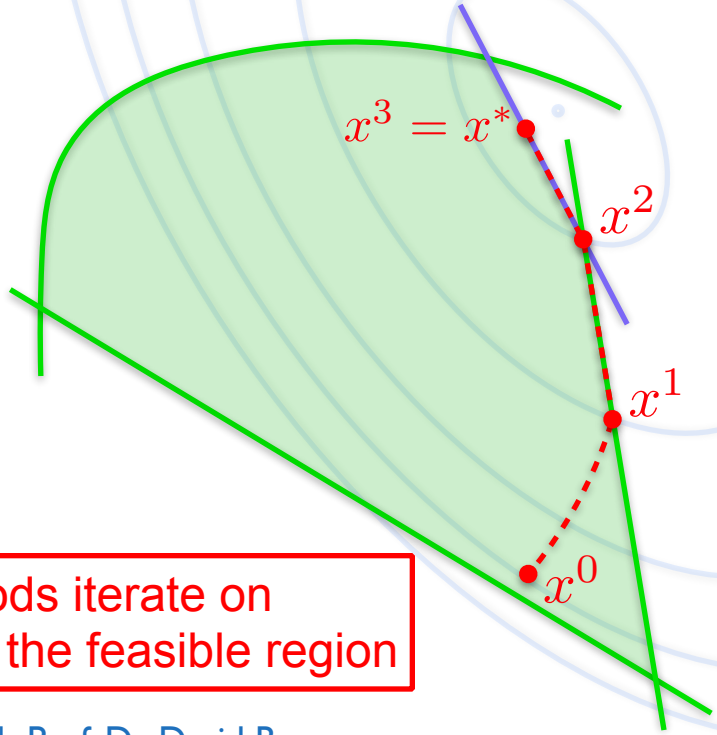
- While KKT not satisfied
  - Optimize with equality constraints of A
  - if constraint gets violated, **activate** it



Active Set Method

- Assume feasible x^0
- Initialize Active Set $A = \{\}$

- While KKT not satisfied
  - Optimize with equality constraints of A
  - if constraint gets violated, **activate** it
  - if Lagrange multiplier gets negative **deactivate** corresponding constraint

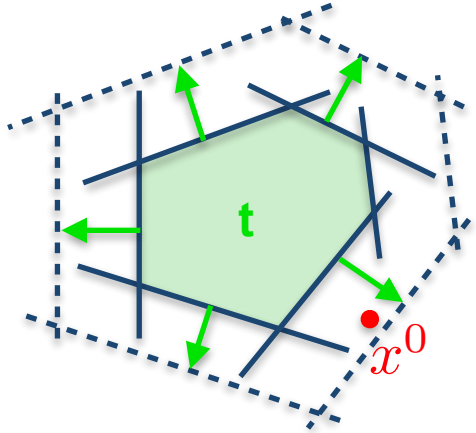


active set methods iterate on the boundary of the feasible region

How to find feasible starting point?

- Given

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0 \qquad i = 1 \dots m \end{aligned}$$



- Corresponding (Phase 1) **Feasibility Problem**:

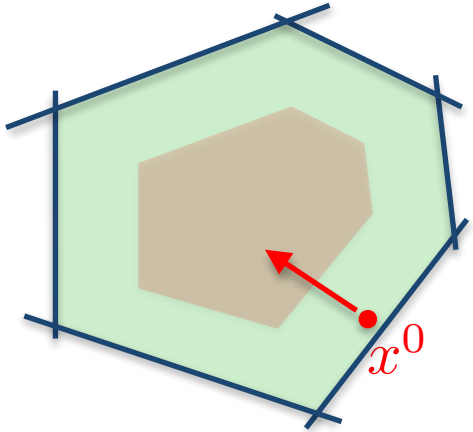
$$\begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & g_i(x) \leq t \qquad i = 1 \dots m \end{aligned}$$

also constrained problem
but for arbitrary $x^0 \in \mathbb{R}^n$
choose $t^0 = \max_i g_i(x^0)$

How to find feasible starting point?

- Given

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0 \qquad i = 1 \dots m \end{aligned}$$



- Corresponding (Phase 1) **Feasibility Problem**:

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & g_i(x) \leq t \qquad i = 1 \dots m \end{aligned}$$

also constrained problem
but for arbitrary $x^0 \in \mathbb{R}^n$
choose $t^0 = \max_i g_i(x^0)$

terminate as soon as t < 0 !!!

Active Set Method

- Many (sophisticated) variants
  - primal, dual, primal-dual, …
- Many specializations
  - LP (e.g. simplex algorithm), QP, NLP, …
- Good choice if
  - few constraints
  - warmstart desired
- Disadvantage
  - sometimes requires huge number of iterations

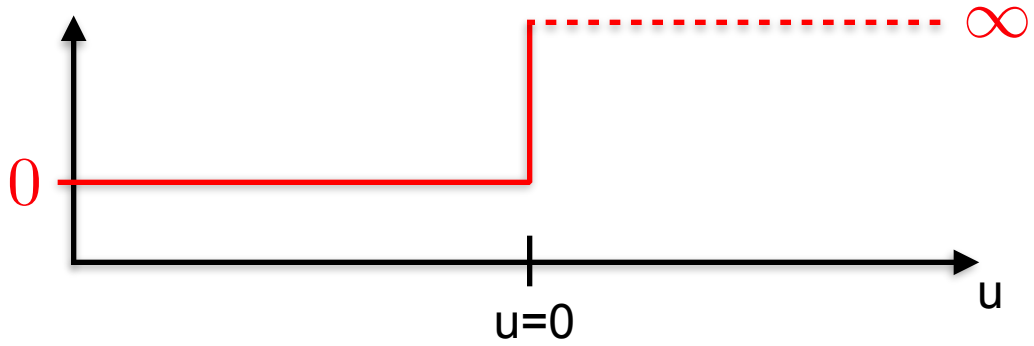
Interior Point Method

Interior Point Method

- Sometimes called **Barrier Method**

- **Barrier function**

$$I_{-}(u) = \begin{cases} 0 & \text{if } u \leq 0 \\ \infty & \text{if } u > 0 \end{cases}$$

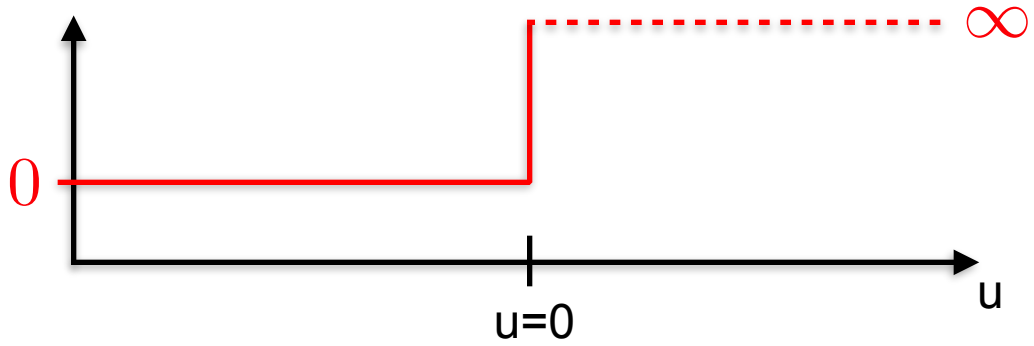


Interior Point Method

- Sometimes called **Barrier Method**

- **Barrier function**

$$I_-(u) = \begin{cases} 0 & \text{if } u \leq 0 \\ \infty & \text{if } u > 0 \end{cases}$$



- **Idea:**

constrained

$$\begin{aligned} &\text{minimize} && f(x) && \\ &\text{subject to} && g_i(x) && \leq 0 \qquad i = 1 \dots m \end{aligned}$$

equivalent

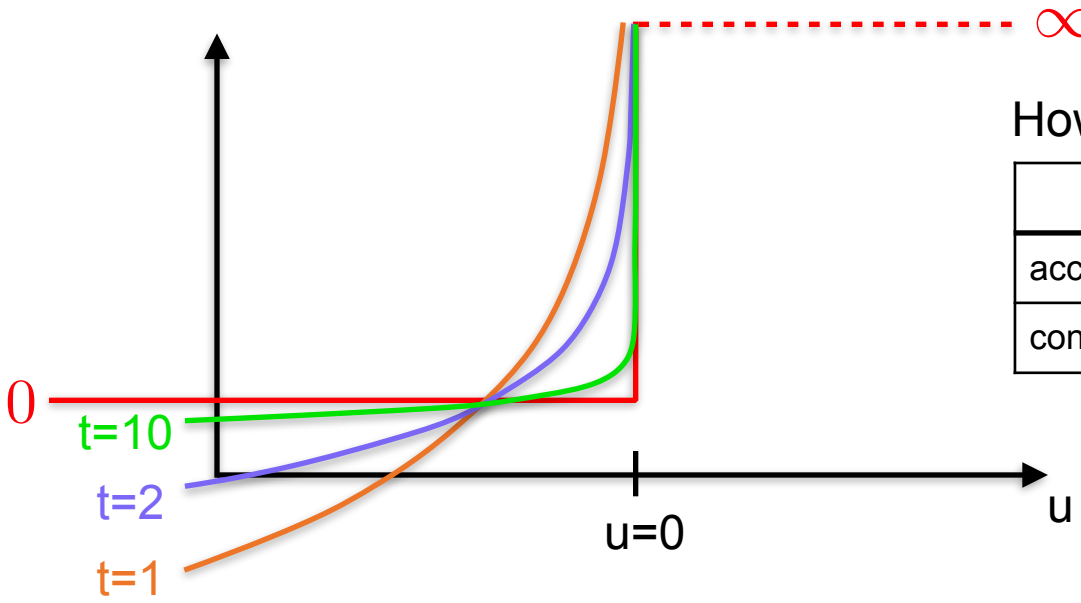
unconstrained

$$\text{minimize} \quad f(x) + \sum_{i=1}^{m} I_-(g_i(x))$$

Barrier Function

- **Logarithmic Barrier** (smooth and convex approximation)

$$\hat{I}_{-}(u) = -\frac{1}{t}\log(-u)$$



How to choose t?

	large t	small t
accuracy	+	-
convergence	-	+

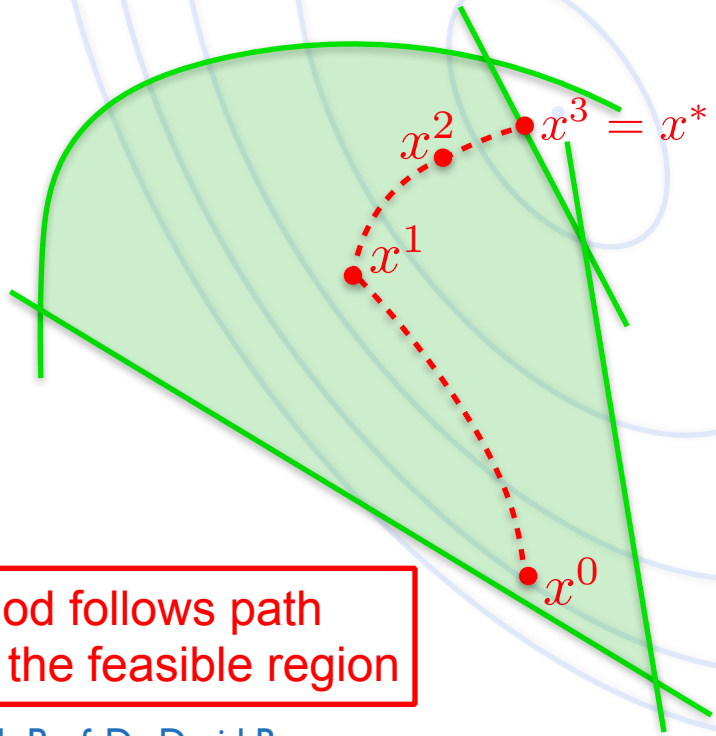
Simple Interior Point Method

- Assume feasible x^0, $t := 1$
- **Iteratively solve** (unconstrained)

$$\text{minimize} \quad f(x) - \frac{1}{t}\sum_{i=1}^{m}\log(-g_i(x))$$

and update $t := 10 \cdot t$

  - stop when $\frac{m}{t} < \epsilon$



interior point method follows path through interior of the feasible region

Interior Point Method

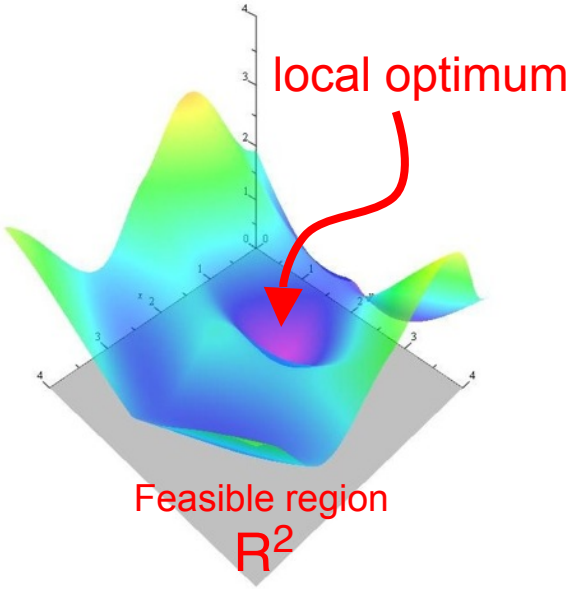
- Many (sophisticated) variants
  - primal, dual, primal-dual, …
  - adaptive choice of t
- Many specializations
  - LP, QP, NLP, …
- Good choice if
  - large scale (sparse) problem with many constraints
  - non-convex feasible region
- Disadvantage
  - limited warmstart capabilities

More Algorithms …

Mixed-Integer Optimization

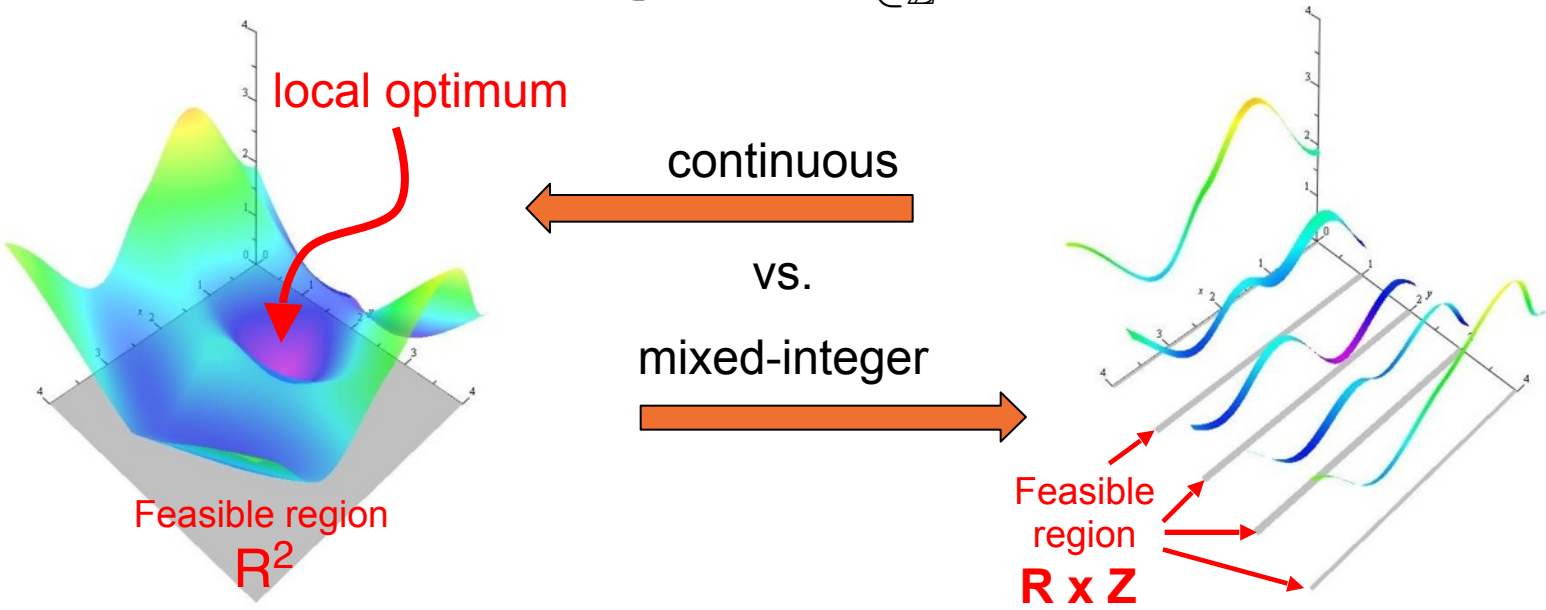
Mixed-Integer Optimization

- Minimize objective $f(\underbrace{x_1 \dots x_n}_{\in \mathbb{R}^n})$



Mixed-Integer Optimization

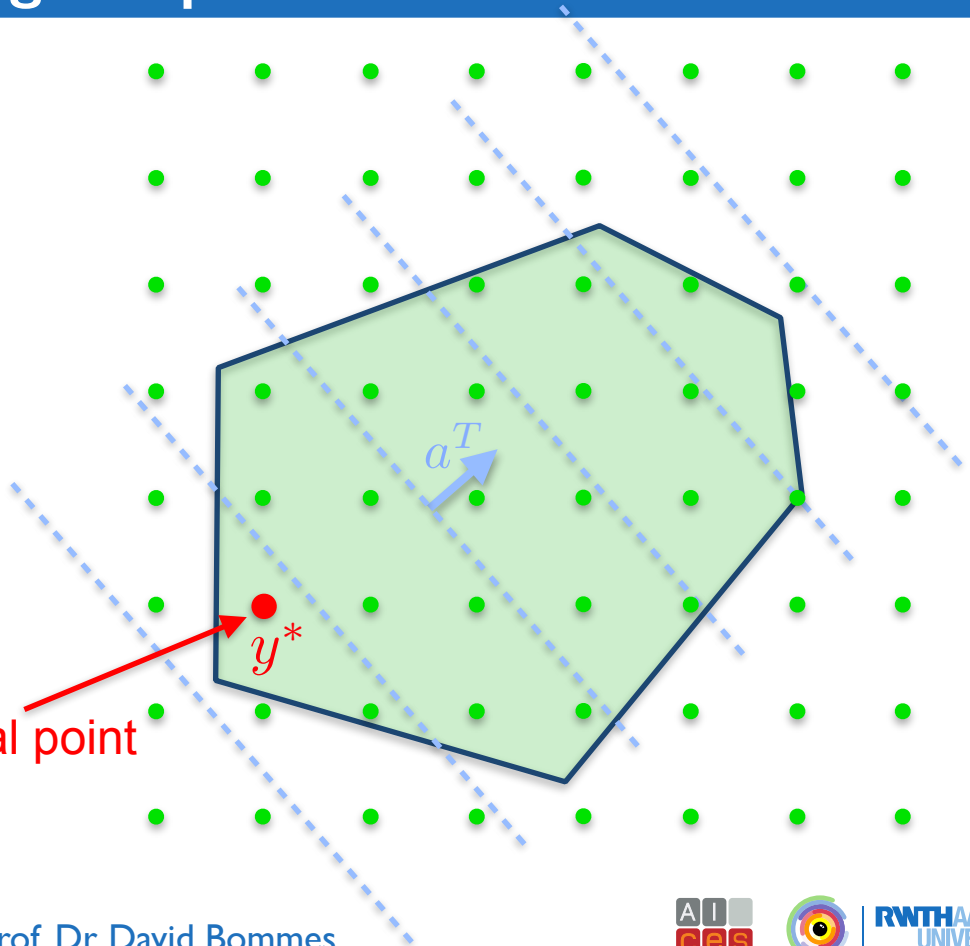
- Minimize objective $f(\underbrace{x_1 \dots x_n}_{\in \mathbb{R}^n}, \underbrace{y_1 \dots y_d}_{\in \mathbb{Z}^d})$



Mixed-Integer Optimization

- Integer Linear Program (ILP)

$$\begin{aligned} \text{minimize} \quad & a^T y \\ \text{subject to} \quad & Ay \leq b \\ & y \in \mathbb{Z}^d \end{aligned}$$



Branch and Bound

Branch and Bound

- **Bounding**: initially we only know $f(x^*, y^*) \in (-\infty, \infty)$

Branch and Bound

- **Bounding**: initially we only know $f(x^*, y^*) \in (-\infty, \infty)$

- **Idea:** improve bounds by series of **continuous problems**

Branch and Bound

- **Bounding**: initially we only know $f(x^*, y^*) \in (-\infty, \infty)$

lower bound upper bound

- **Idea:** improve bounds by series of **continuous problems**
- **Continuous Relaxation:** assume that all variables are continuous

MIP

$$\begin{aligned} \text{minimize} \quad & f(x,y) \\ \text{subject to} \quad & g_i(x,y) \leq 0 \\ & y \in \mathbb{Z}^d \end{aligned}$$

RMIP

$$\begin{aligned} \text{minimize} \quad & f(x,y) \\ \text{subject to} \quad & g_i(x,y) \leq 0 \\ & \cancel{y \in \mathbb{Z}^d} \end{aligned}$$

Branch and Bound

- **Bounding**: initially we only know $f(x^*, y^*) \in (-\infty, \infty)$

lower bound upper bound

- **Idea:** improve bounds by series of **continuous problems**
- **Continuous Relaxation:** assume that all variables are continuous

minimize	$f(x,y)$		**MIP**
subject to	$g_i(x,y)$	≤ 0	
	$y \in \mathbb{Z}^d$		

minimize	$f(x,y)$		**RMIP**
subject to	$g_i(x,y)$	≤ 0	
	~~$y \in \mathbb{Z}^d$~~		

- MIP is restriction of RMIP  lower bound $\min(MIP) \geq \min(RMIP)$

Branch and Bound

- **Bounding**: initially we only know $f(x^*, y^*) \in (-\infty, \infty)$

lower bound upper bound

- **Idea:** improve bounds by series of **continuous problems**
- **Continuous Relaxation:** assume that all variables are continuous

MIP

$$\begin{aligned} \text{minimize} \quad & f(x,y) \\ \text{subject to} \quad & g_i(x,y) \leq 0 \\ & y \in \mathbb{Z}^d \end{aligned}$$

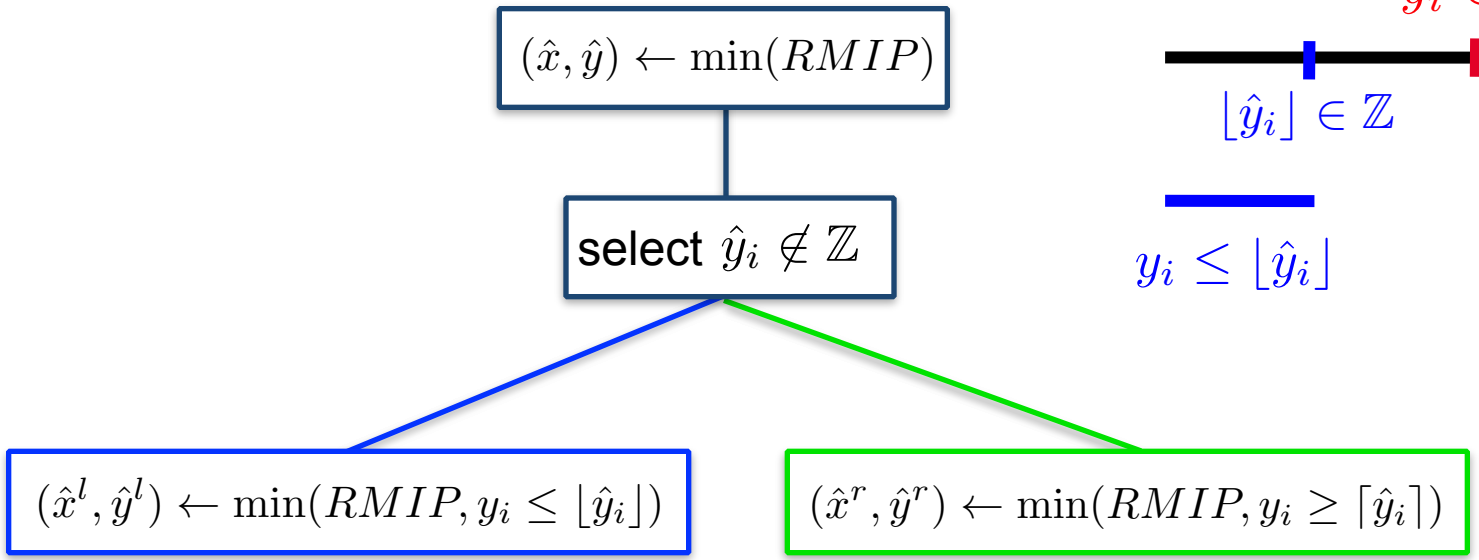
RMIP

$$\begin{aligned} \text{minimize} \quad & f(x,y) \\ \text{subject to} \quad & g_i(x,y) \leq 0 \\ & \cancel{y \in \mathbb{Z}^d} \end{aligned}$$

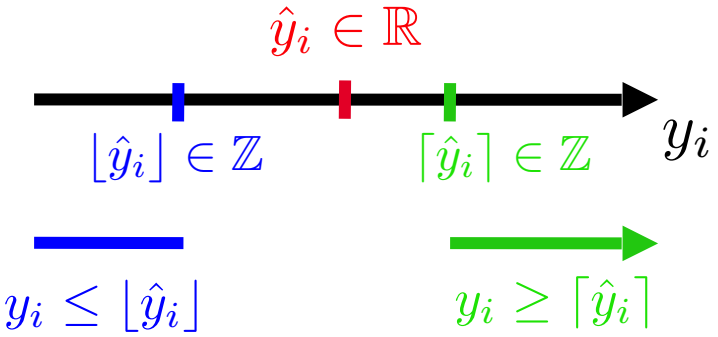
- MIP is restriction of RMIP  lower bound $\min(MIP) \geq \min(RMIP)$
- How to obtain upper bound? How to improve bounds?

Branch and Bound

- **Branching:**



cut off non-integer neighborhood of



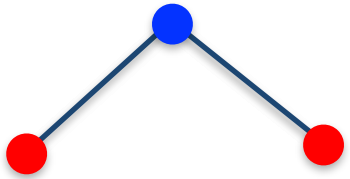
improve lower bound ⇨ $\min\{f(\hat{x}^l, \hat{y}^l), f(\hat{x}^r, \hat{y}^r)\} \geq f(\hat{x}, \hat{y})$

Branch and Bound

- active node

Branch and Bound

- active node
- finished node

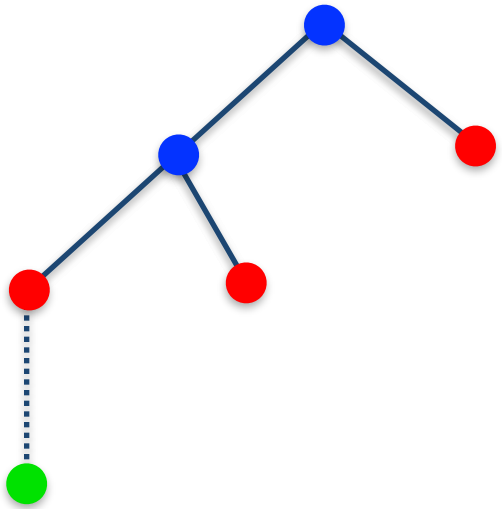


- lower bound: min{ }

Branch and Bound

- active node
- finished node
- feasible node

- lower bound: min{ }

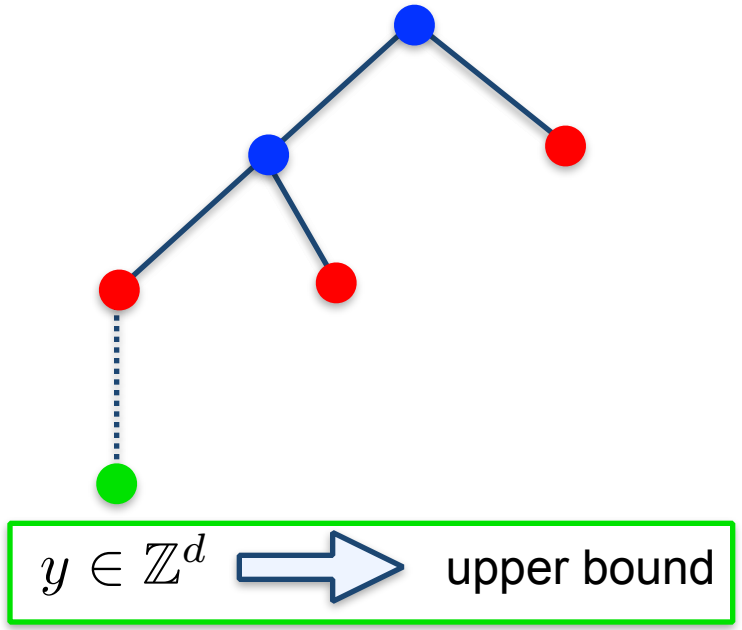


$$y \in \mathbb{Z}^d$$

Branch and Bound

- active node
- finished node
- feasible node

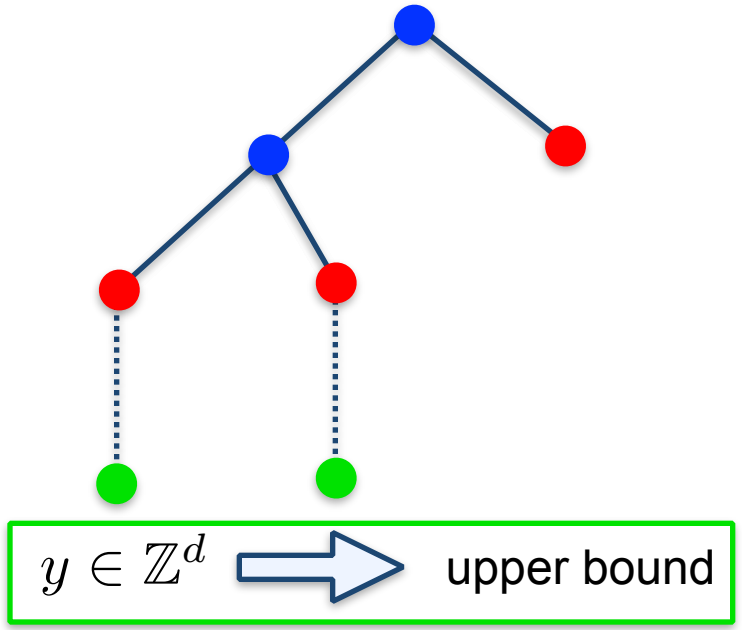
- lower bound: min{●}



Branch and Bound

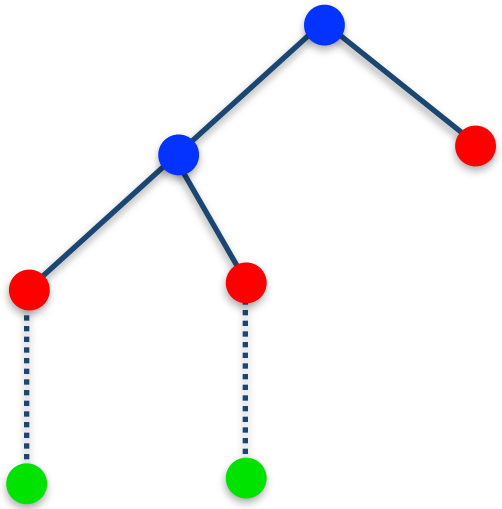
- active node
- finished node
- feasible node

- lower bound: min{}
- upper bound: min{}



Branch and Bound

- active node
- finished node
- feasible node

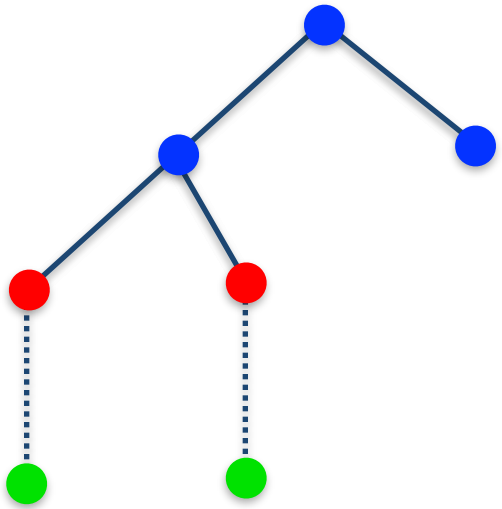


pruning
if ● > upper bound
(or infeasible)

- lower bound: min{●}
- upper bound: min{●}

Branch and Bound

- active node
- finished node
- feasible node

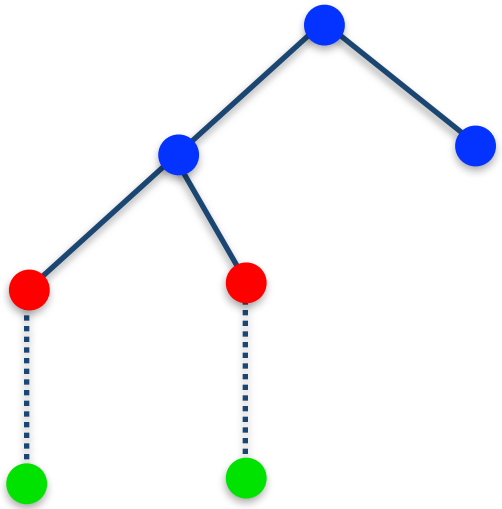


pruning
if ● > upper bound
(or infeasible)

- lower bound: min{●}
- upper bound: min{●}

Branch and Bound

- active node
- finished node
- feasible node



- lower bound: min{●}
- upper bound: min{●}

- terminate when min{●}-min{●} < epsilon

global optimum if RMIP convex!

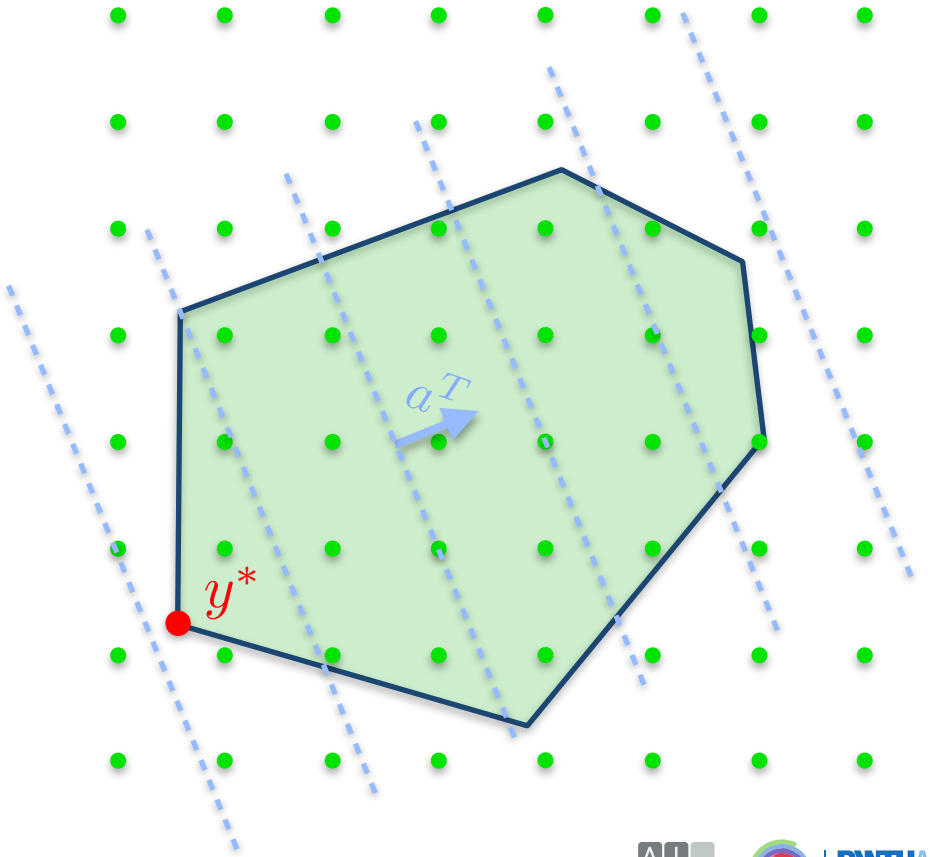
Branch and Bound

- Node Selection Strategies
  - Depth-First Search
  - Best-First Search (w.r.t. lower bound)
  - Dynamic heuristics

- Branching Strategies
  - Maximum Infeasibility — most undecided first (close to .5)
  - Strong Branching — best improvement of lower bound
  - Pseudo Reduced Cost — approximation of strong branching
  - Dynamic heuristics

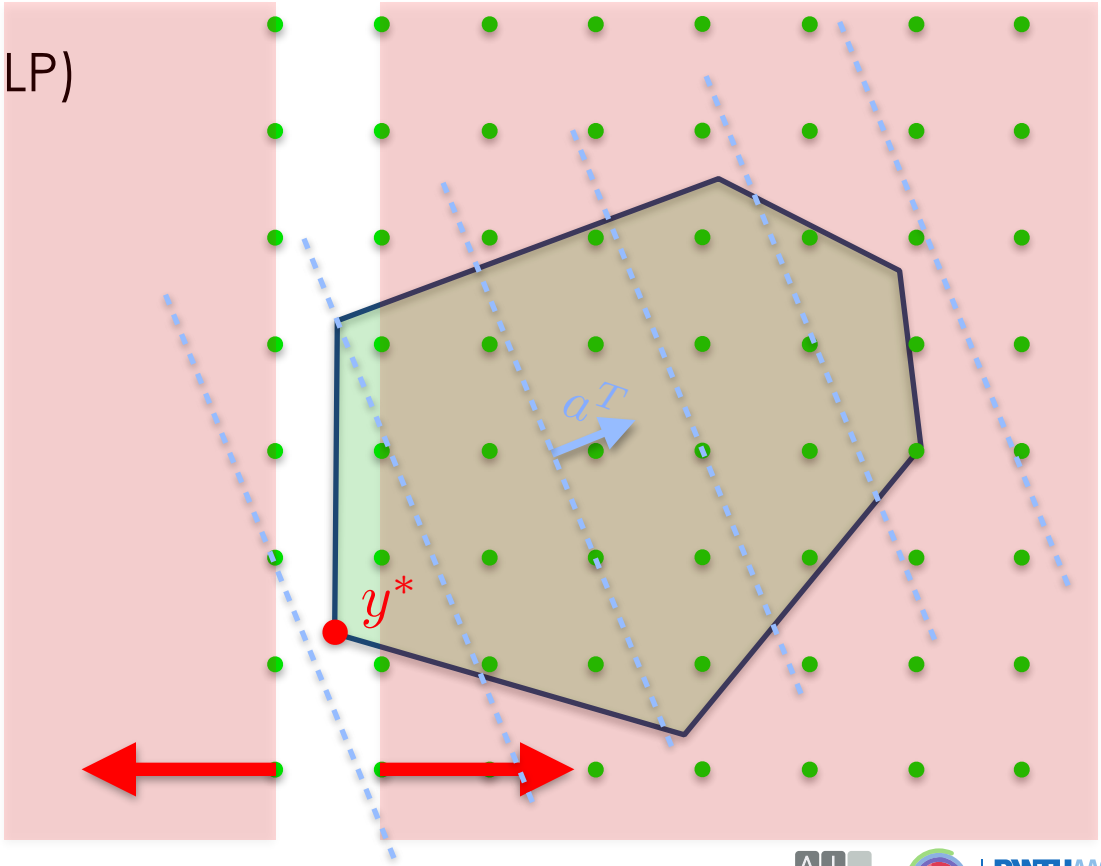
Example: Branch and Bound

- Integer Linear Program (ILP)



Example: Branch and Bound

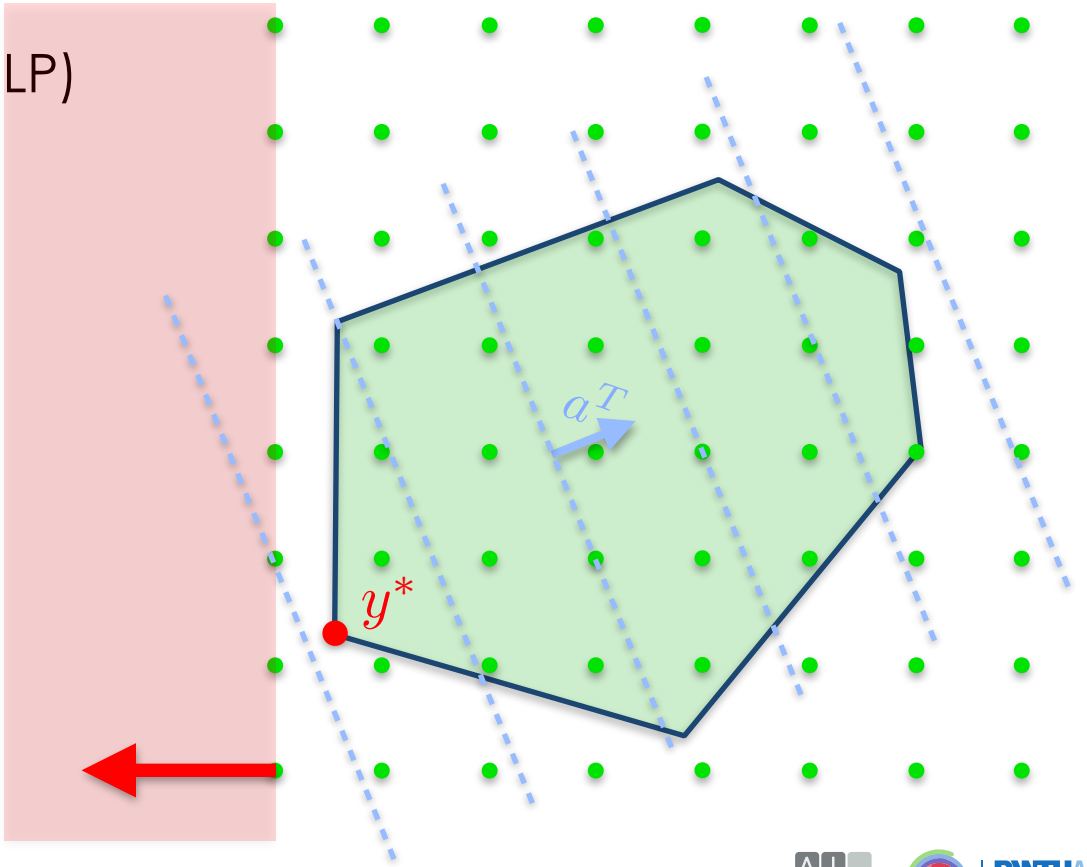
- Integer Linear Program (ILP)



Example: Branch and Bound

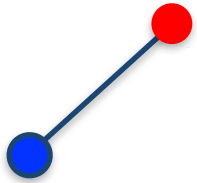
- Integer Linear Program (ILP)

infeasible

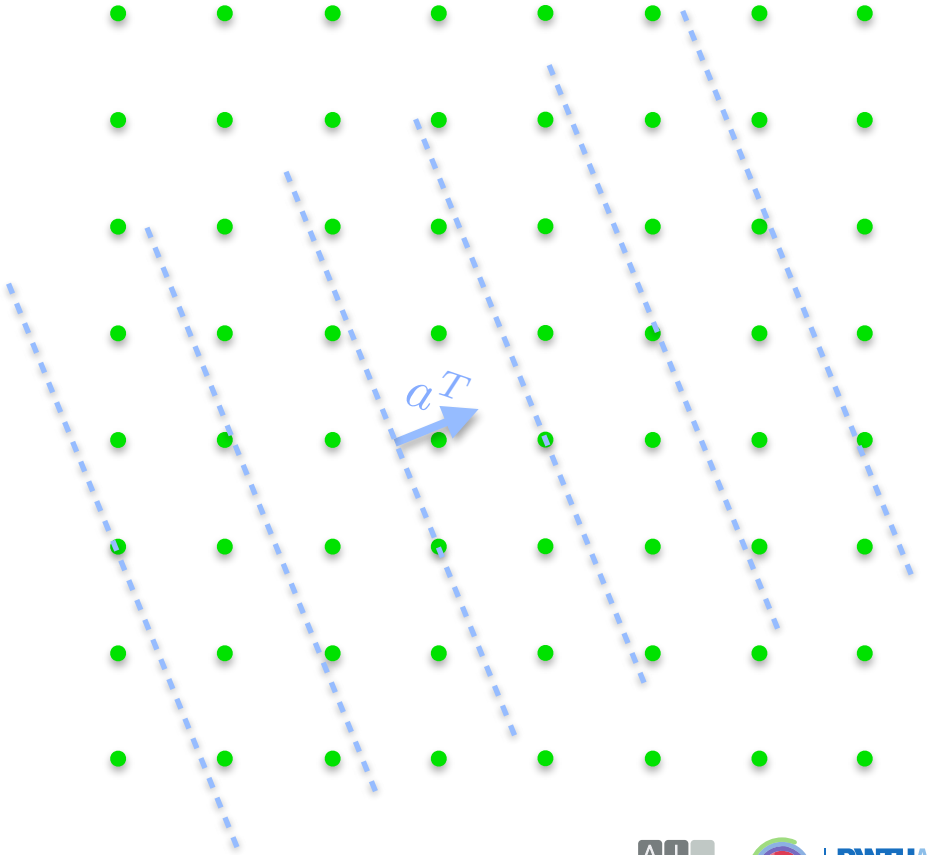


Example: Branch and Bound

- Integer Linear Program (ILP)

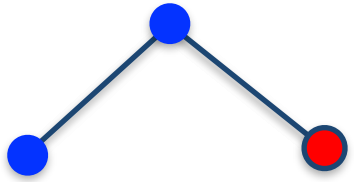


infeasible

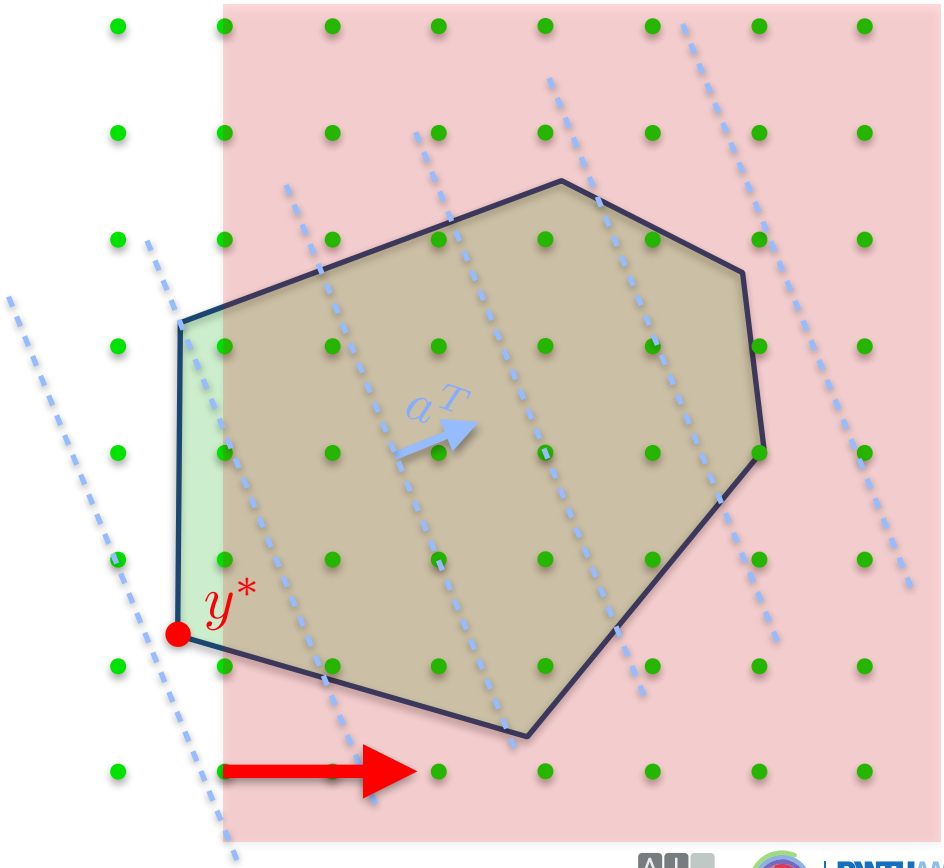


Example: Branch and Bound

- Integer Linear Program (ILP)

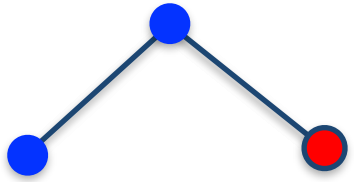


infeasible

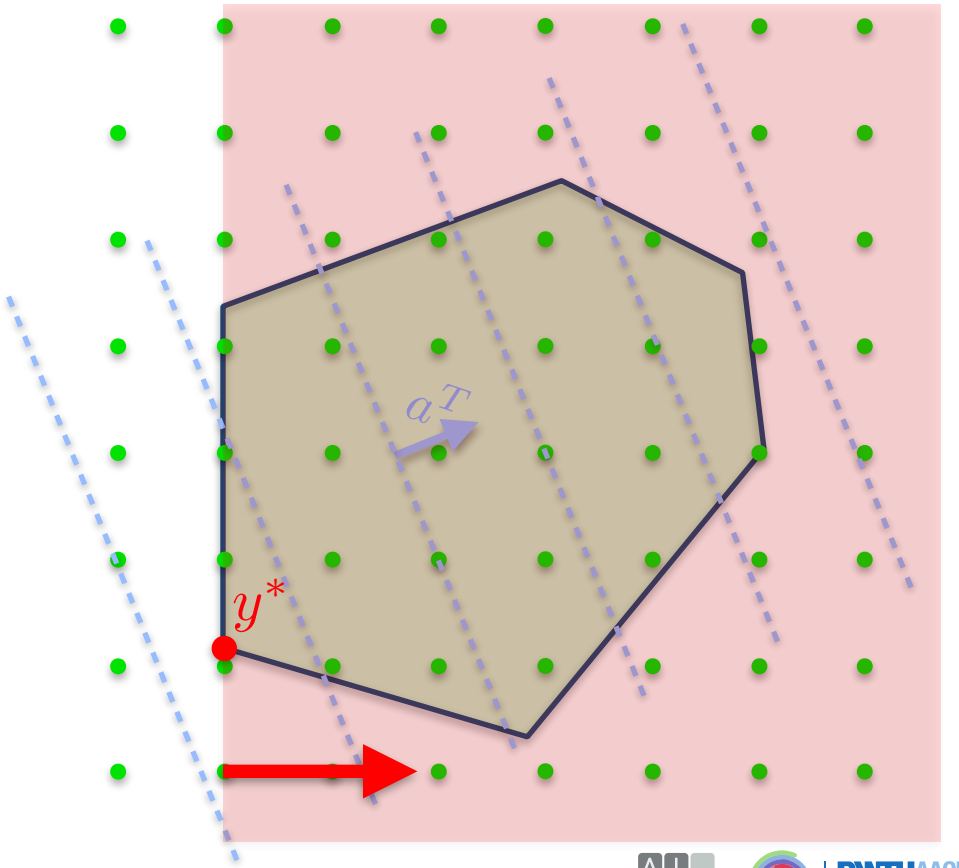


Example: Branch and Bound

- Integer Linear Program (ILP)

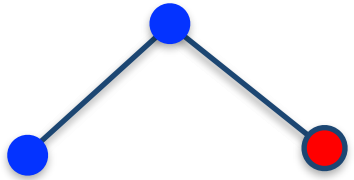


infeasible

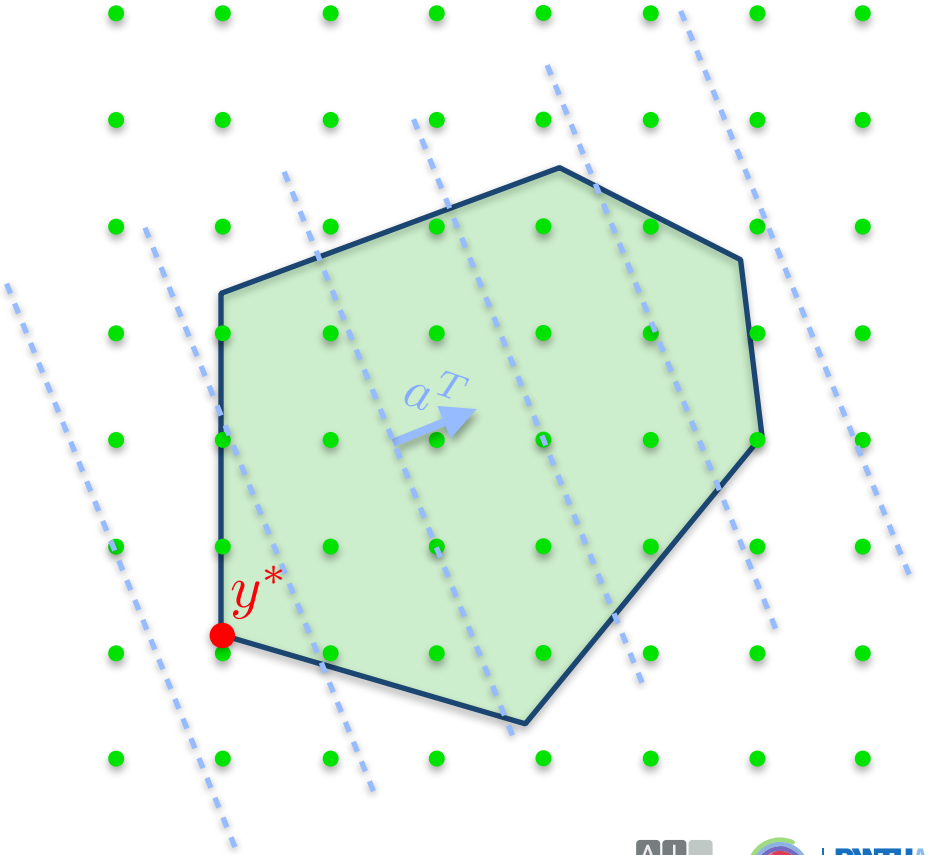


Example: Branch and Bound

- Integer Linear Program (ILP)

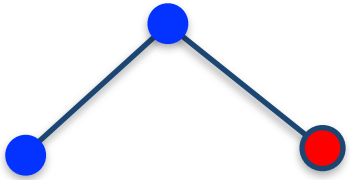


infeasible

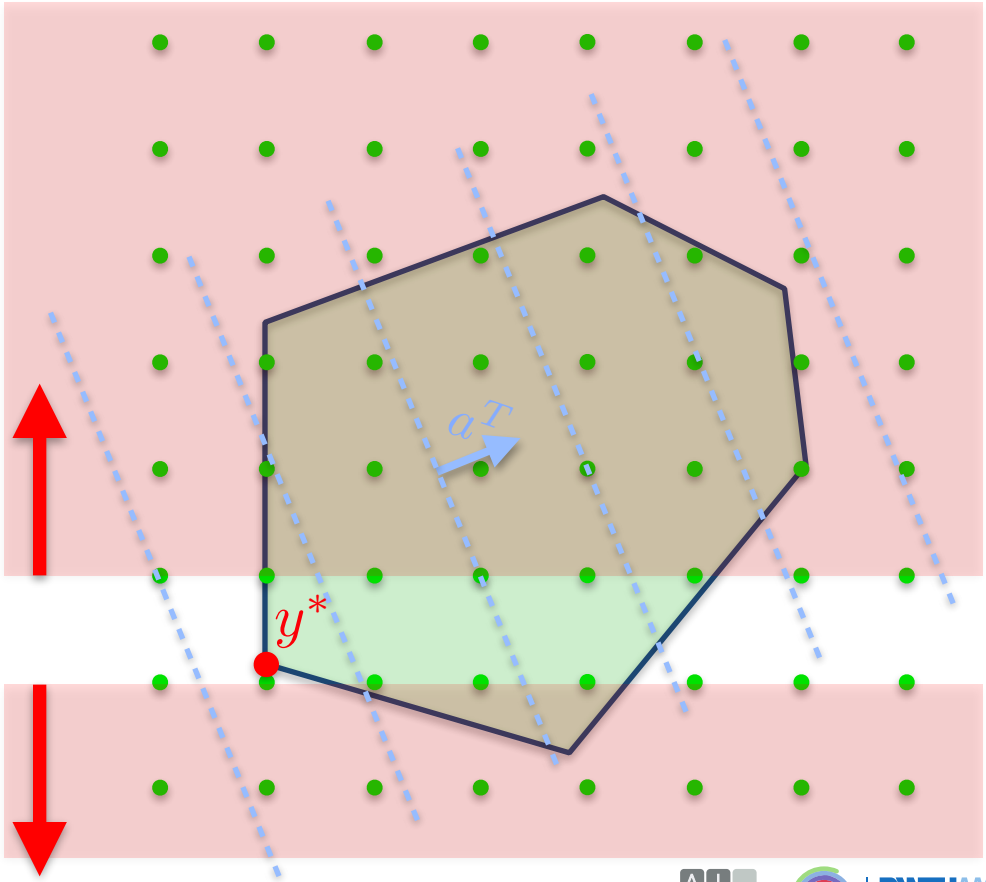


Example: Branch and Bound

- Integer Linear Program (ILP)

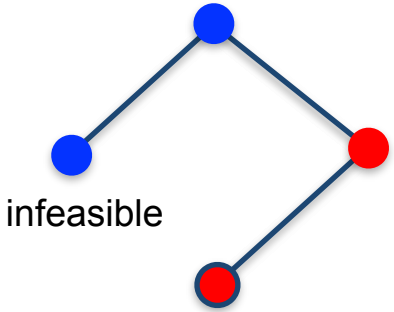


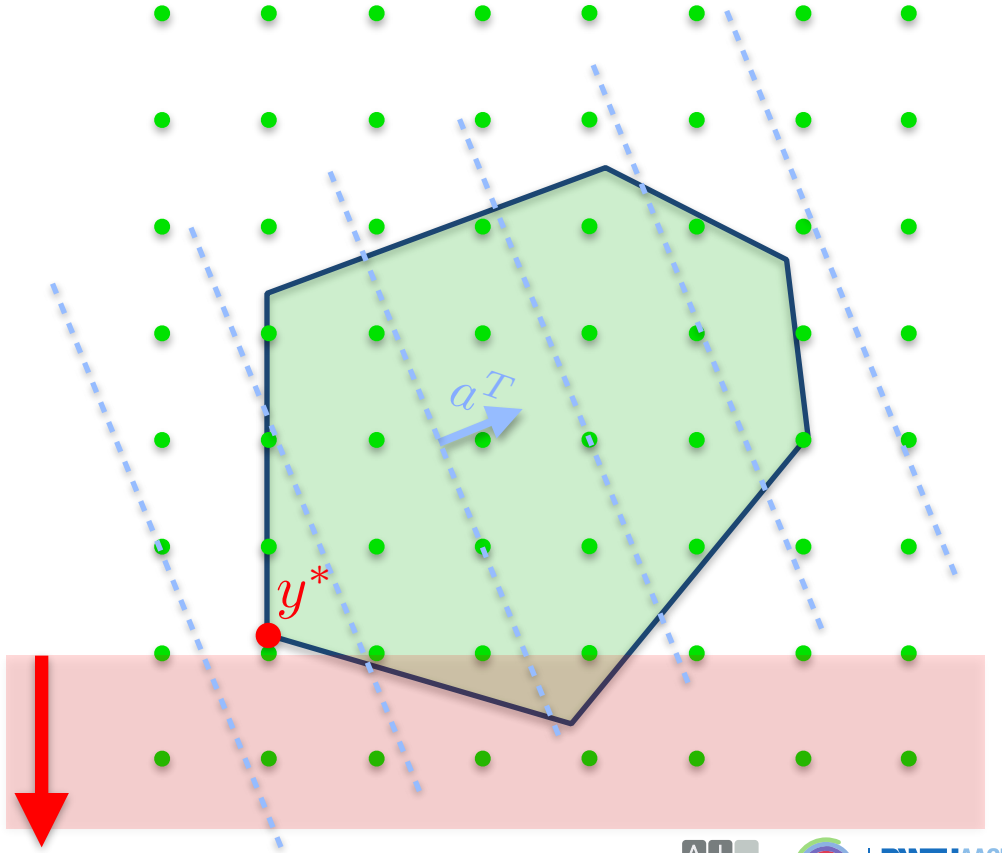
infeasible



Example: Branch and Bound

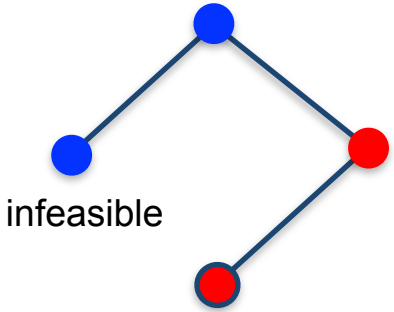
- Integer Linear Program (ILP)

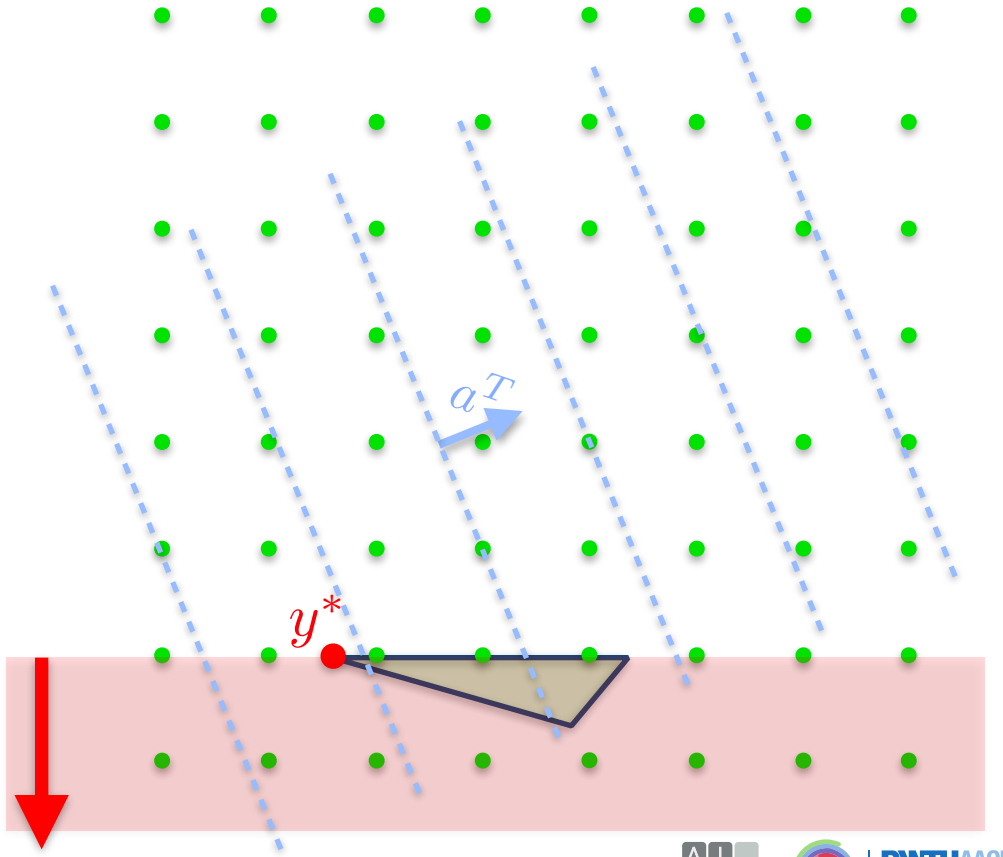




Example: Branch and Bound

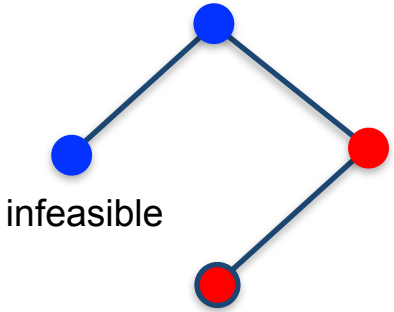
- Integer Linear Program (ILP)

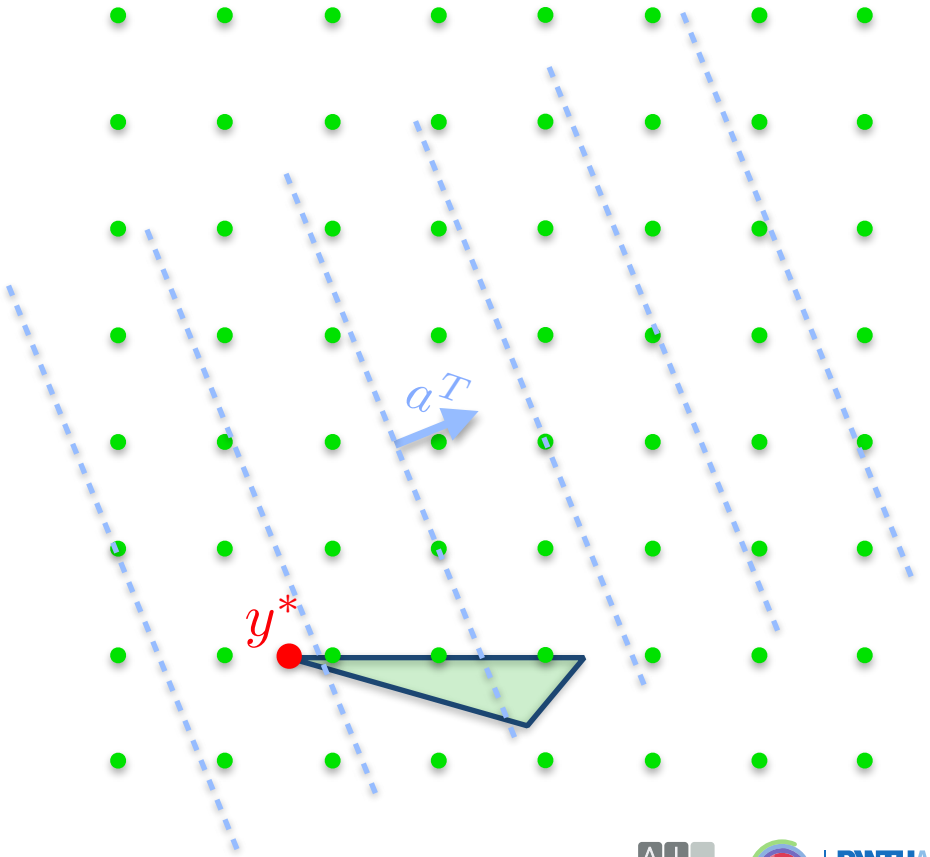




Example: Branch and Bound

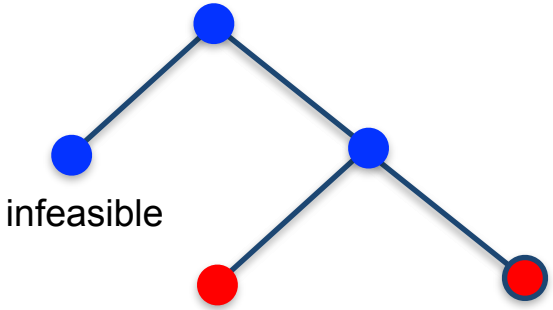
- Integer Linear Program (ILP)

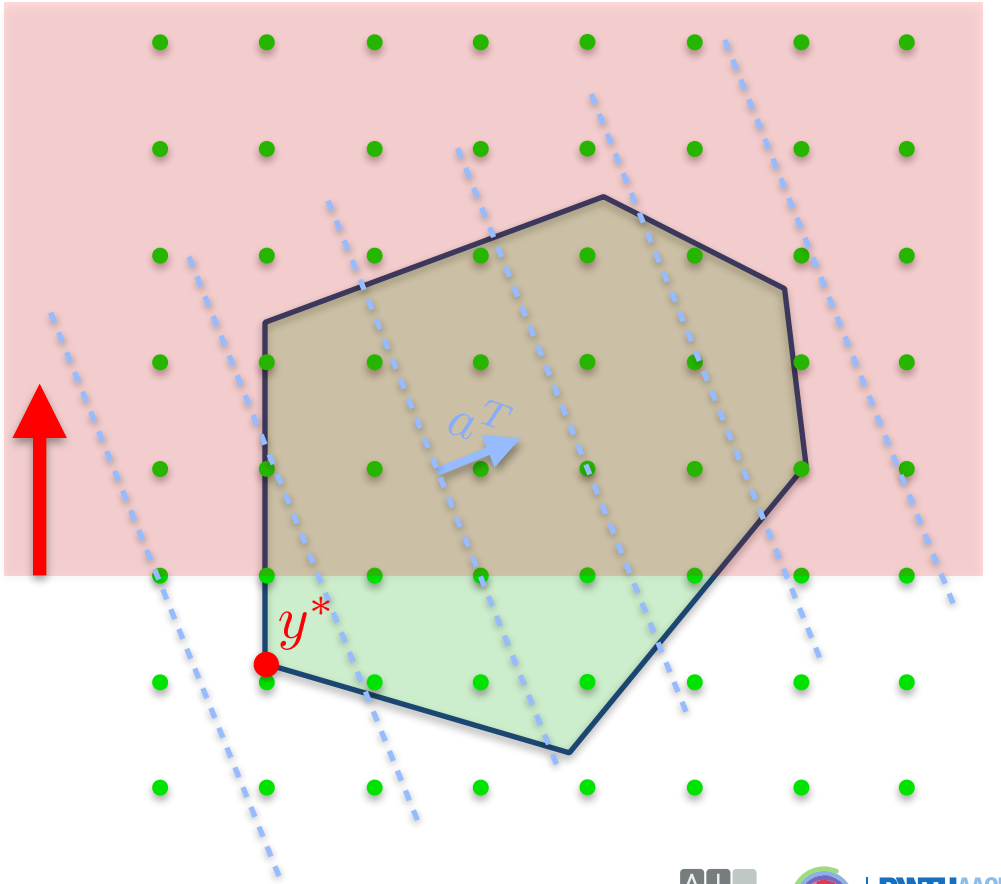




Example: Branch and Bound

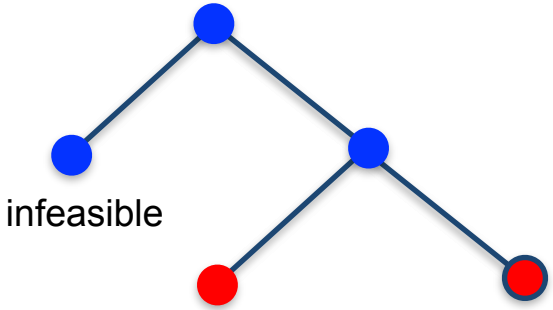
- Integer Linear Program (ILP)

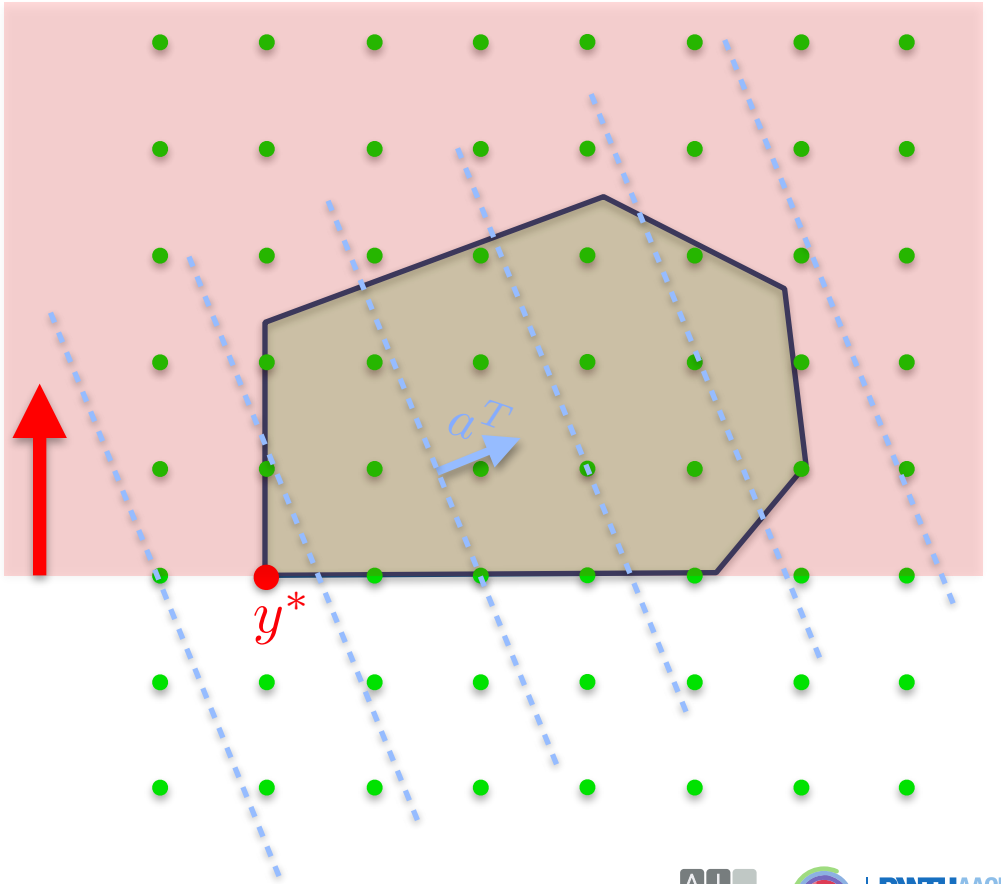




Example: Branch and Bound

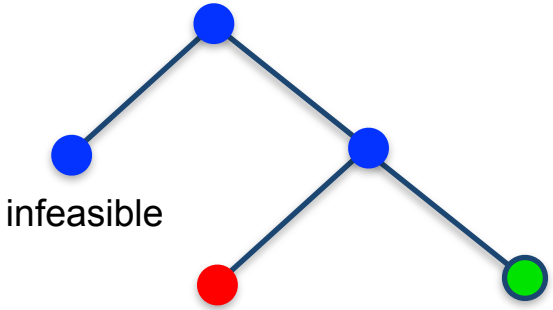
- Integer Linear Program (ILP)

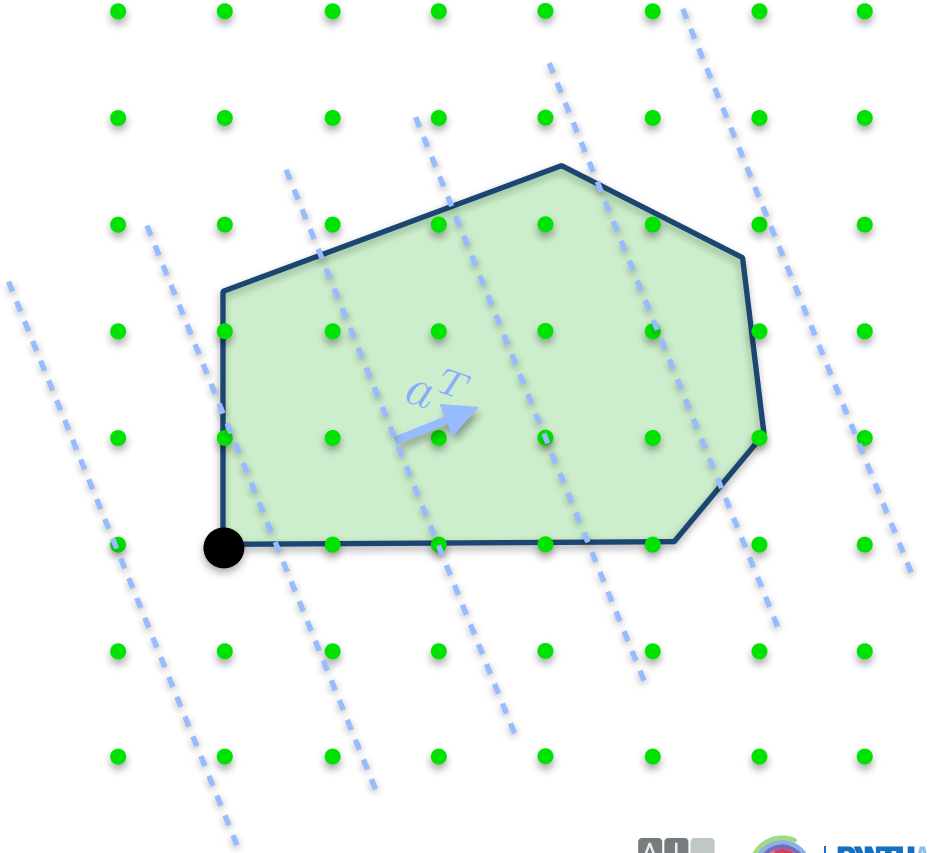




Example: Branch and Bound

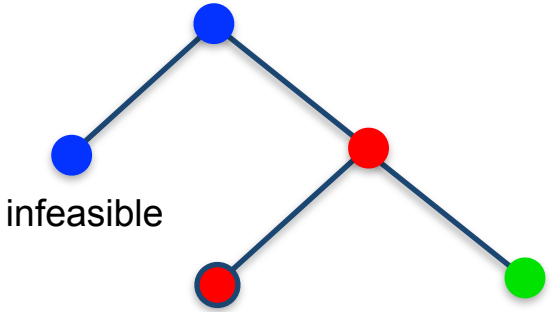
- Integer Linear Program (ILP)

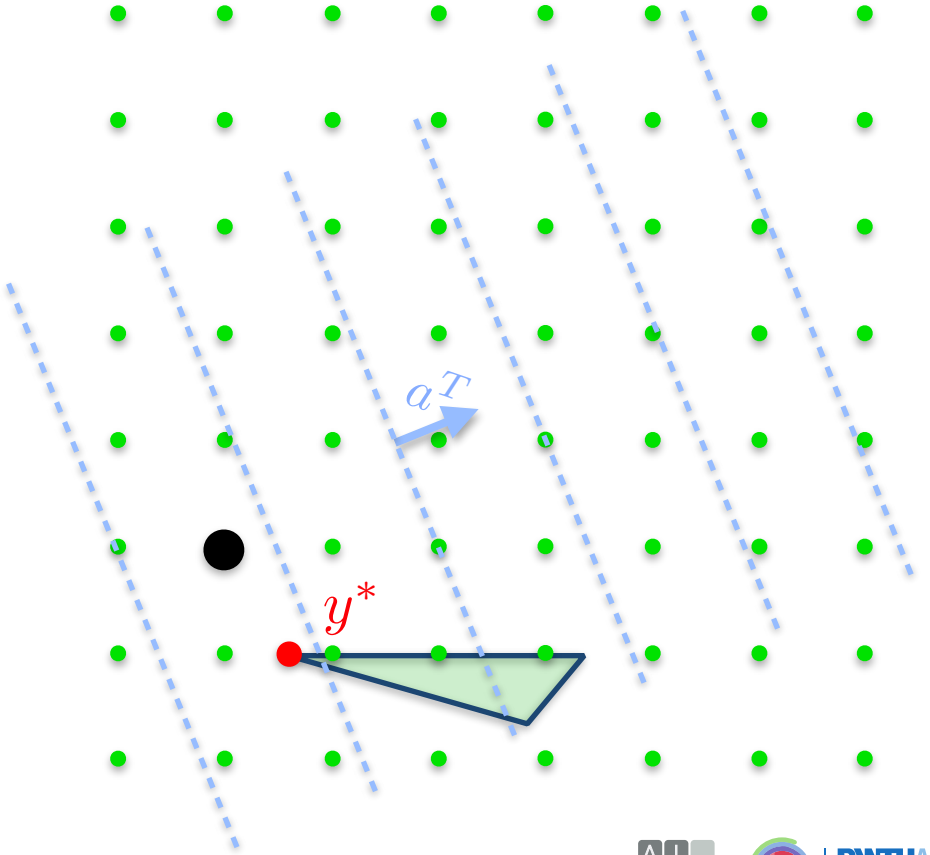




Example: Branch and Bound

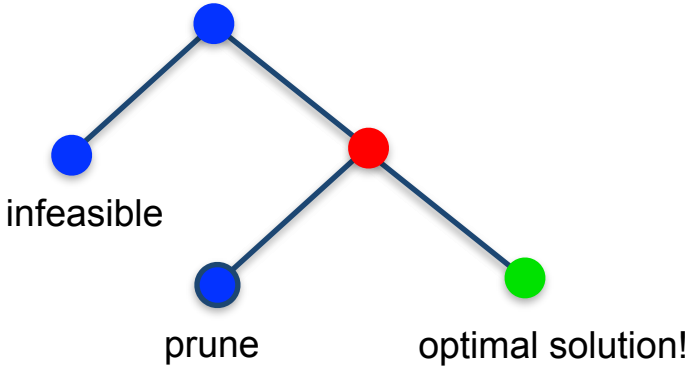
- Integer Linear Program (ILP)

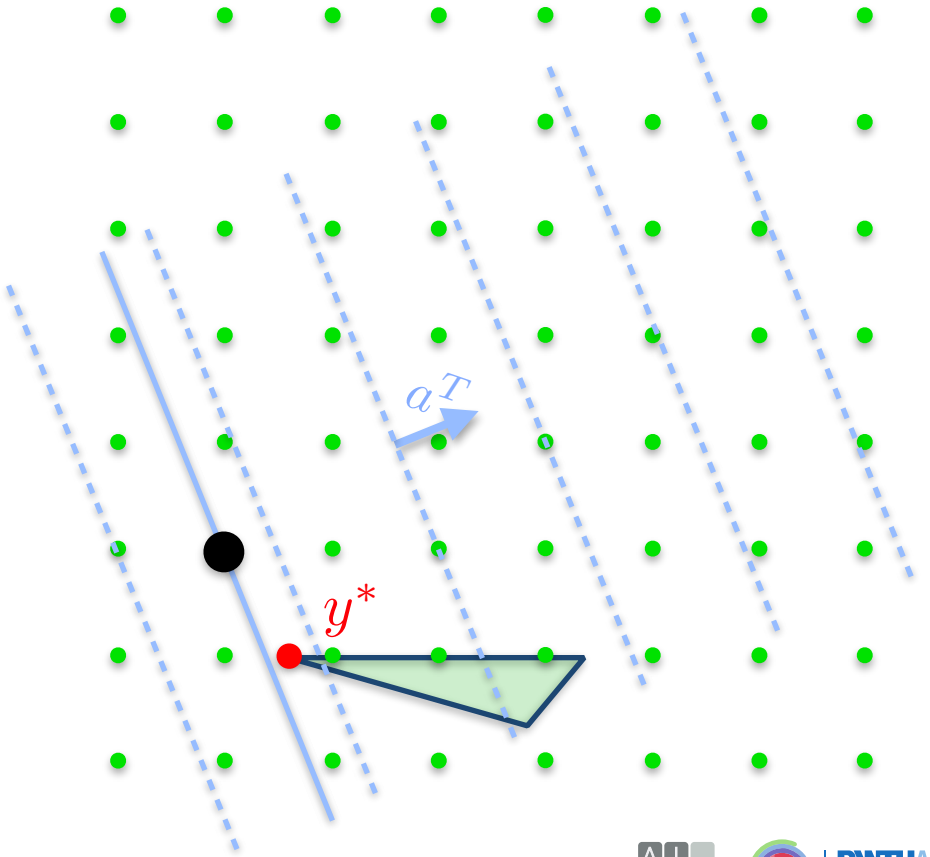




Example: Branch and Bound

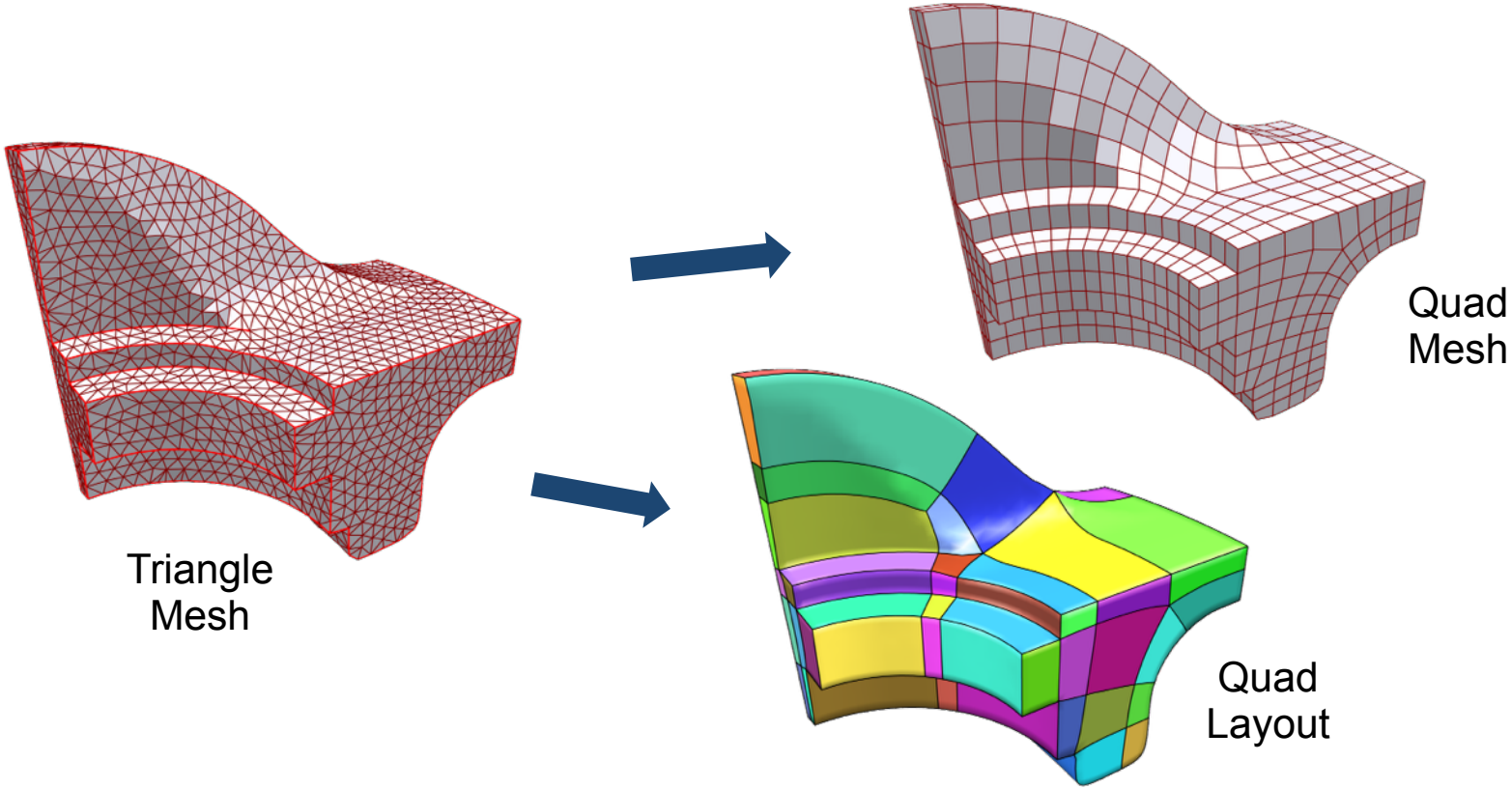
- Integer Linear Program (ILP)



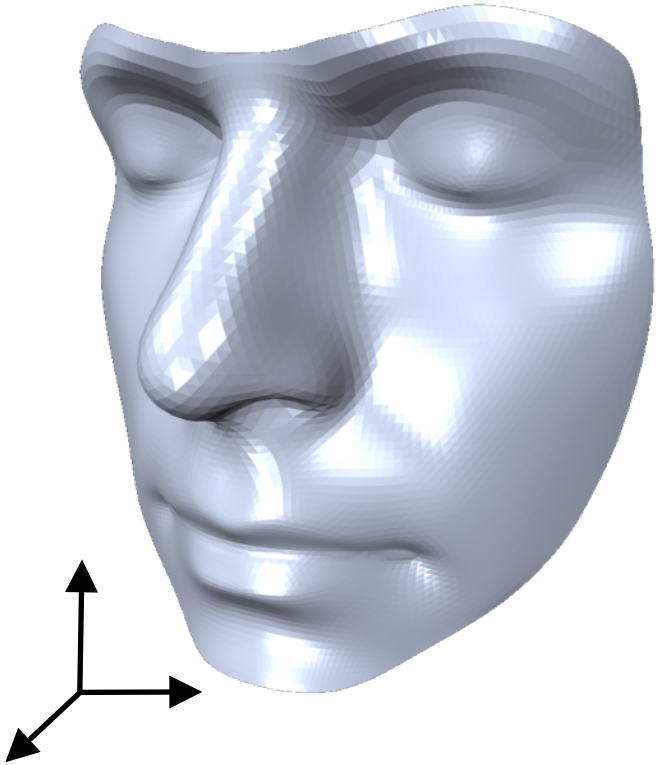


GP Example
Parametrization Based Quad Meshing

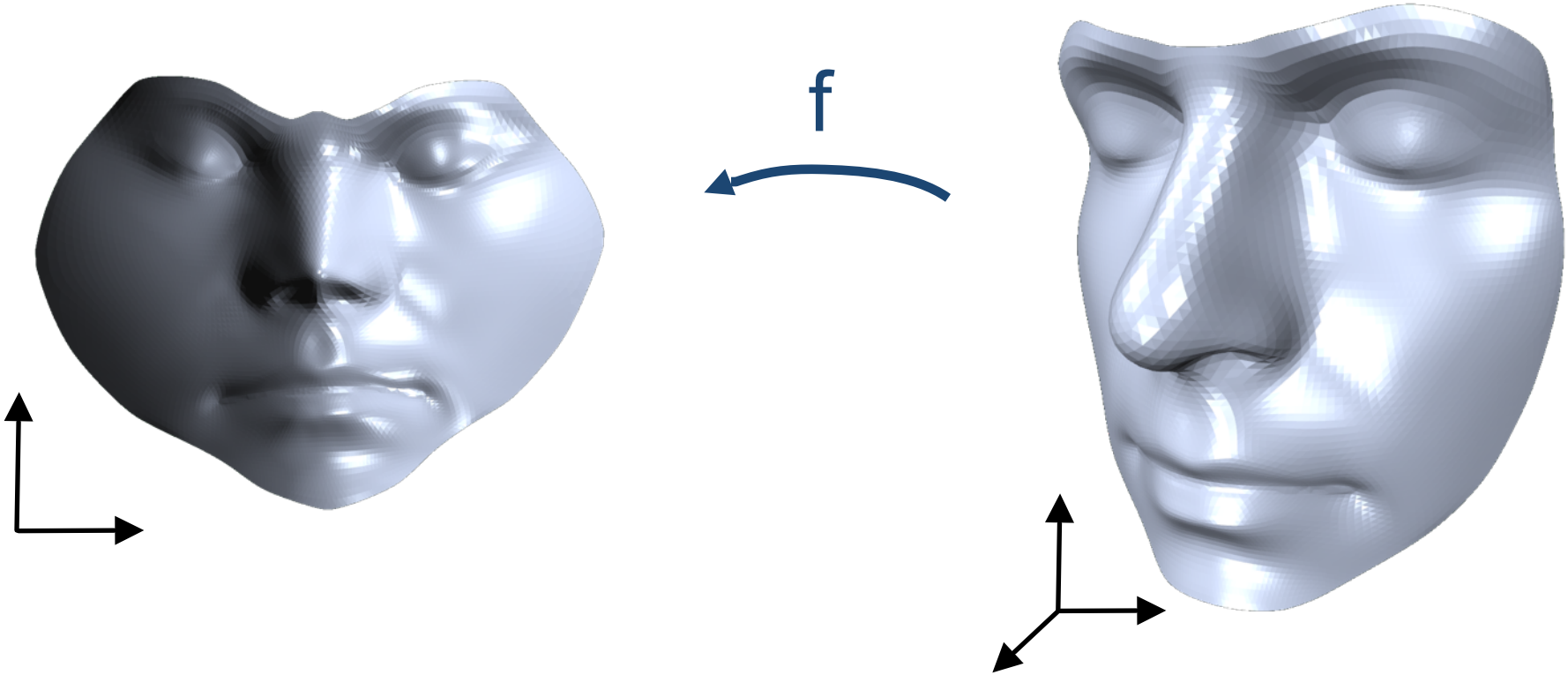
Quad Meshing / Layouting



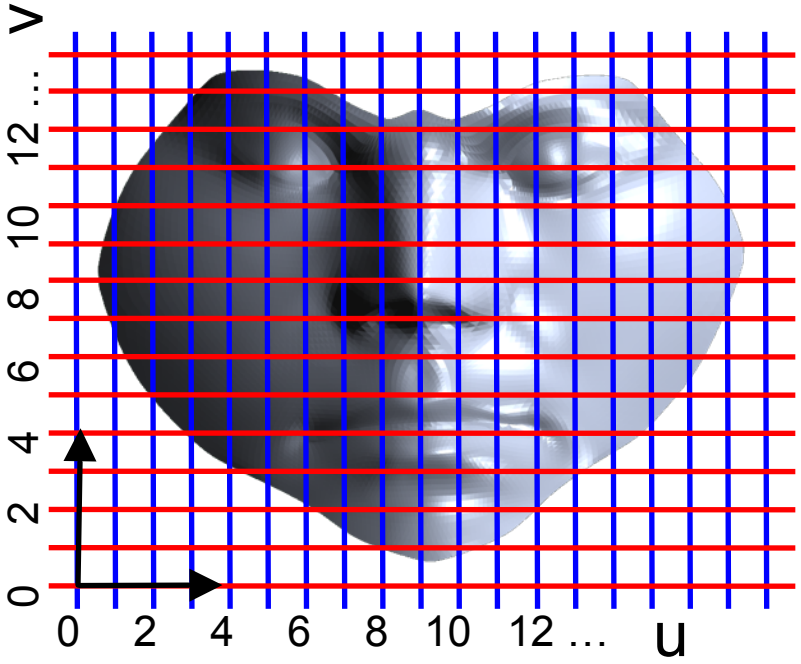
Base Idea



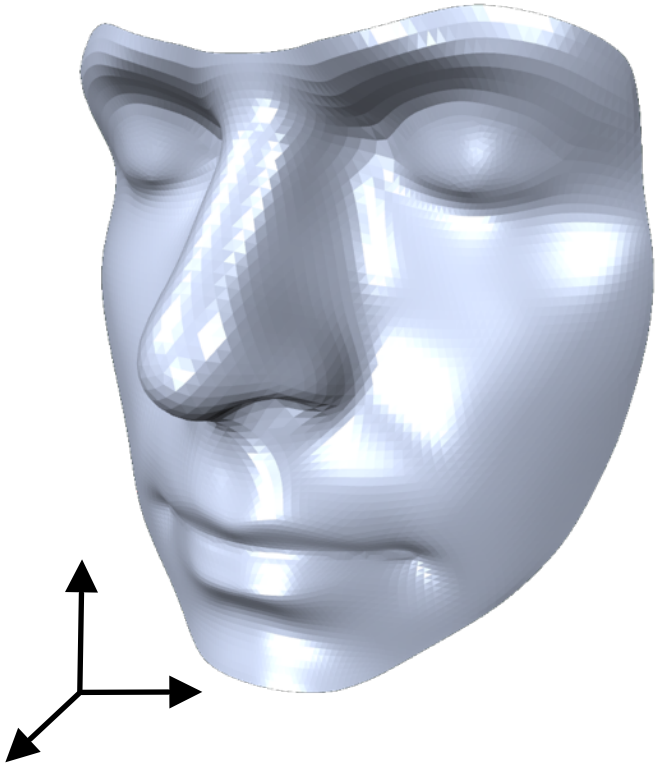
Base Idea



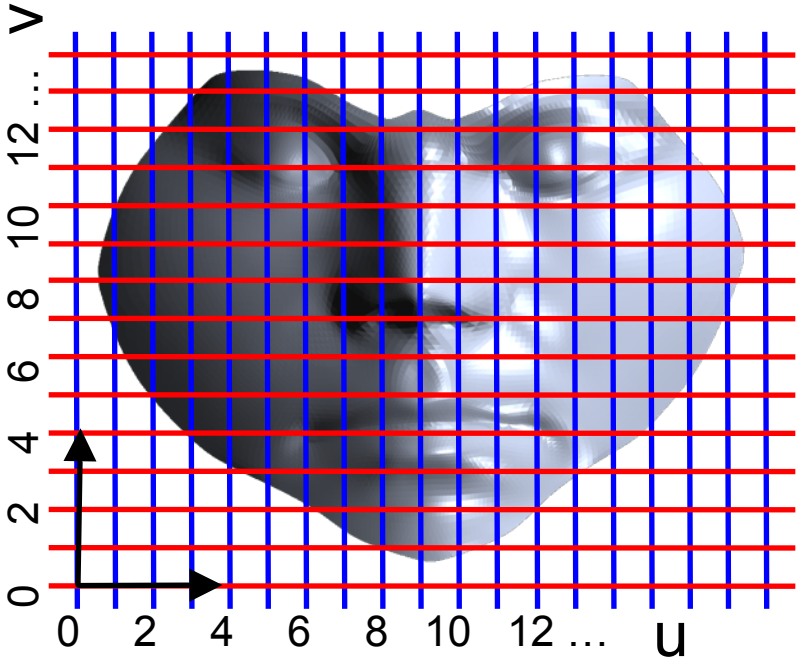
Base Idea



f

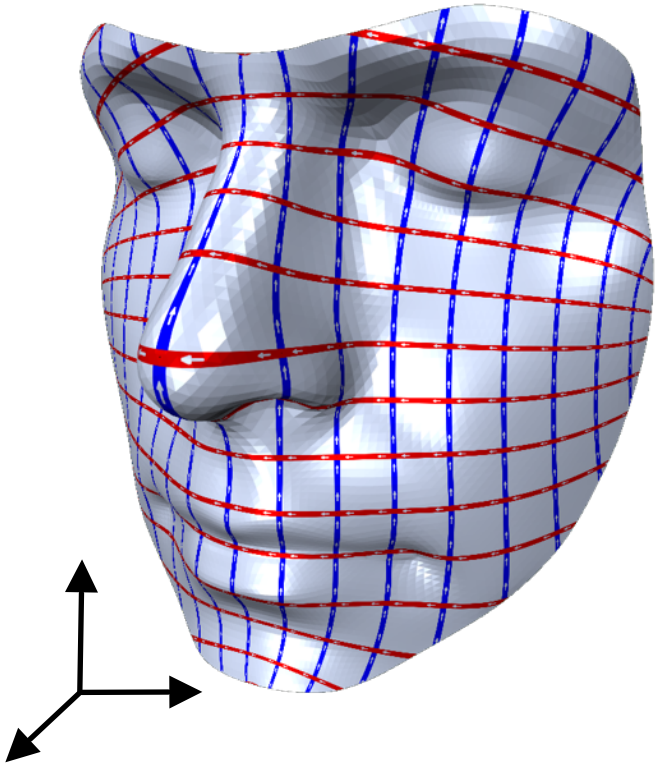


Base Idea



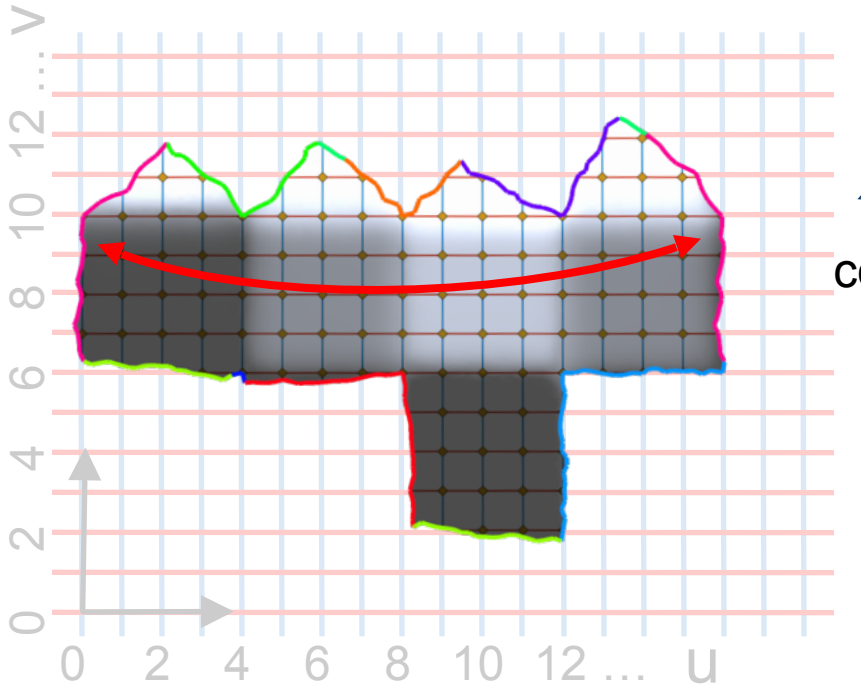
f

f^{-1}



Integer-Grid Maps (IGM)

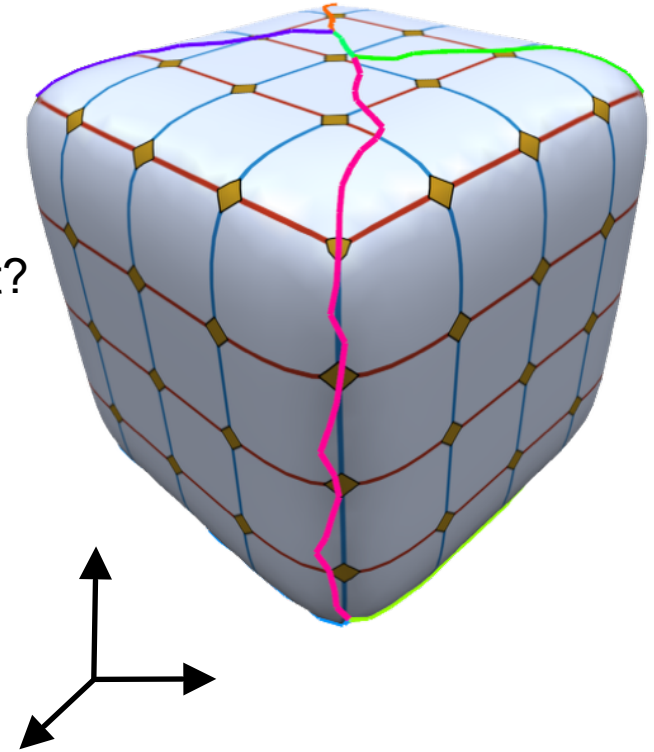
Integer-Grid Maps



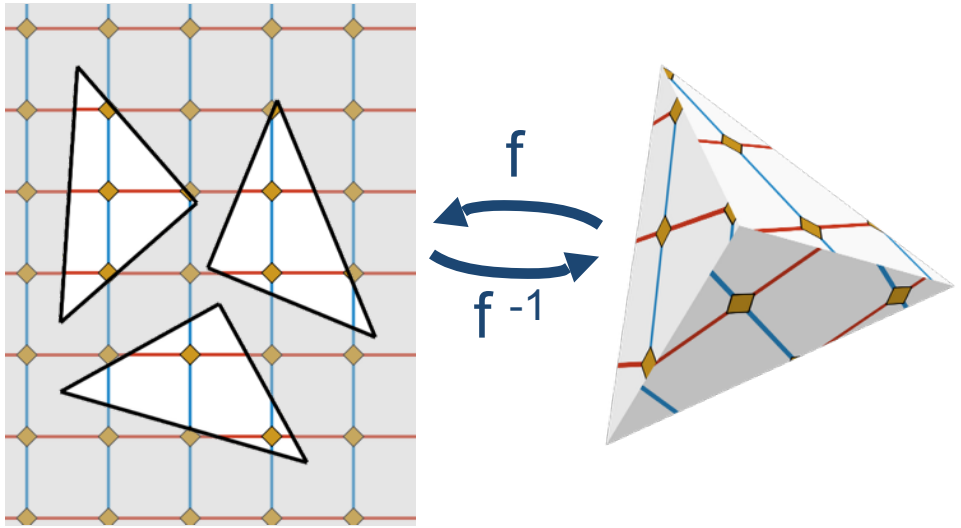
f

continuity at cut?

f^{-1}



Local Conditions of IGM

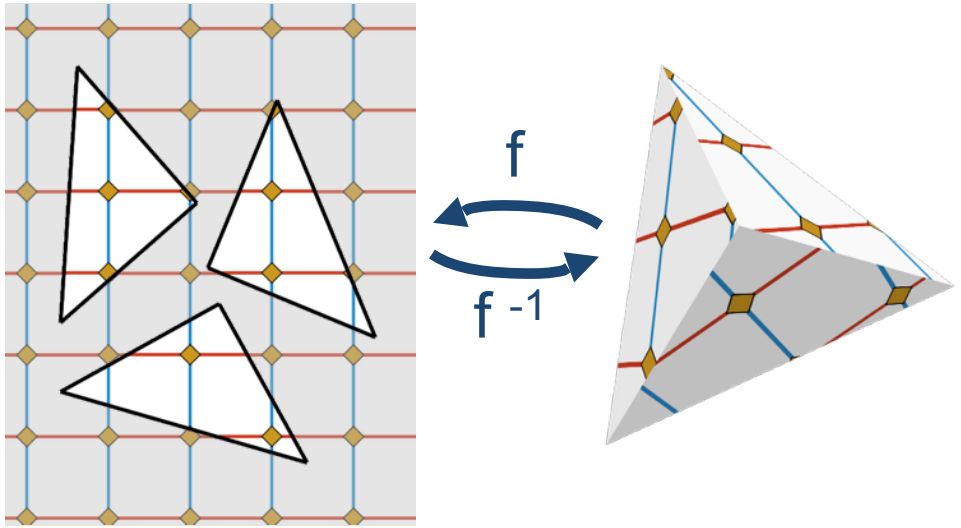


Local Conditions of IGM

1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2$$

$$r_{ij} \in \{0,1,2,3\}$$

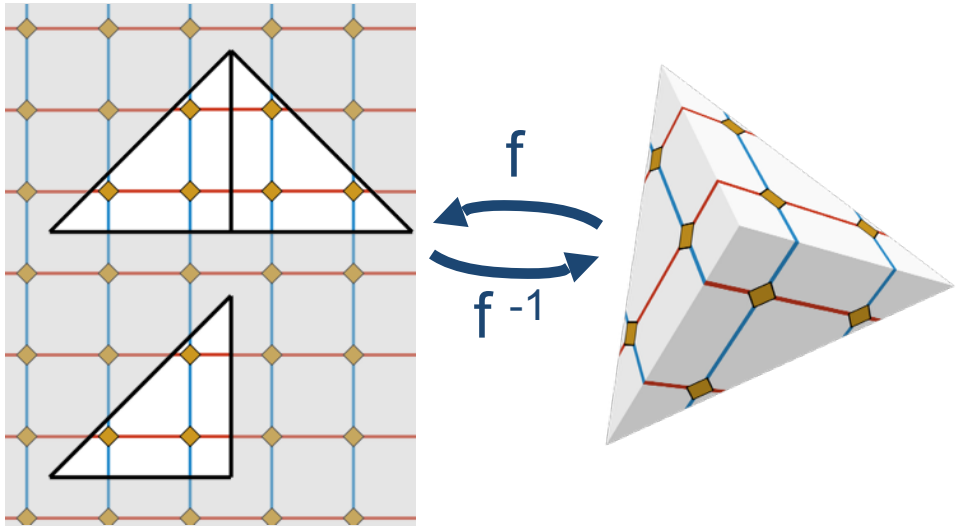


Local Conditions of IGM

1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2$$

$$r_{ij} \in \{0,1,2,3\}$$

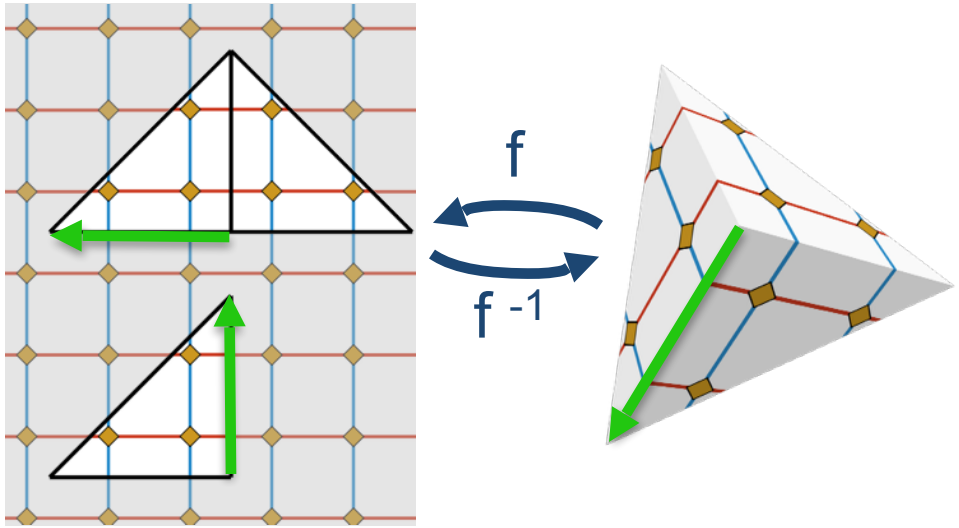


Local Conditions of IGM

1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2$$

$$r_{ij} \in \{0,1,2,3\}$$

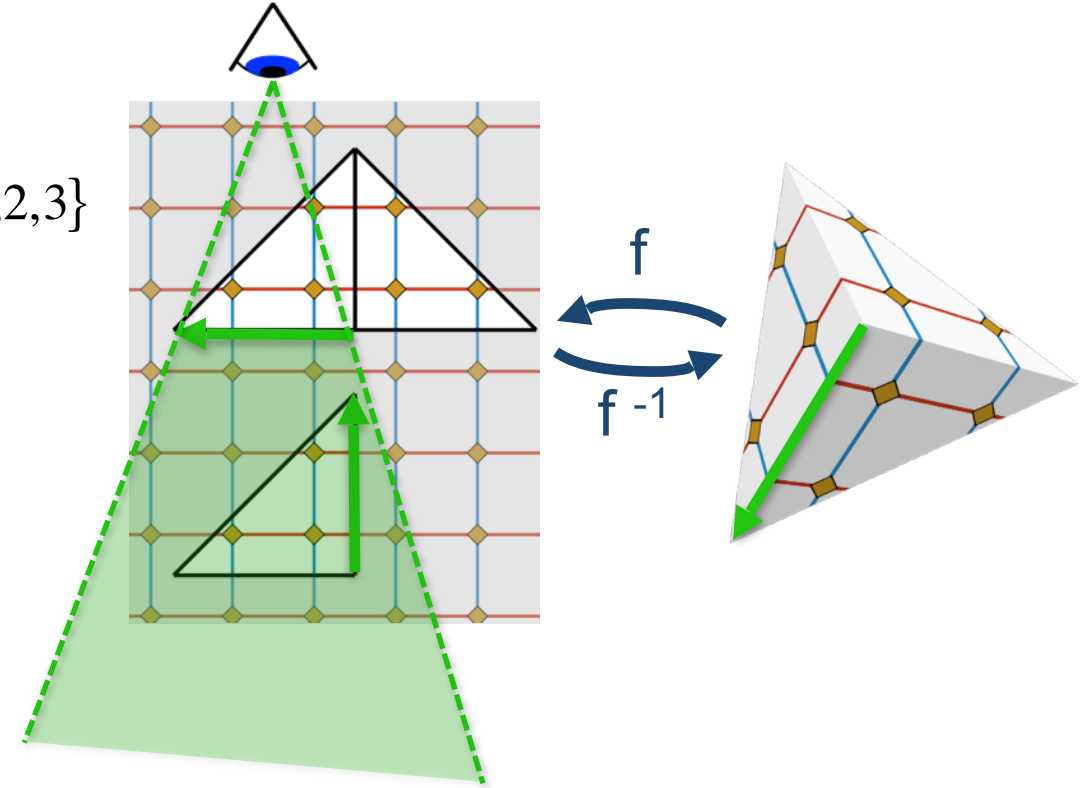


Local Conditions of IGM

1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2$$

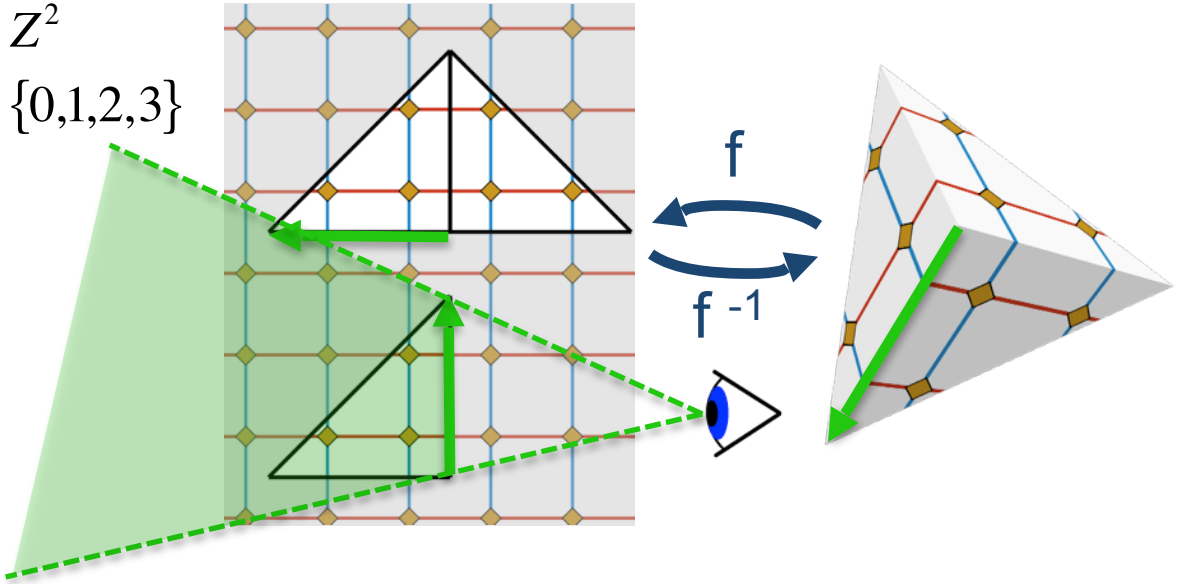
$$r_{ij} \in \{0,1,2,3\}$$



Local Conditions of IGM

1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2 \qquad r_{ij} \in \{0,1,2,3\}$$

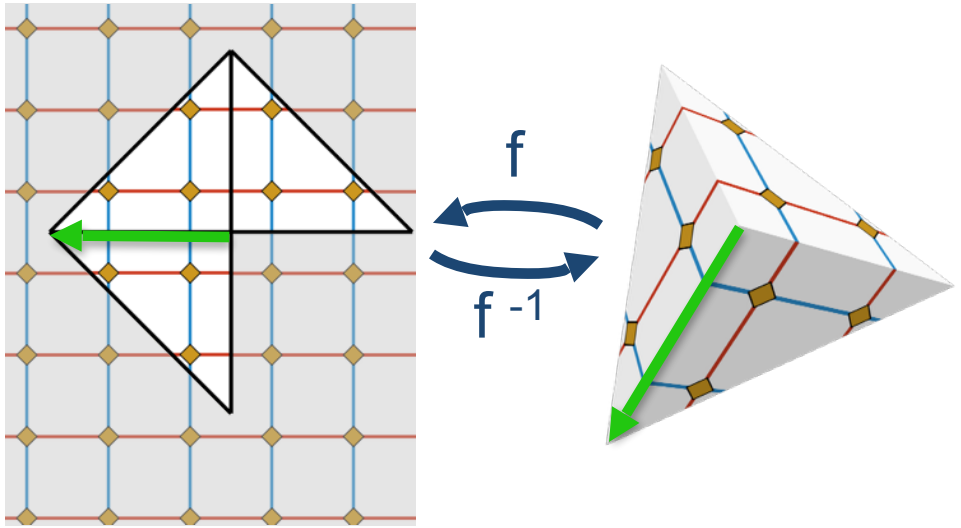


Local Conditions of IGM

1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2$$

$$r_{ij} \in \{0,1,2,3\}$$

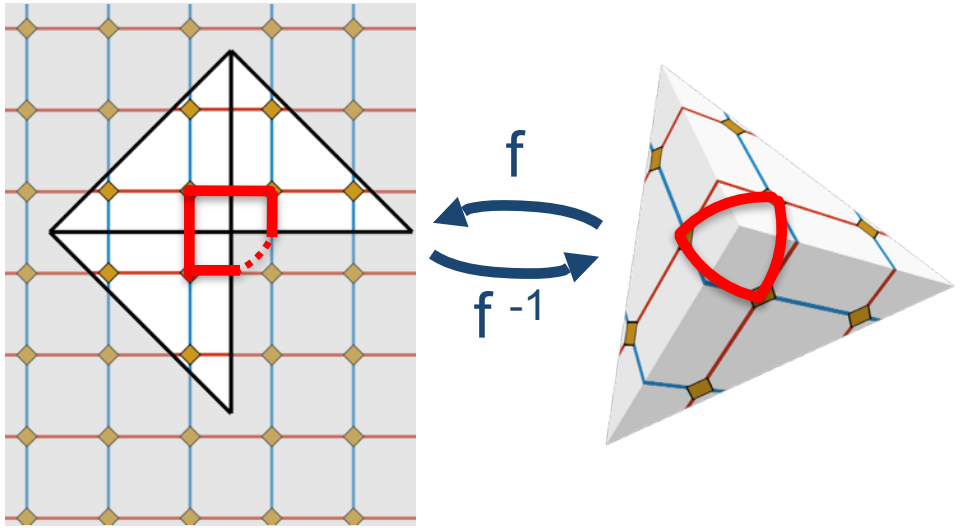


Local Conditions of IGM

1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2$$

$$r_{ij} \in \{0,1,2,3\}$$



Local Conditions of IGM

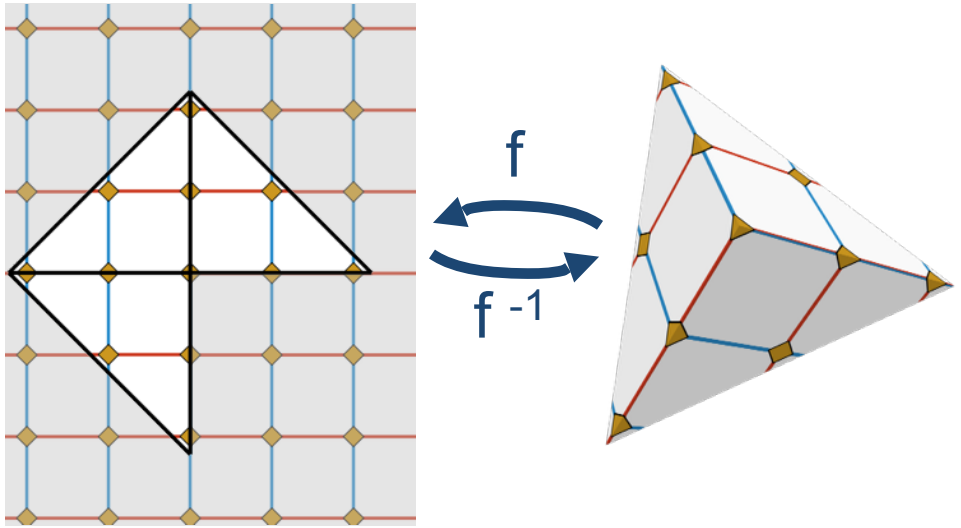
1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2$$

$$r_{ij} \in \{0,1,2,3\}$$

2. Singularities at integer positions:

$$f(\vec{s}_i) \in Z^2 \qquad \forall \vec{s}_i$$



Local Conditions of IGM

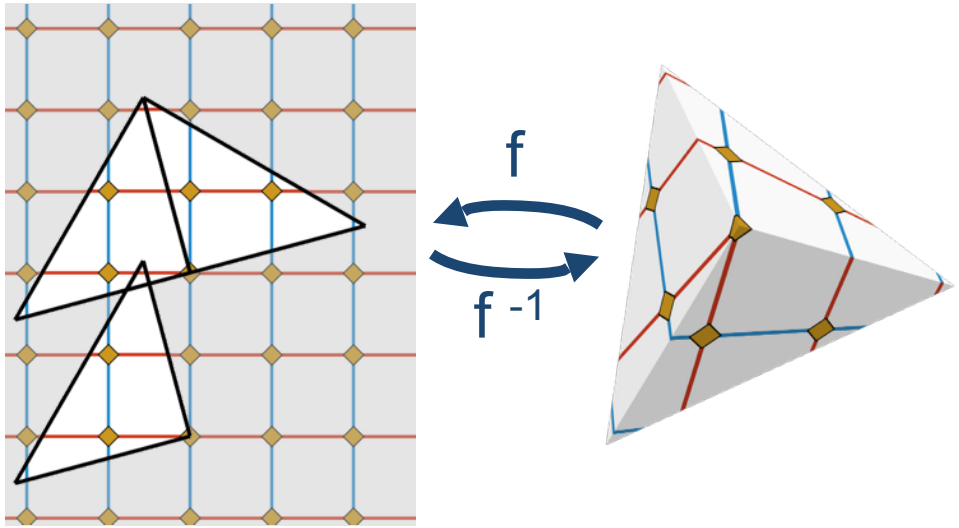
1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2$$

$$r_{ij} \in \{0,1,2,3\}$$

2. Singularities at integer positions:

$$f(\vec{s}_i) \in Z^2 \qquad \forall \vec{s}_i$$



Local Conditions of IGM

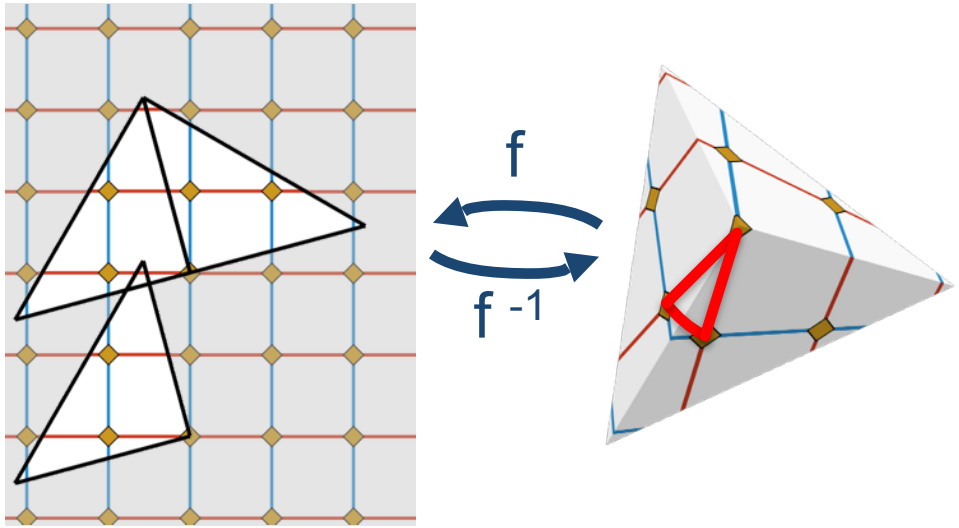
1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2$$

$$r_{ij} \in \{0,1,2,3\}$$

2. Singularities at integer positions:

$$f(\vec{s}_i) \in Z^2 \qquad \forall \vec{s}_i$$



Local Conditions of IGM

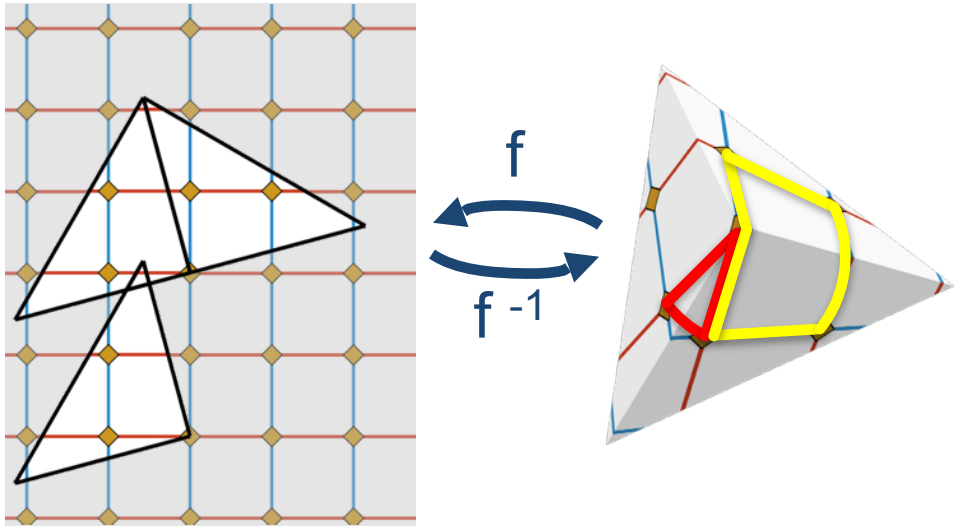
1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2$$

$$r_{ij} \in \{0,1,2,3\}$$

2. Singularities at integer positions:

$$f(\vec{s}_i) \in Z^2 \qquad \forall \vec{s}_i$$



Local Conditions of IGM

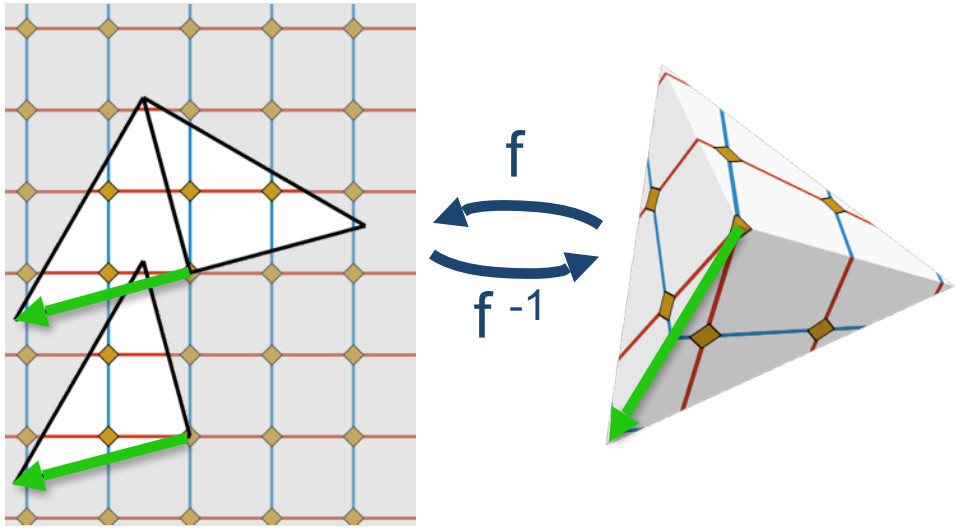
1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2$$

$$r_{ij} \in \{0,1,2,3\}$$

2. Singularities at integer positions:

$$f(\vec{s}_i) \in Z^2 \qquad \forall \vec{s}_i$$



Local Conditions of IGM

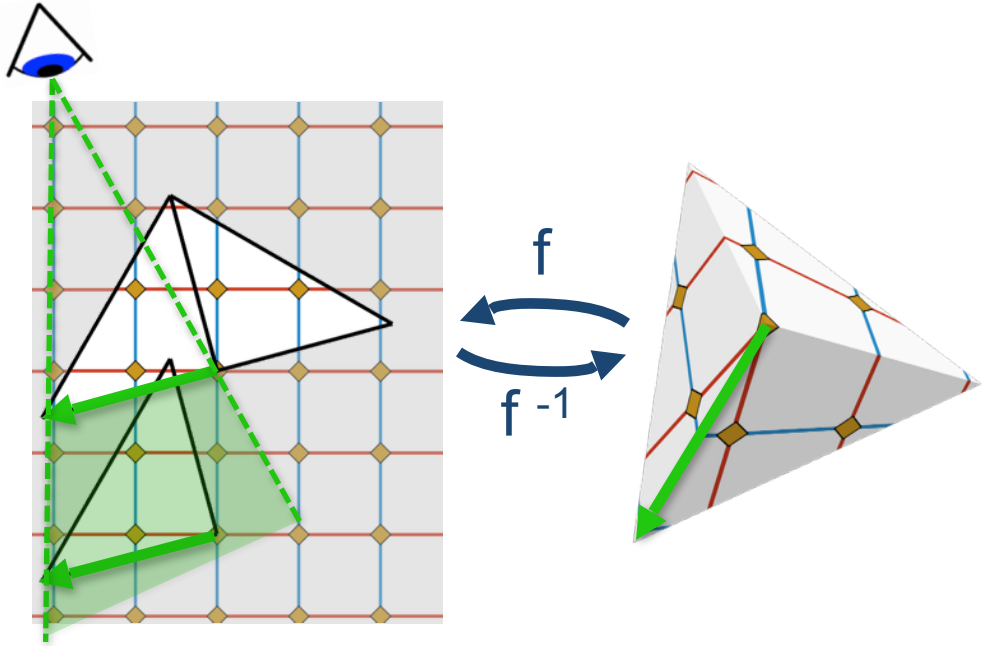
1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2$$

$$r_{ij} \in \{0,1,2,3\}$$

2. Singularities at integer positions:

$$f(\vec{s}_i) \in Z^2 \qquad \forall \vec{s}_i$$



Local Conditions of IGM

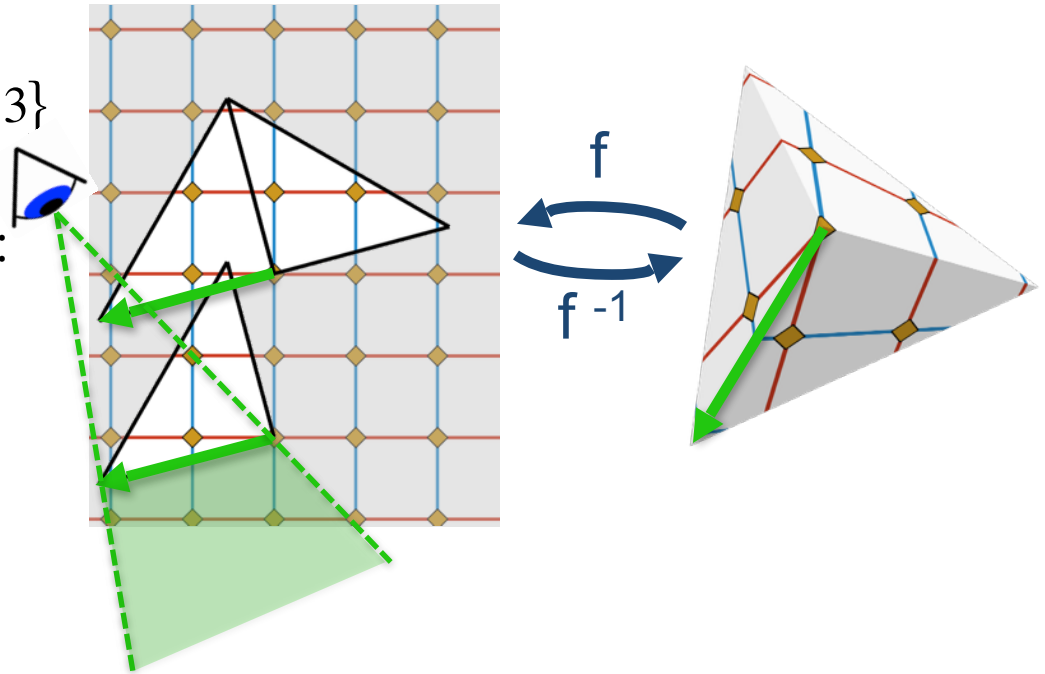
1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2$$

$$r_{ij} \in \{0,1,2,3\}$$

2. Singularities at integer positions:

$$f(\vec{s}_i) \in Z^2 \qquad \forall \vec{s}_i$$



Local Conditions of IGM

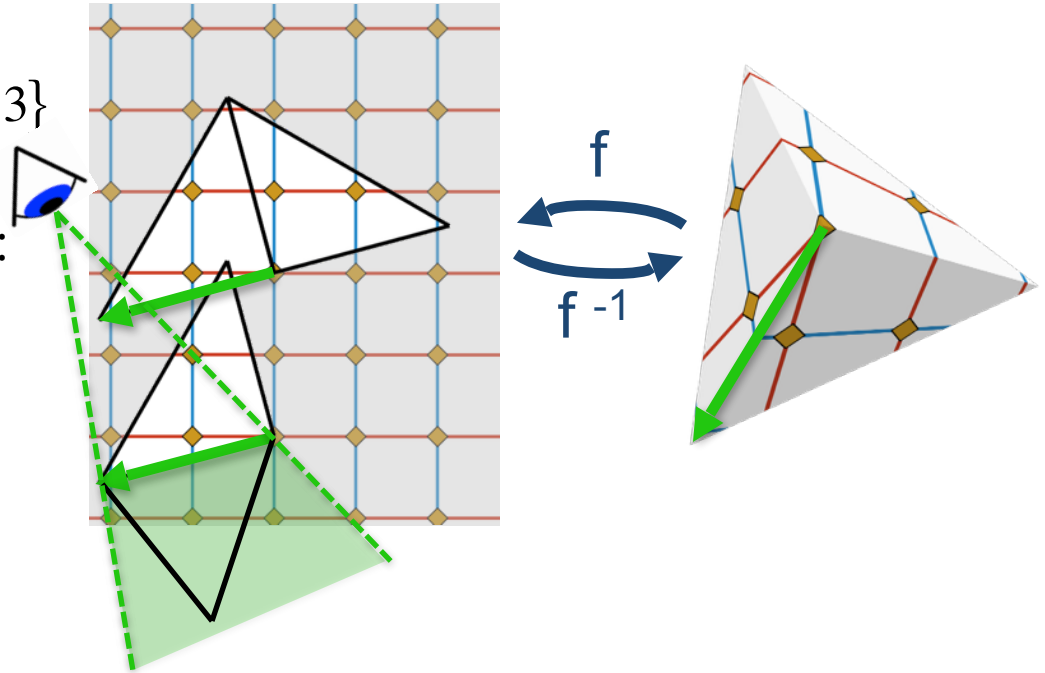
1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2$$

$$r_{ij} \in \{0,1,2,3\}$$

2. Singularities at integer positions:

$$f(\vec{s}_i) \in Z^2 \qquad \forall \vec{s}_i$$



Local Conditions of IGM

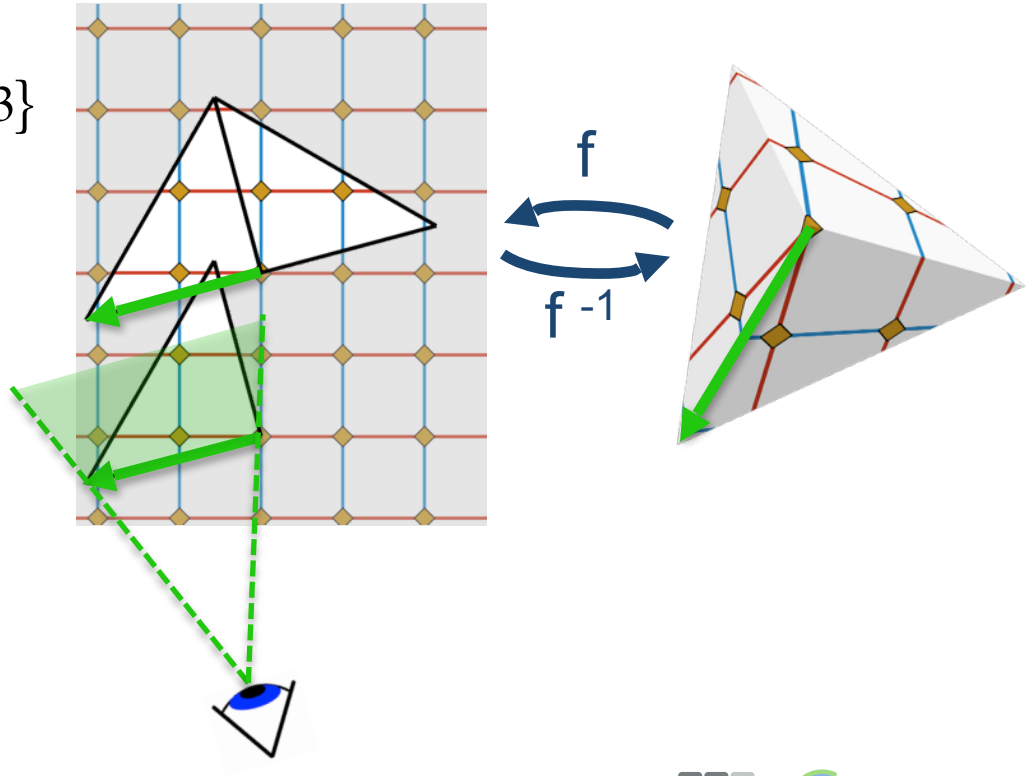
1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2$$

$$r_{ij} \in \{0,1,2,3\}$$

2. Singularities at integer positions:

$$f(\vec{s}_i) \in Z^2 \qquad \forall \vec{s}_i$$



Local Conditions of IGM

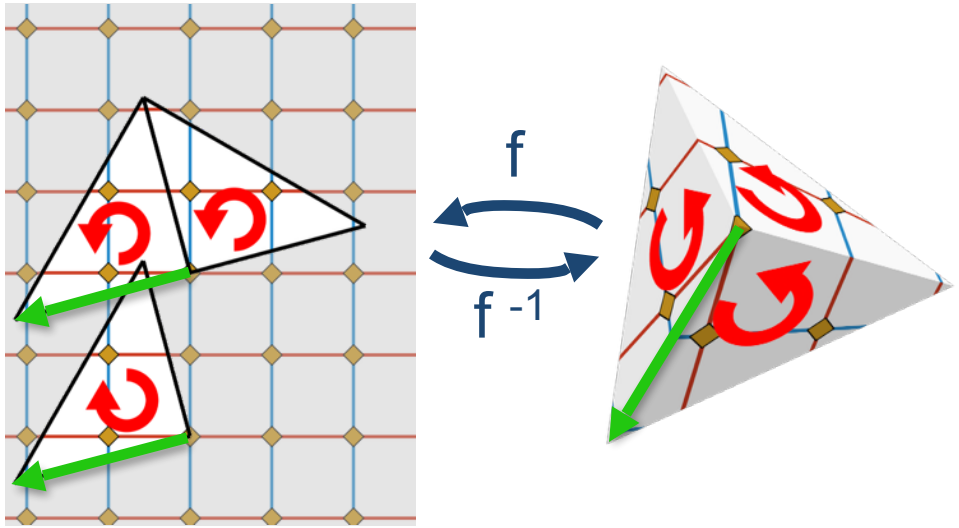
1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2$$

$$r_{ij} \in \{0,1,2,3\}$$

2. Singularities at integer positions:

$$f(\vec{s}_i) \in Z^2 \qquad \forall \vec{s}_i$$



Local Conditions of IGM

1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2$$

$$r_{ij} \in \{0,1,2,3\}$$

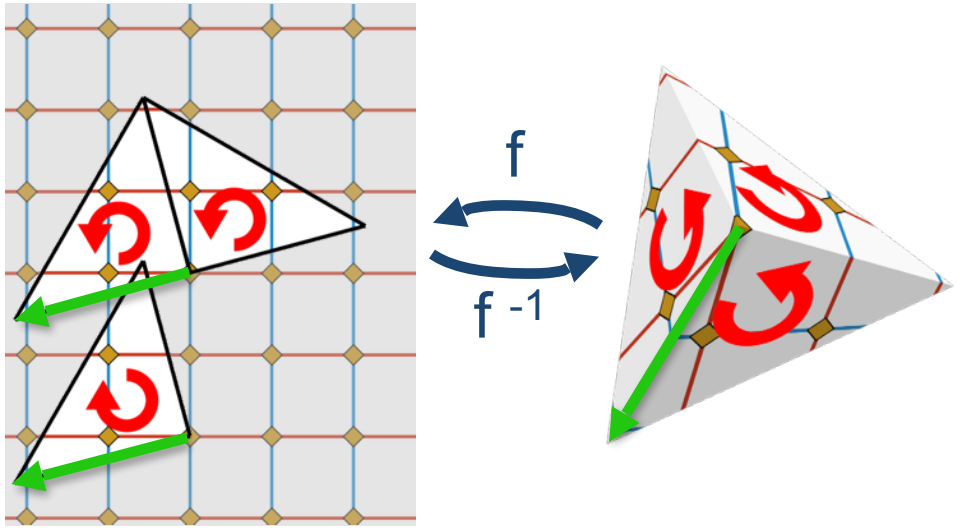
2. Singularities at integer positions:

$$f(\vec{s}_i) \in Z^2 \qquad \forall \vec{s}_i$$

3. Consistent orientation:

for all triangle images (u,v,w)

$$\det(\vec{v}-\vec{u}, \vec{w}-\vec{u}) > 0$$



Local Conditions of IGM

1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2$$

non-linear + integer $\qquad r_{ij} \in \{0,1,2,3\}$

2. Singularities at integer positions:

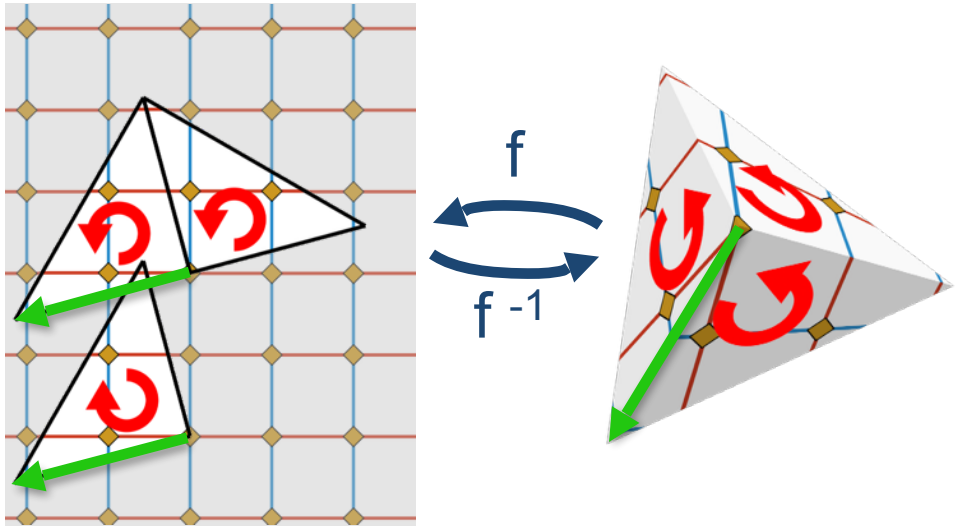
$$f(\vec{s}_i) \in Z^2 \qquad \forall \vec{s}_i$$

3. Consistent orientation:

for all triangle images (u,v,w)

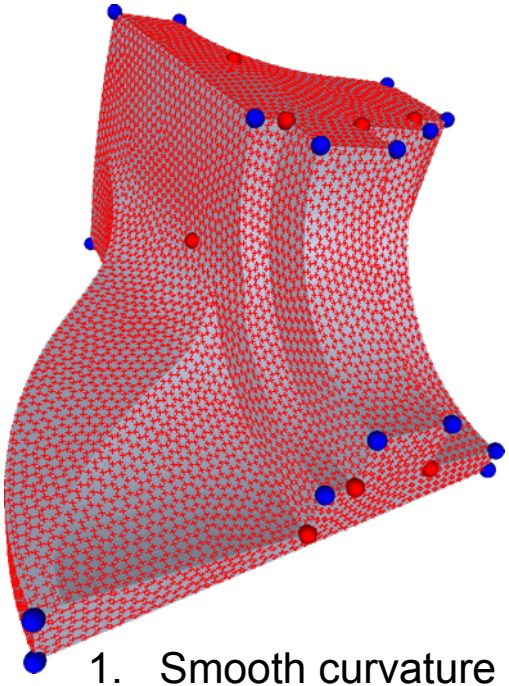
$$\det(\vec{v} - \vec{u}, \vec{w} - \vec{u}) > 0$$

non-linear + non-convex



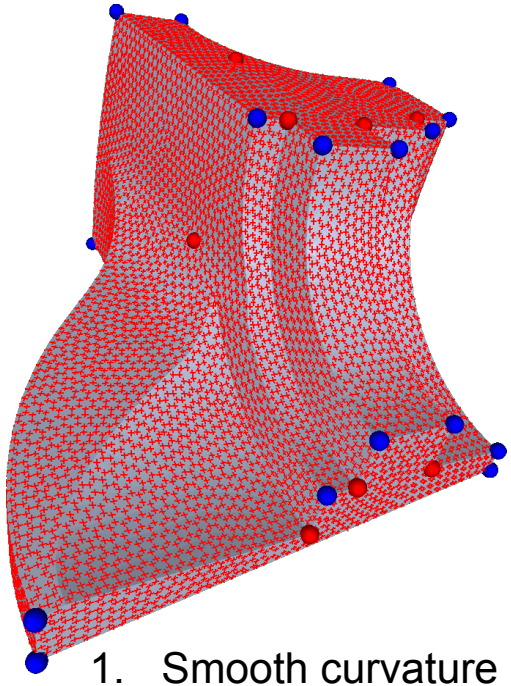
Efficient Search for Integer-Grid Maps

Splitting Approach

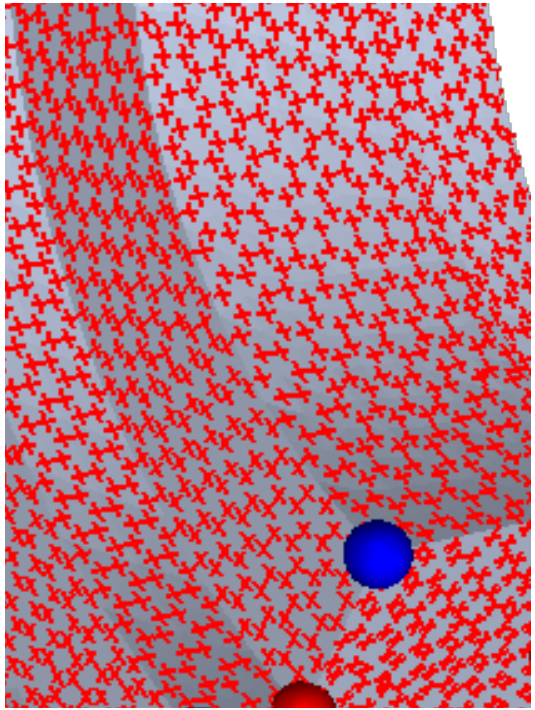


1. Smooth curvature aligned cross field (Approximation + Regularity)

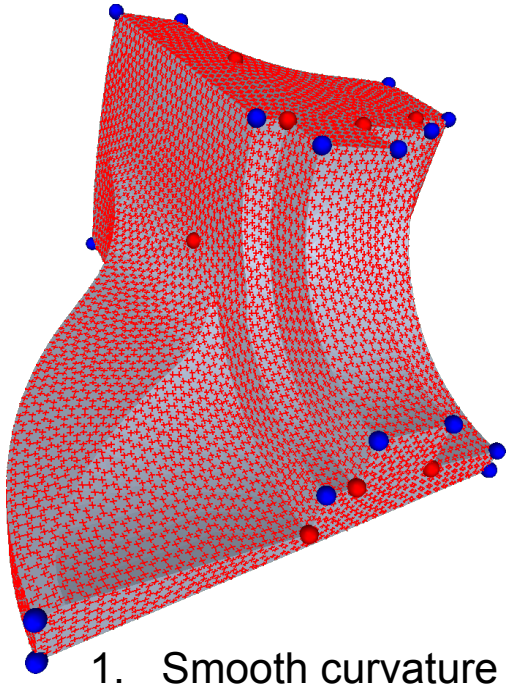
Splitting Approach



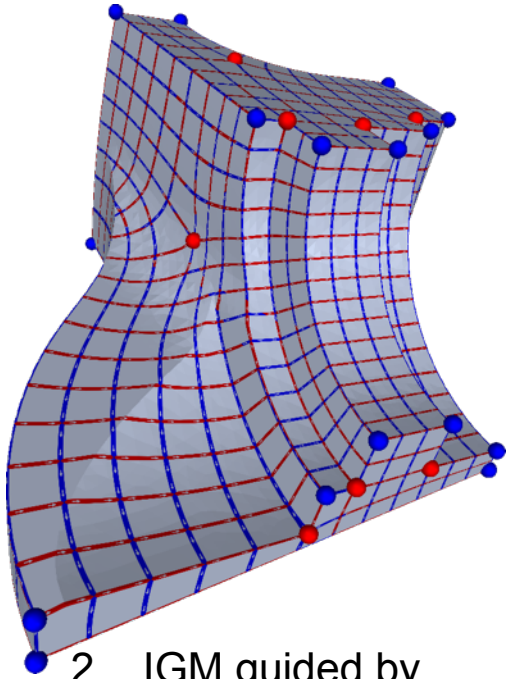
1. Smooth curvature aligned cross field (Approximation + Regularity)



Splitting Approach

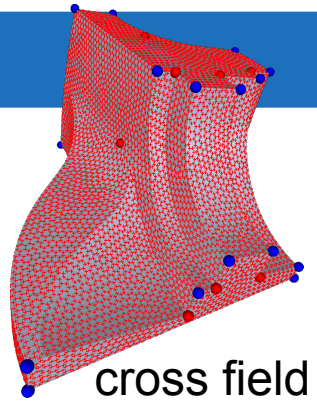


1. Smooth curvature aligned cross field (Approximation + Regularity)



2. IGM guided by cross field (Quad Consistency)

Local Conditions of IGM



1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2$$

non-linear + integer $\qquad r_{ij} \in \{0,1,2,3\}$

2. Singularities at integer positions:

$$f(\vec{s}_i) \in Z^2 \qquad \forall \vec{s}_i$$

3. Consistent orientation:

for all triangle images (u,v,w)

$$\det(\vec{v} - \vec{u}, \vec{w} - \vec{u}) > 0$$

non-linear + non-convex

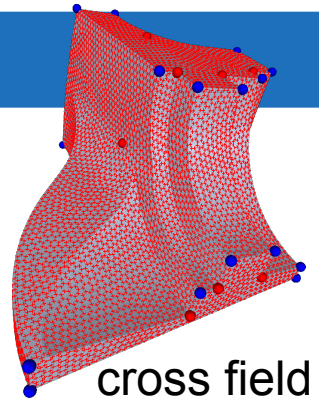
Local Conditions of IGM

1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2$$

non-linear + integer $\qquad r_{ij} \in \{0,1,2,3\}$

→ linear + integer



cross field

2. Singularities at integer positions:

$$f(\vec{s}_i) \in Z^2 \qquad \forall \vec{s}_i$$

3. Consistent orientation:

for all triangle images (u,v,w)

$$\det(\vec{v} - \vec{u}, \vec{w} - \vec{u}) > 0$$

non-linear + non-convex

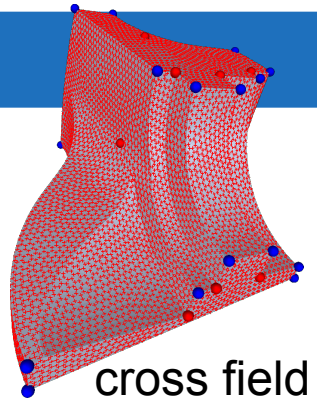
Local Conditions of IGM

1. Transition function per edge:

$$g_{i\to j}(\vec{u}) = R_{90}^{r_{ij}}(\vec{u}) + \vec{t}_{ij} \qquad \vec{t}_{ij} \in Z^2$$

non-linear + integer $\qquad r_{ij} \in \{0,1,2,3\}$

→ linear + integer 



2. Singularities at integer positions:

$$f(\vec{s}_i) \in Z^2 \qquad \forall \vec{s}_i$$

→ known 

3. Consistent orientation:

for all triangle images (u,v,w)

$$\det(\vec{v} - \vec{u}, \vec{w} - \vec{u}) > 0$$

non-linear + non-convex

Local Conditions of IGM

1. Transition function per edge:

$g_{i \to j}(\vec{u})$

non-lin

2. Singul

3. Consis

for all

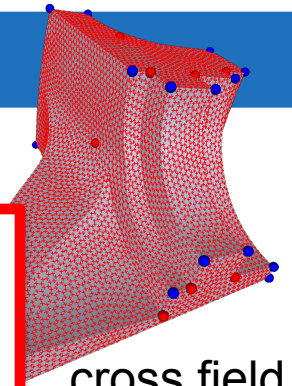
de

non-line

Integer-Grid Map

via

Mixed-Integer Quadratic Programming



cross field

Efficient Search for Integer-Grid Maps

- Branch and cut successful if
  1. convex relaxation 
  2. low-dimensional
  3. well-behaved relaxation

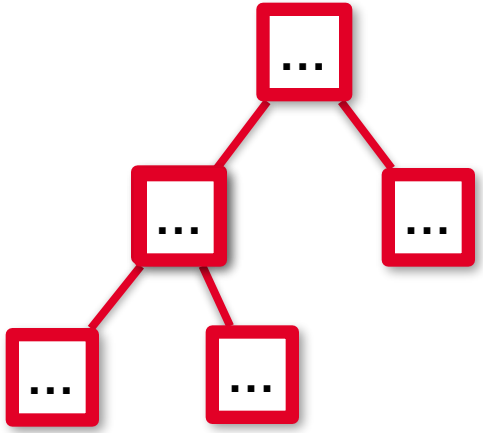
Efficient Search for Integer-Grid Maps

- Branch and cut successful if
  1. convex relaxation 
  2. **low-dimensional**
  3. well-behaved relaxation

Efficient Search for Integer-Grid Maps

- Branch and cut successful if
  1. convex relaxation
  2. **low-dimensional**
  3. well-behaved relaxation

30k triangles



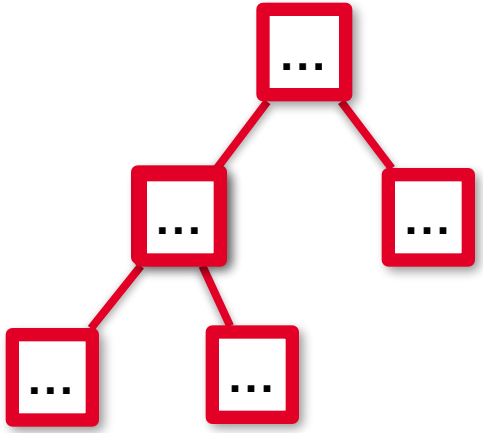
Each node is instance of Quadratic Programming!

Efficient Search for Integer-Grid Maps

- Branch and cut successful if
  1. convex relaxation
  2. **low-dimensional**
  3. well-behaved relaxation

cross field singularities are sufficient for discrete search space

30k triangles

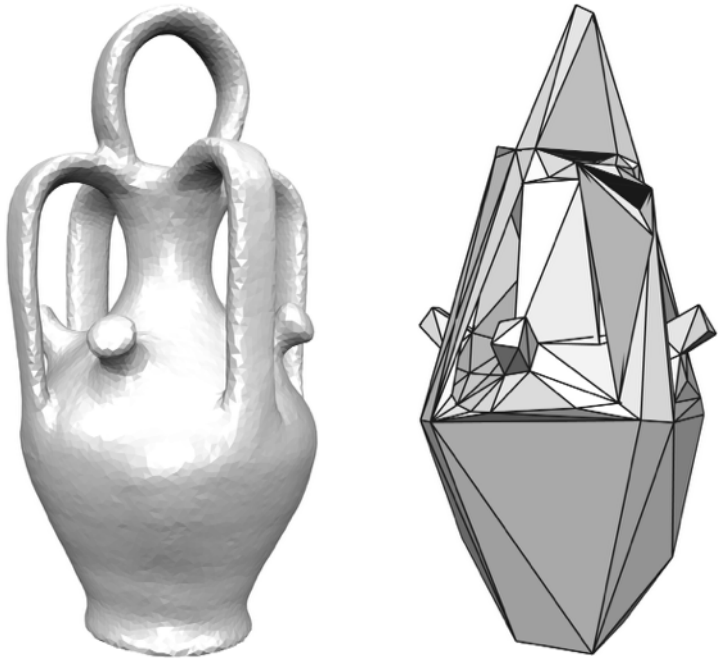


Each node is instance of Quadratic Programming!

Efficient Search for Integer-Grid Maps

- Branch and cut successful if
  1. convex relaxation
  2. **low-dimensional**
  3. well-behaved relaxation

cross field singularities are sufficient for discrete search space



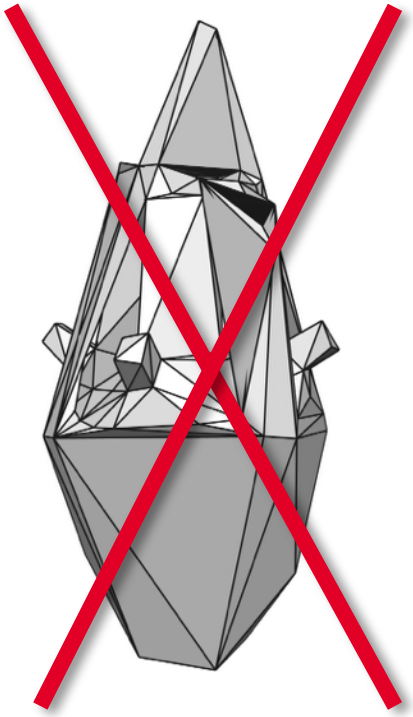
30k triangles

Efficient Search for Integer-Grid Maps

- Branch and cut successful if
  1. convex relaxation
  2. **low-dimensional**
  3. well-behaved relaxation

cross field singularities are sufficient for discrete search space

30k triangles

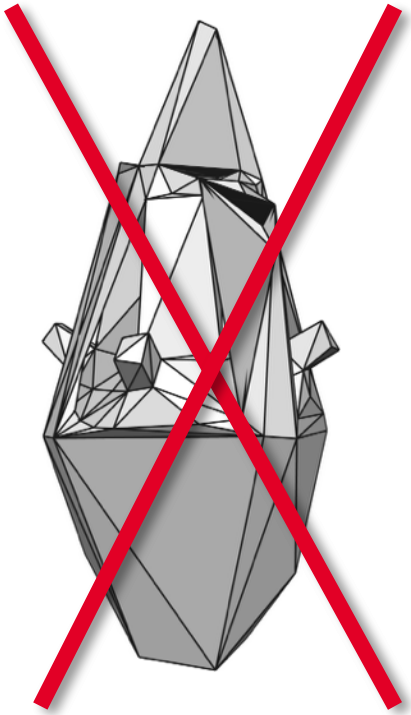


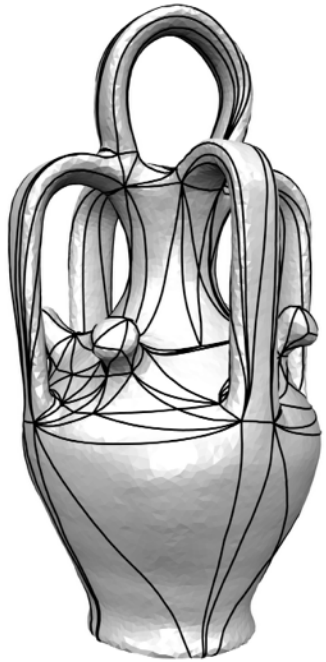
Efficient Search for Integer-Grid Maps

- Branch and cut successful if
  1. convex relaxation
  2. **low-dimensional**
  3. well-behaved relaxation

cross field singularities are sufficient for discrete search space

30k triangles





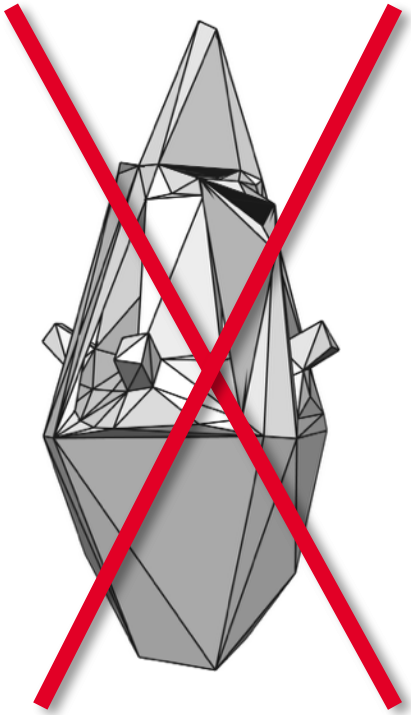
226 triangles

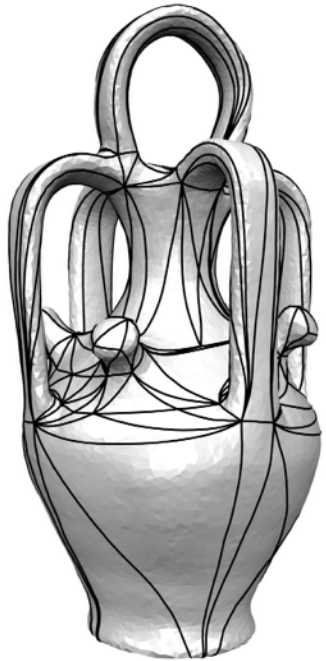
Efficient Search for Integer-Grid Maps

- Branch and cut successful if
  1. convex relaxation
  2. **low-dimensional**
  3. well-behaved relaxation

cross field singularities are sufficient for discrete search space

30k triangles





226 triangles

Efficient Search for Integer-Grid Maps

- Branch and cut successful if
  1. convex relaxation 
  2. low-dimensional 
  3. **well-behaved relaxation**

Efficient Search for Integer-Grid Maps

- Branch and cut successful if
  1. convex relaxation
  2. low-dimensional
  3. **well-behaved relaxation**

Mixed-Integer minimizer : $(X^*, Y^*) = \min\{E(x,y)\}, \quad X \in R^n, Y \in Z^m$

Minimizer of relaxation: $(x^*, y^*) = \min\{E(x,y)\}, \quad x \in R^n, y \in R^m$

$\|Y^* - y^*\|$ small

Efficient Search for Integer-Grid Maps

- Branch and cut successful if
  1. convex relaxation
  2. low-dimensional
  3. **well-behaved relaxation**

Mixed-Integer minimizer : $(X^*, Y^*) = \min\{E(x,y)\}, \quad X \in R^n, Y \in Z^m$

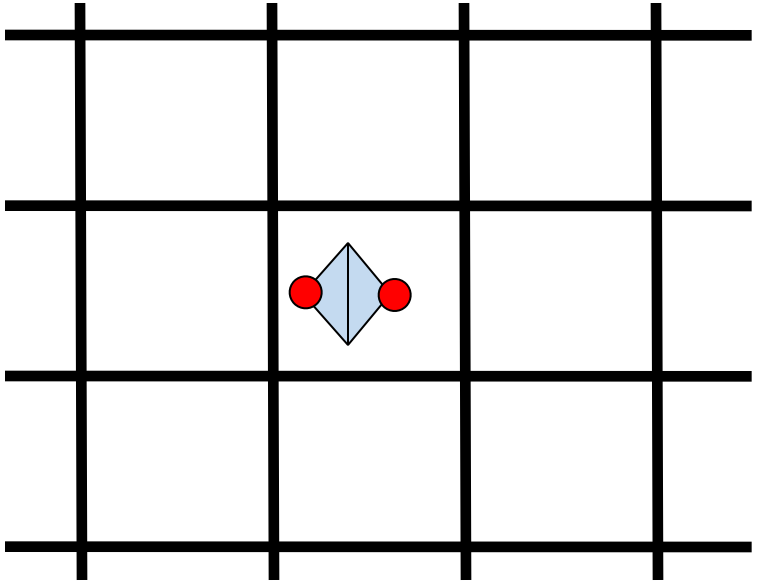
Minimizer of relaxation: $(x^*, y^*) = \min\{E(x,y)\}, \quad x \in R^n, y \in R^m$

$\|Y^* - y^*\|$ small

Recall that singularities get mapped to integer positions!

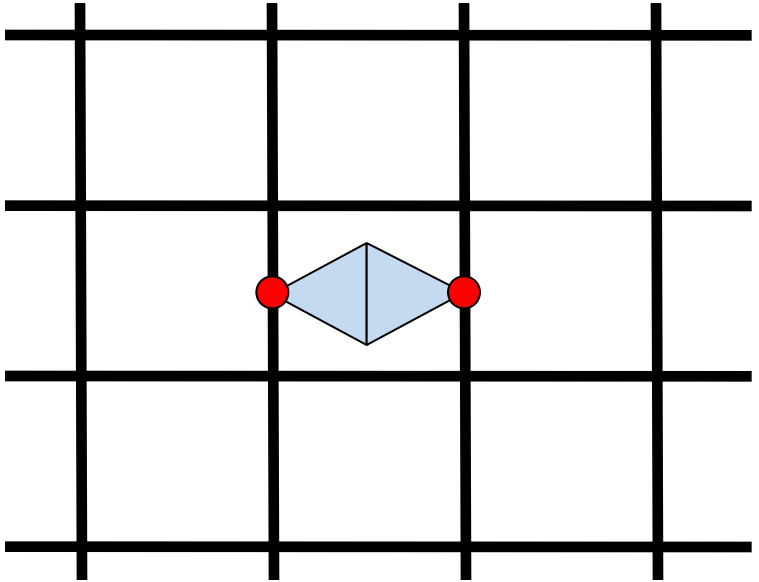
Efficient Search for Integer-Grid Maps

- Branch and cut successful if
  1. convex relaxation
  2. low-dimensional
  3. **well-behaved relaxation**



Efficient Search for Integer-Grid Maps

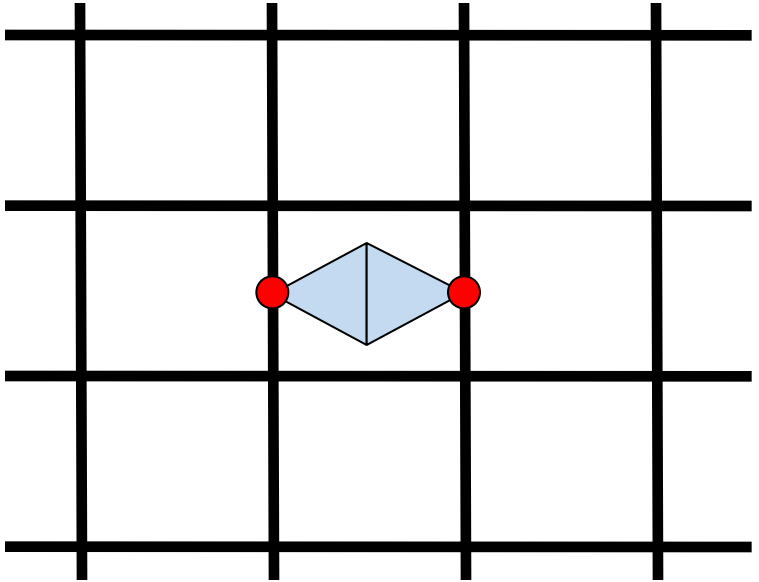
- Branch and cut successful if
  1. convex relaxation
  2. low-dimensional
  3. **well-behaved relaxation**



Efficient Search for Integer-Grid Maps

- Branch and cut successful if
  1. convex relaxation
  2. low-dimensional
  3. **well-behaved relaxation**

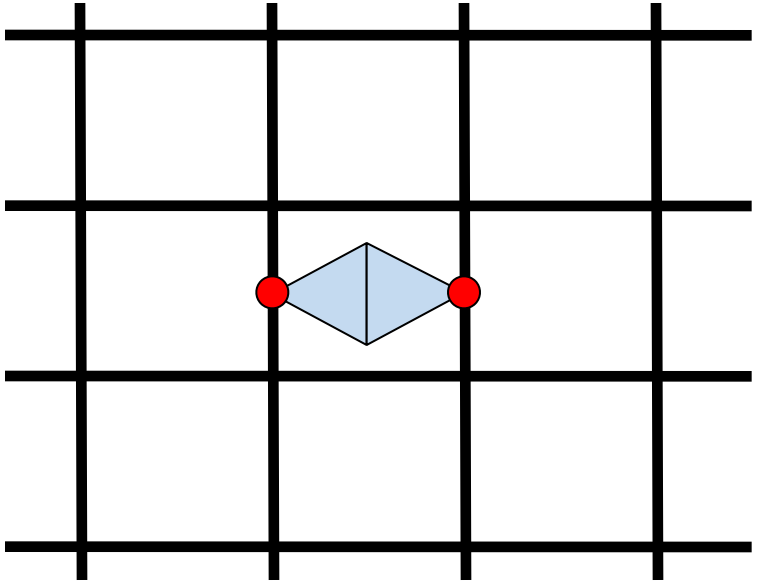
  - start with initial relaxation
  - add spacer constraint if geodesic distance between two singularities is less than 1



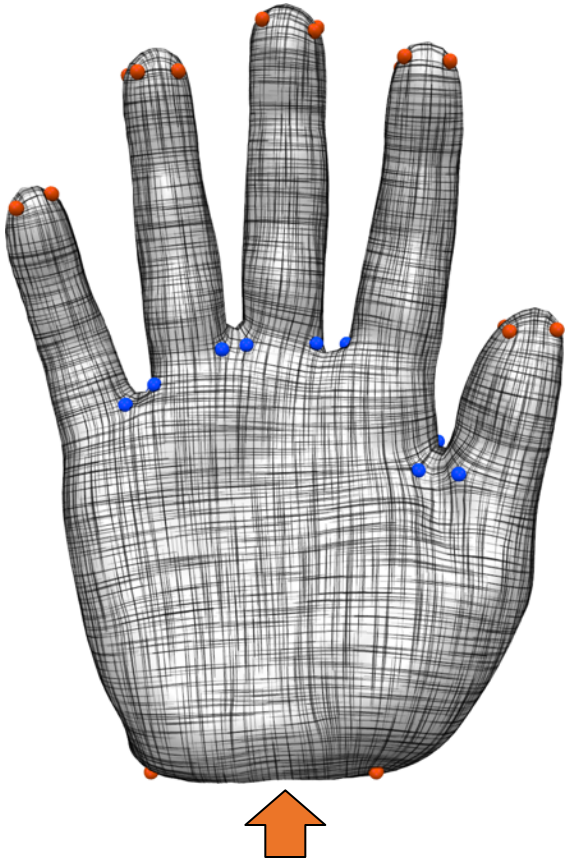
Efficient Search for Integer-Grid Maps

- Branch and cut successful if
  1. convex relaxation
  2. low-dimensional
  3. **well-behaved relaxation**

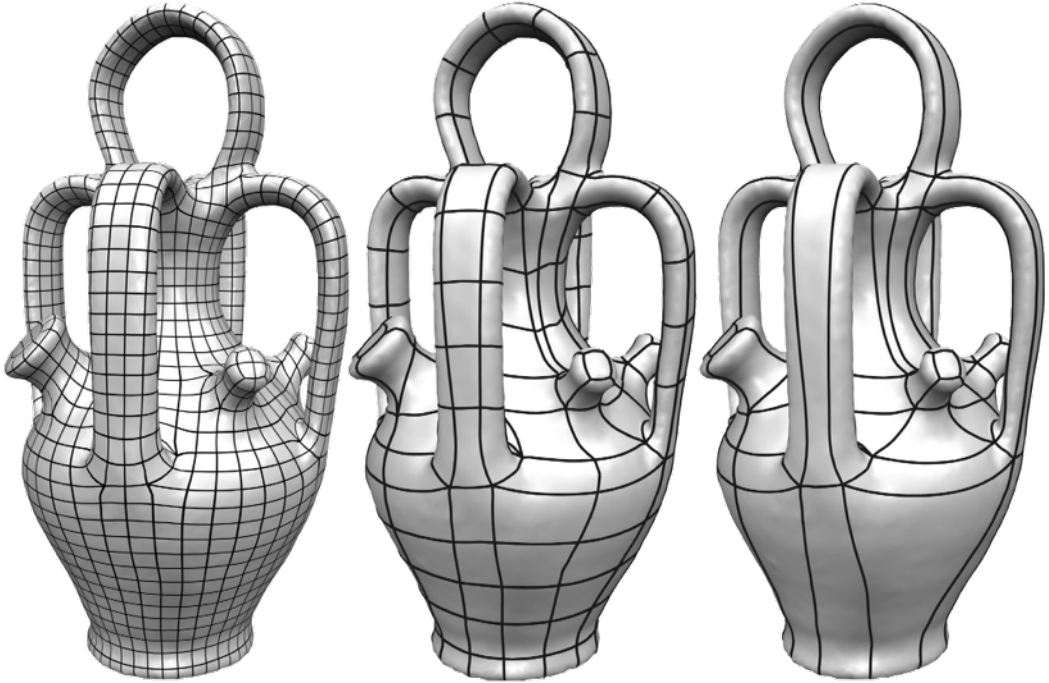
  - start with initial relaxation
  - add spacer constraint if geodesic distance between two singularities is less than 1

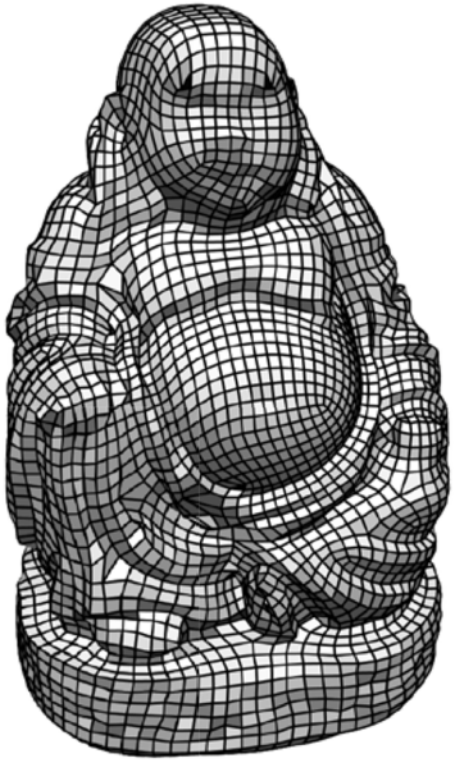


Algorithm Overview



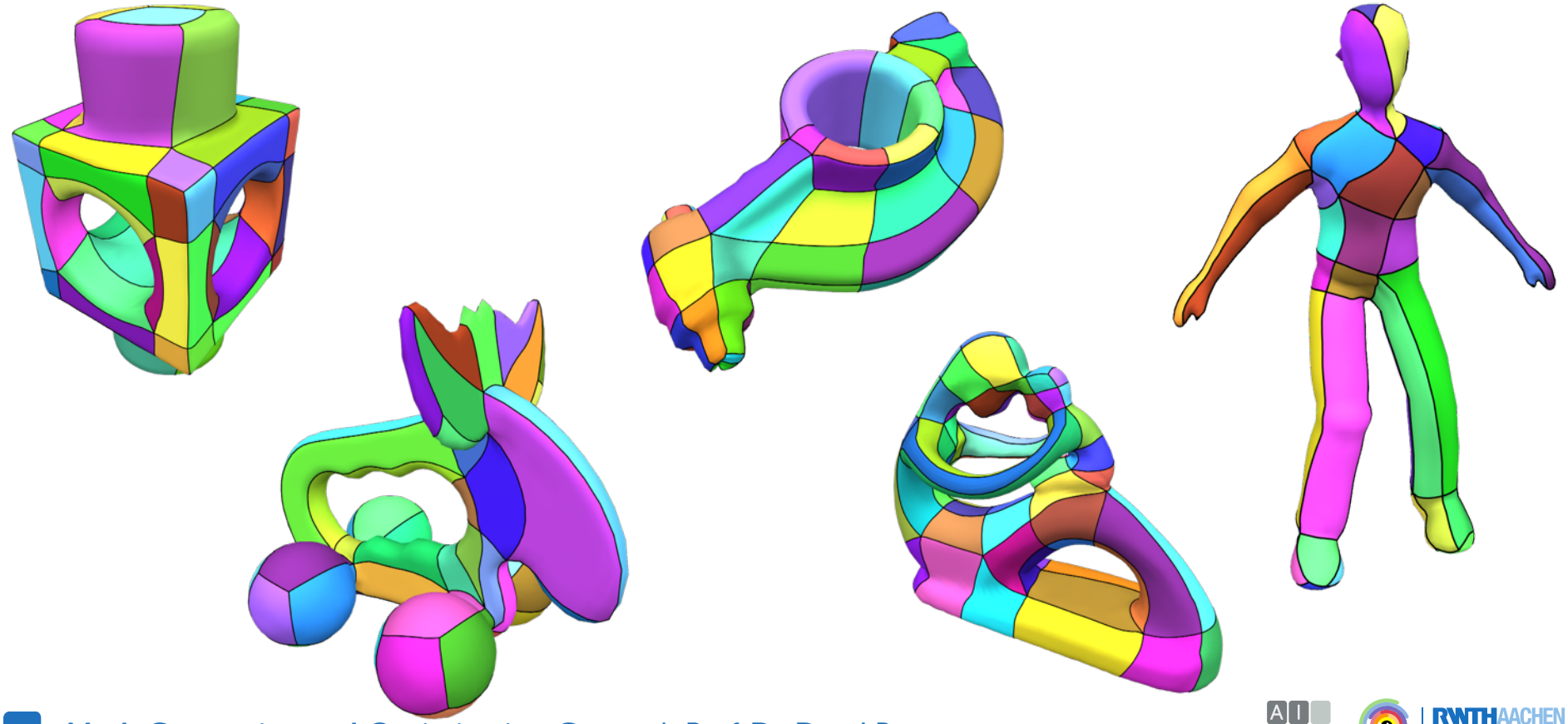
Results





200k triangles
= 4 minutes

Quad Layouts



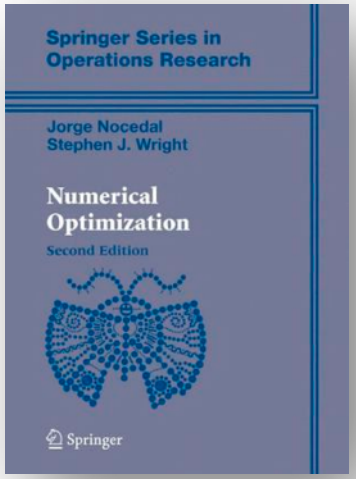
Summary

- Omnipresence of Optimization in Geometry Processing
- Unconstrained Optimization
- Equality Constrained Optimization
- Inequality Constrained Optimization
- Mixed-Integer Optimization
- Importance of Convexity

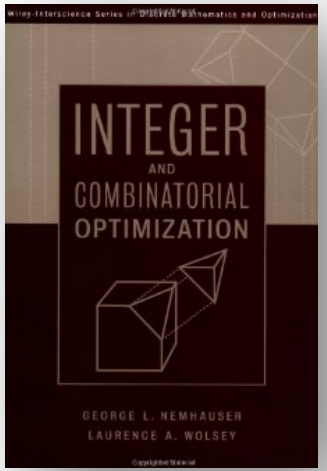
Outlook

- Further Reading

S. Boyd and L. Vandenberghe
Convex Optimization
Cambridge University Press, 2004.



J. Nocedal and S. J. Wright
Numerical Optimization
Springer, 2006.

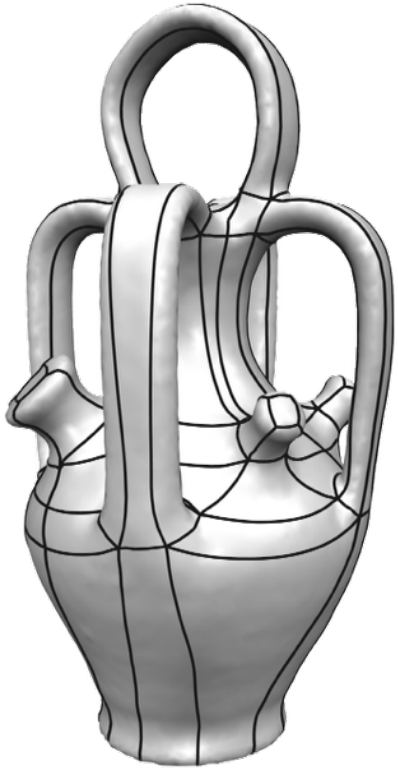


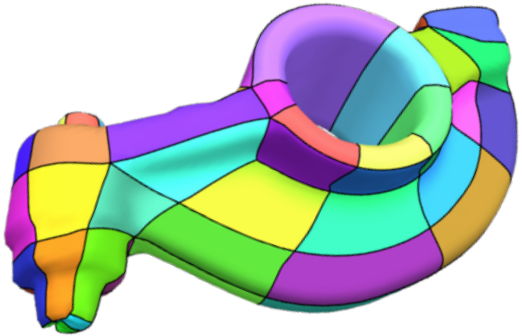
G. L. Nemhauser and L. A. Wolsey
Integer and Combinatorial Optimization
John Wiley & Sons, 1999.

Get PDF online:
http://stanford.edu/~boyd/cvxbook/

Software

- **Eigen** — linear algebra
- **IPOPT** — fast opensource C++ interior point method
- **Mosek** — commercial (convex) optimization in C, Java, Python…
- **Gurobi** — commercial mixed-integer optimization
- **CPLEX** — commercial mixed-integer optimization
- **Matlab** — many algorithms, good for prototyping
- **CVX** — prototyping for convex optimization
- **CoMISo** — unified interface to above algorithms





Thank You!