Mappings
Representation and Distortion

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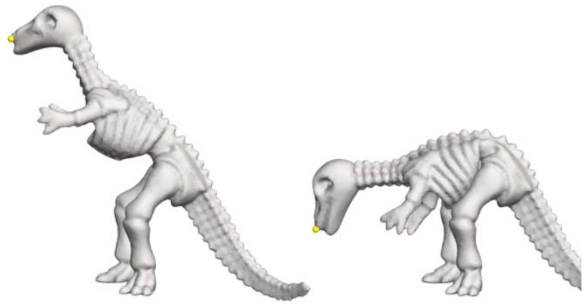
WEIZMANN INSTITUTE OF SCIENCE

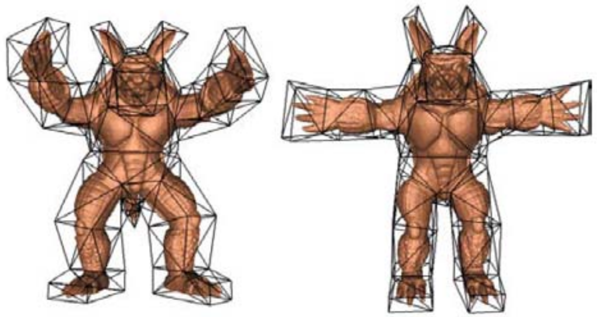
Mappings

Representation and Distortion

Modeling

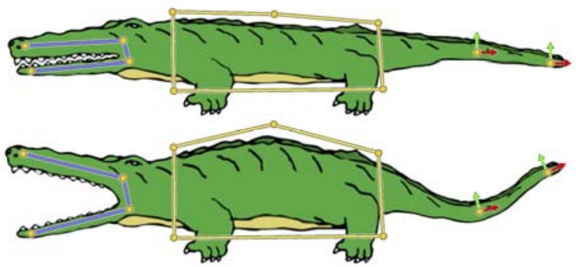


[Sorkine & Alexa 07]



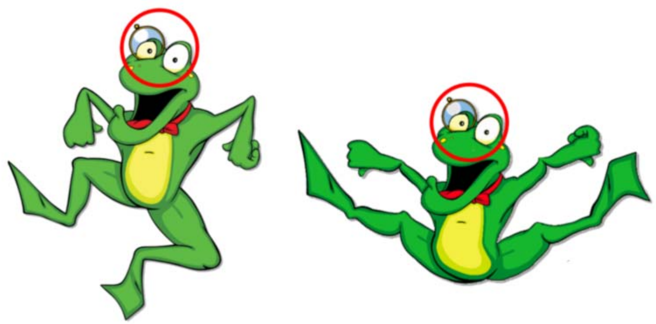
[Ju et al. 05]

[Poranne & Lipman 14]



[Jacobson 07]

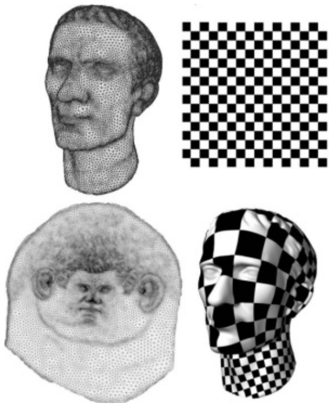
[Weber et al. 09]



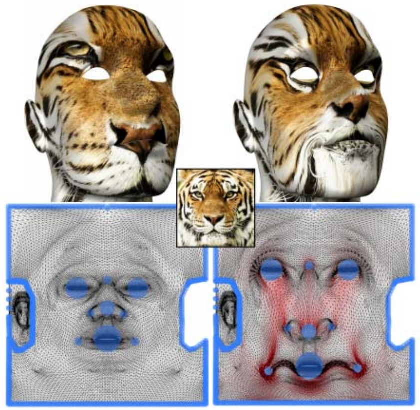
[Lipman et al. 07]

Mappings
Representation and Distortion

Parameterization



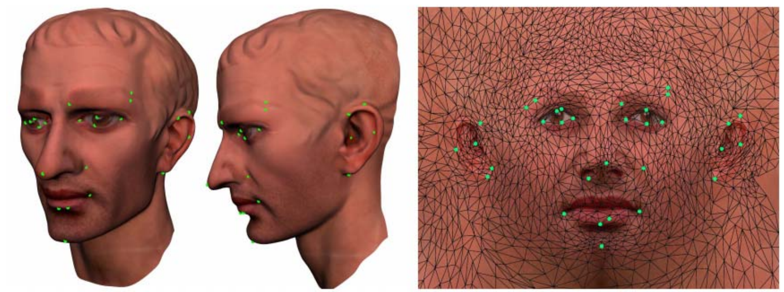
[Lévy et al. 02]



[Schuler et al. 13]

[Fu et al. 15]

[Mullen et al. 08]



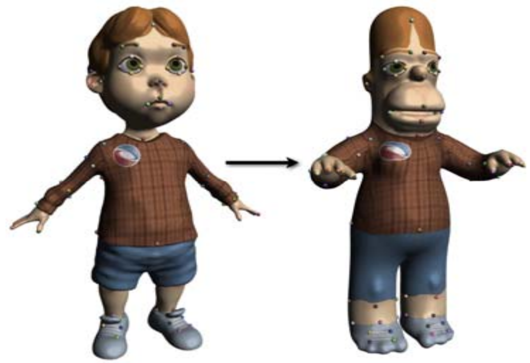
[Weber et al. 12]

Mappings
Representation and Distortion

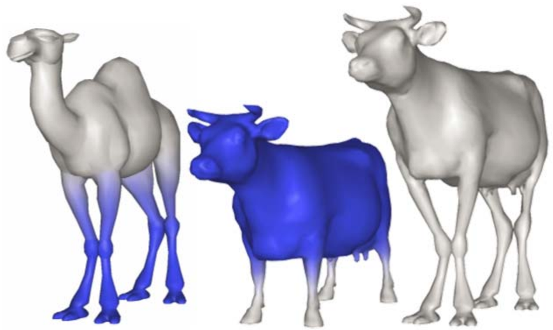
Surface mappings

[Kim et al. 11]

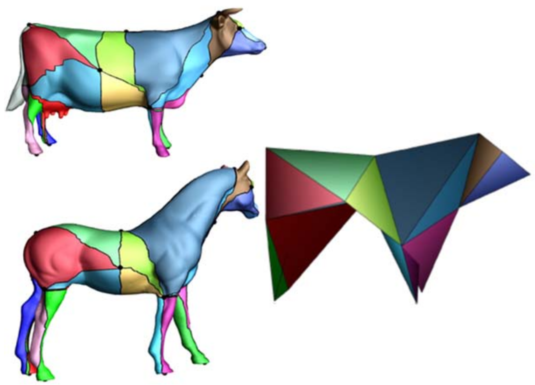
[Ovsjanikov et al. 12]



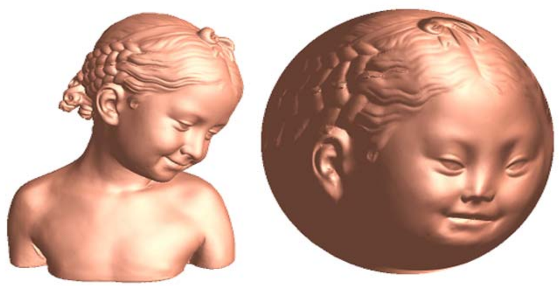
[Panozzo et al. 13]



[Kraevoy and Sheffer 04]



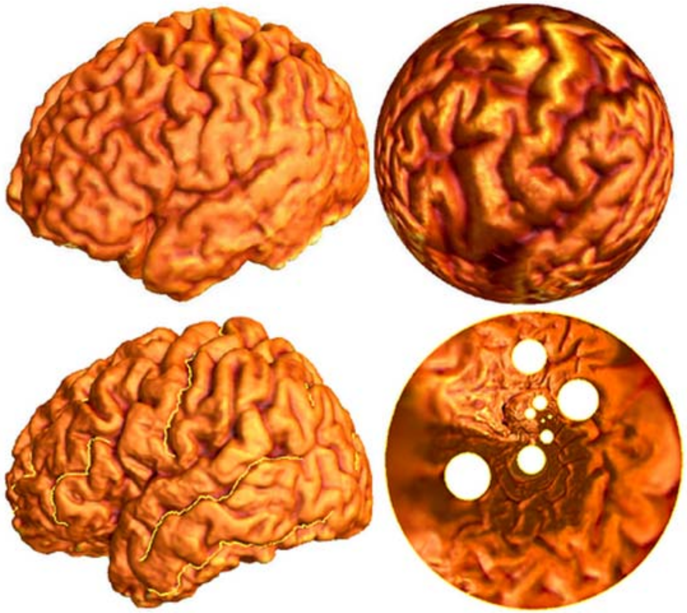
[Schreiner et al. 04]

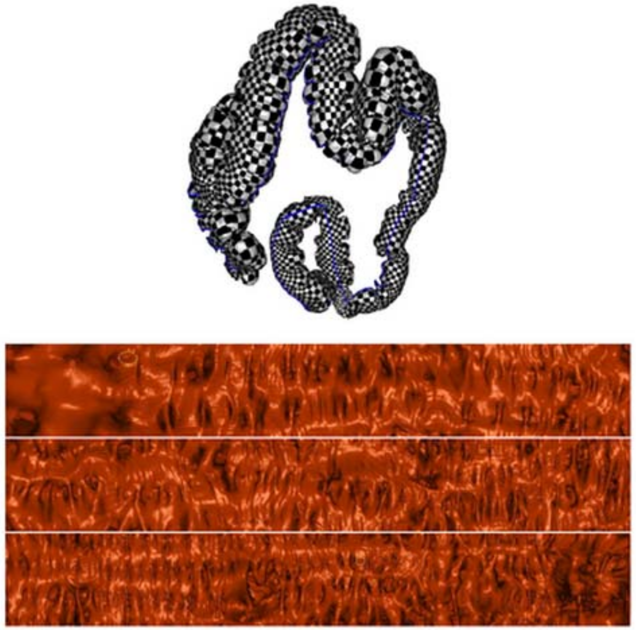


[Jin et al. 08]

“Real” Applications

Brain/colon mapping

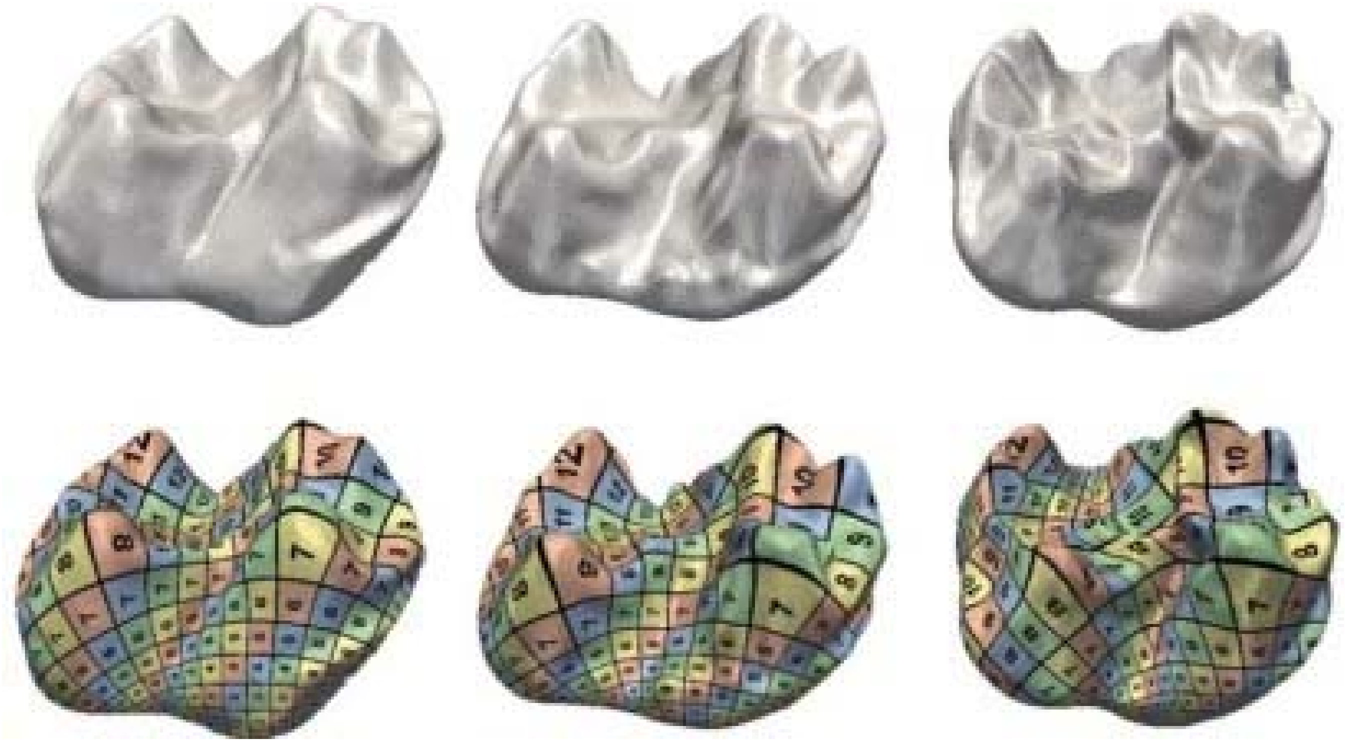




[Gu et al.]

“Real” Applications

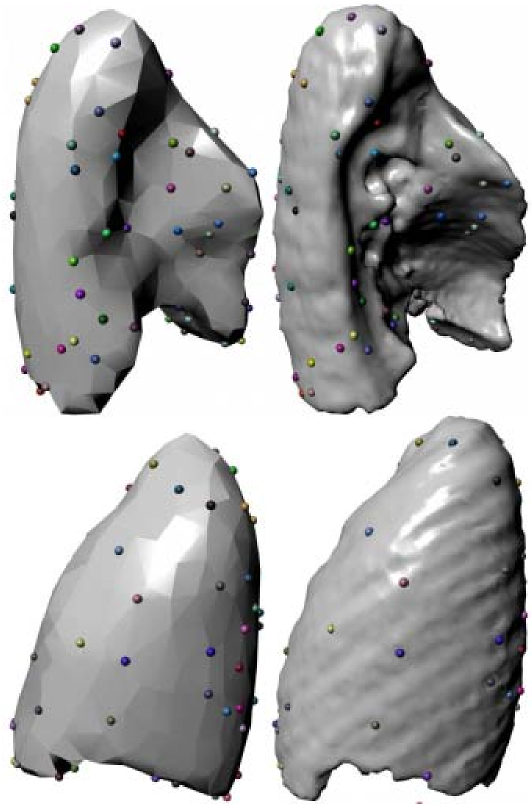
Biological Morphology

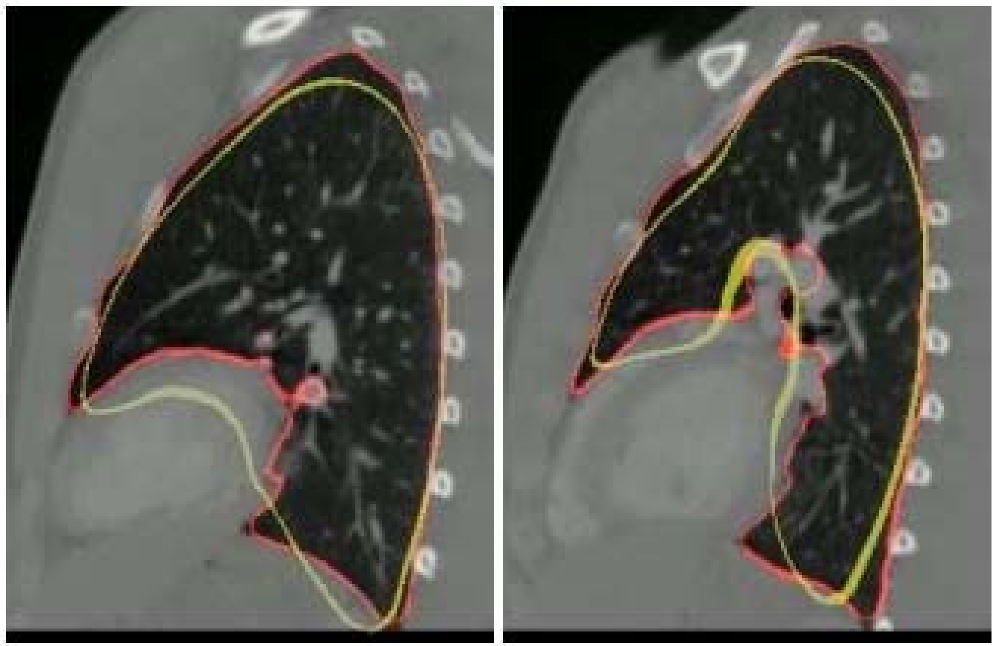


[Boyer et al. 2012]

“Real” Applications

Medical segmentation/registration





[Levi and Gotsman 12]

Definition

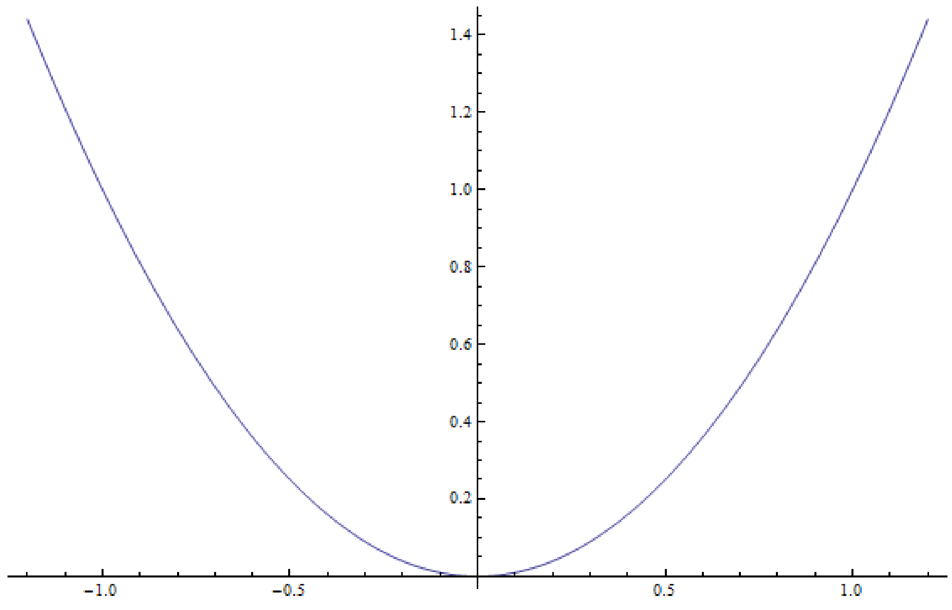
Mapping / Map :

A smooth function between shapes / spaces

Examples

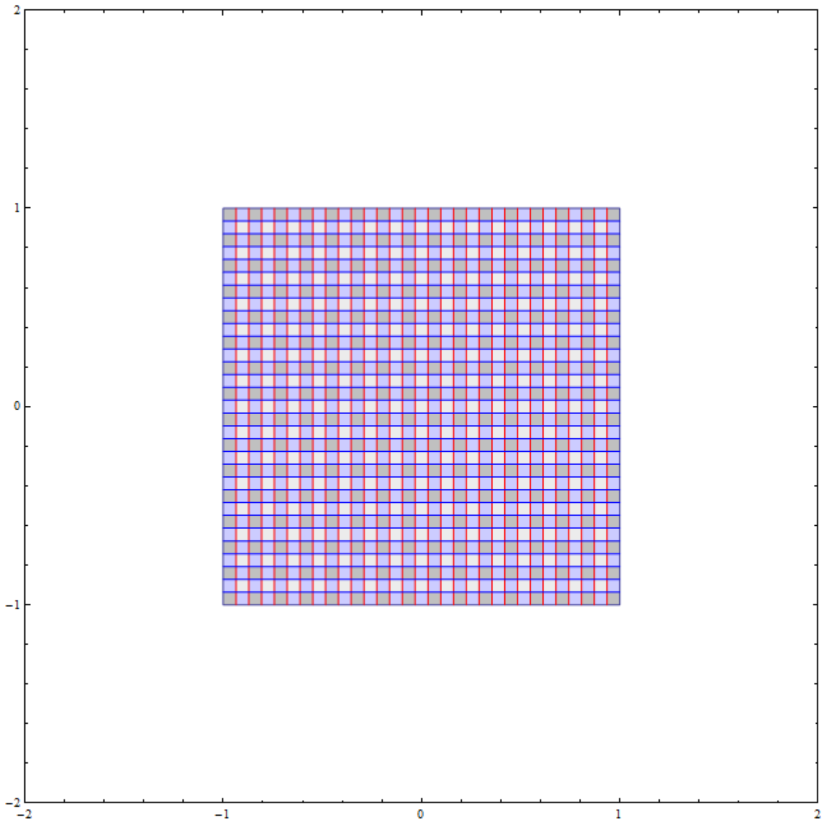
$$f: \mathbb{R} \to \mathbb{R}$$

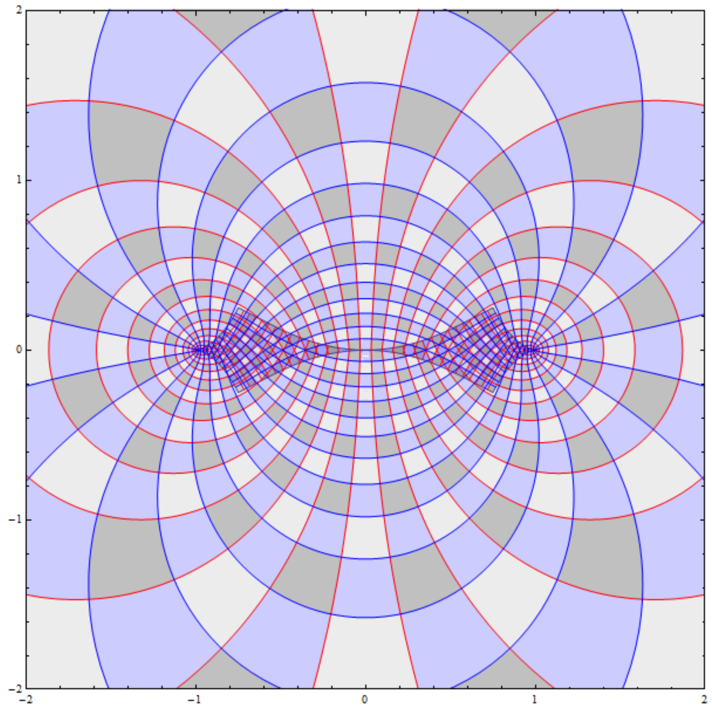
$$f(x) = x^2$$



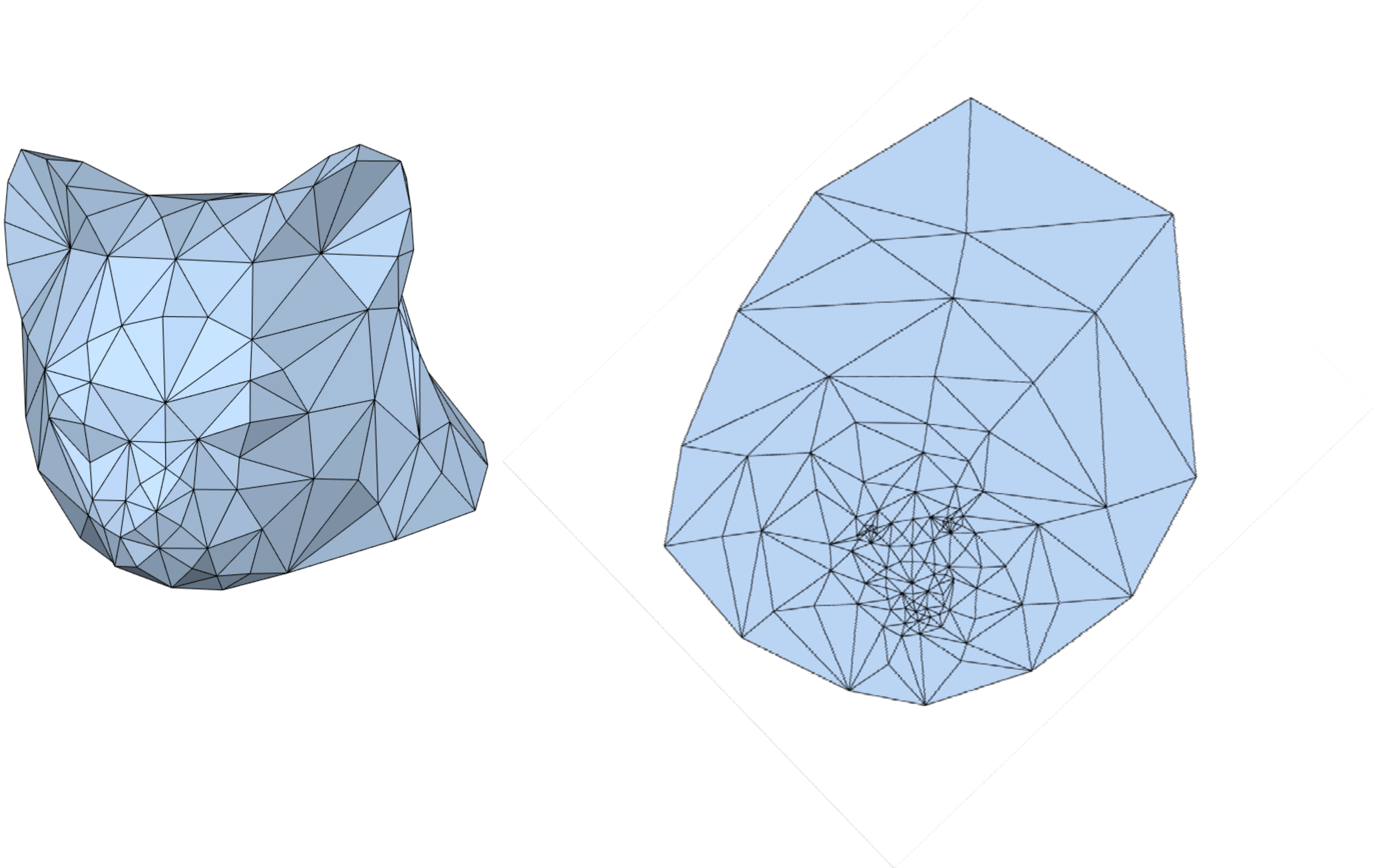
$$f: \mathbb{R}^2 \to \mathbb{R}^2$$

$$f(z) = z + \frac{1}{z}$$

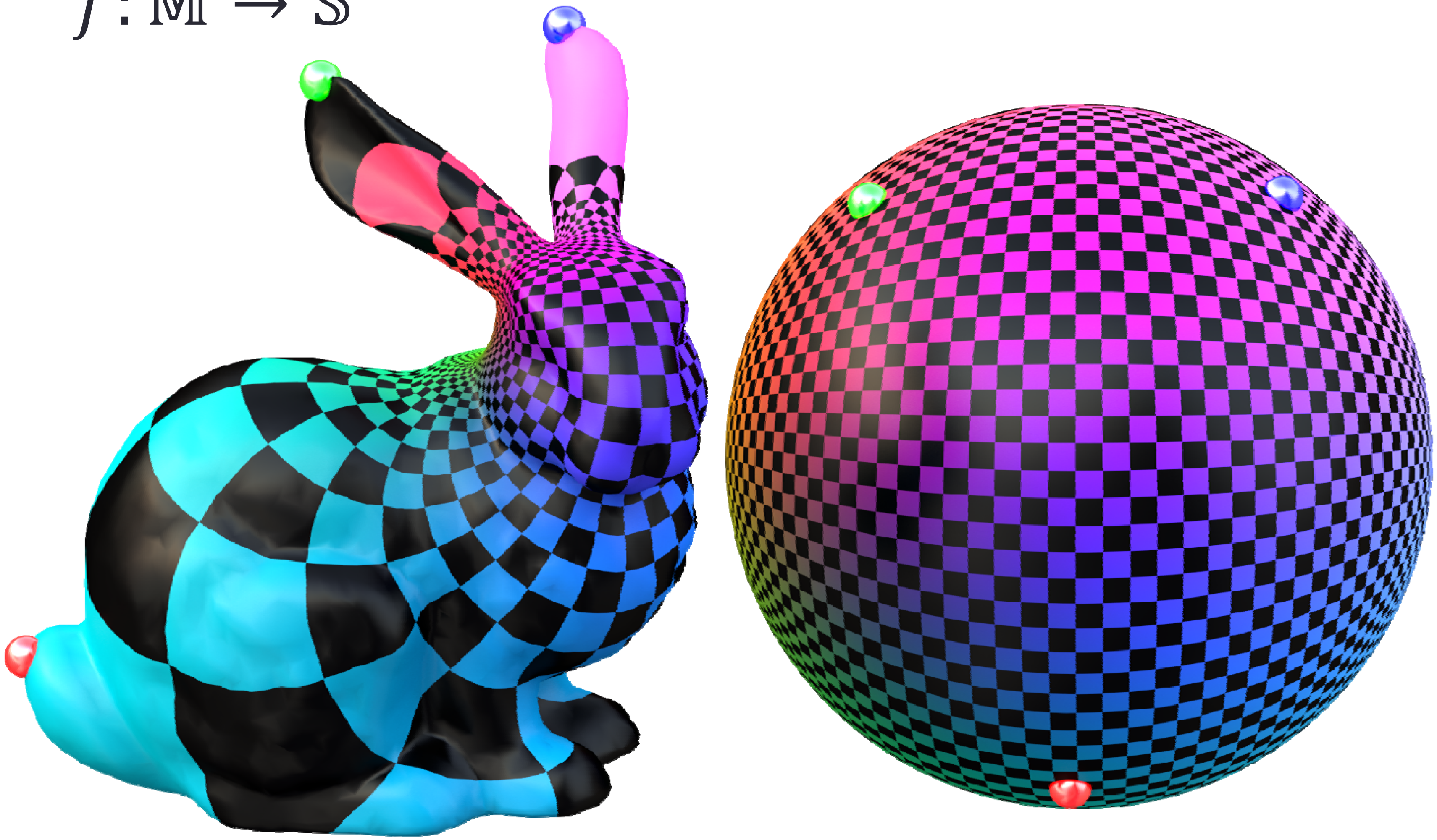




$f : \mathbb{M} \rightarrow \mathbb{R}^2$

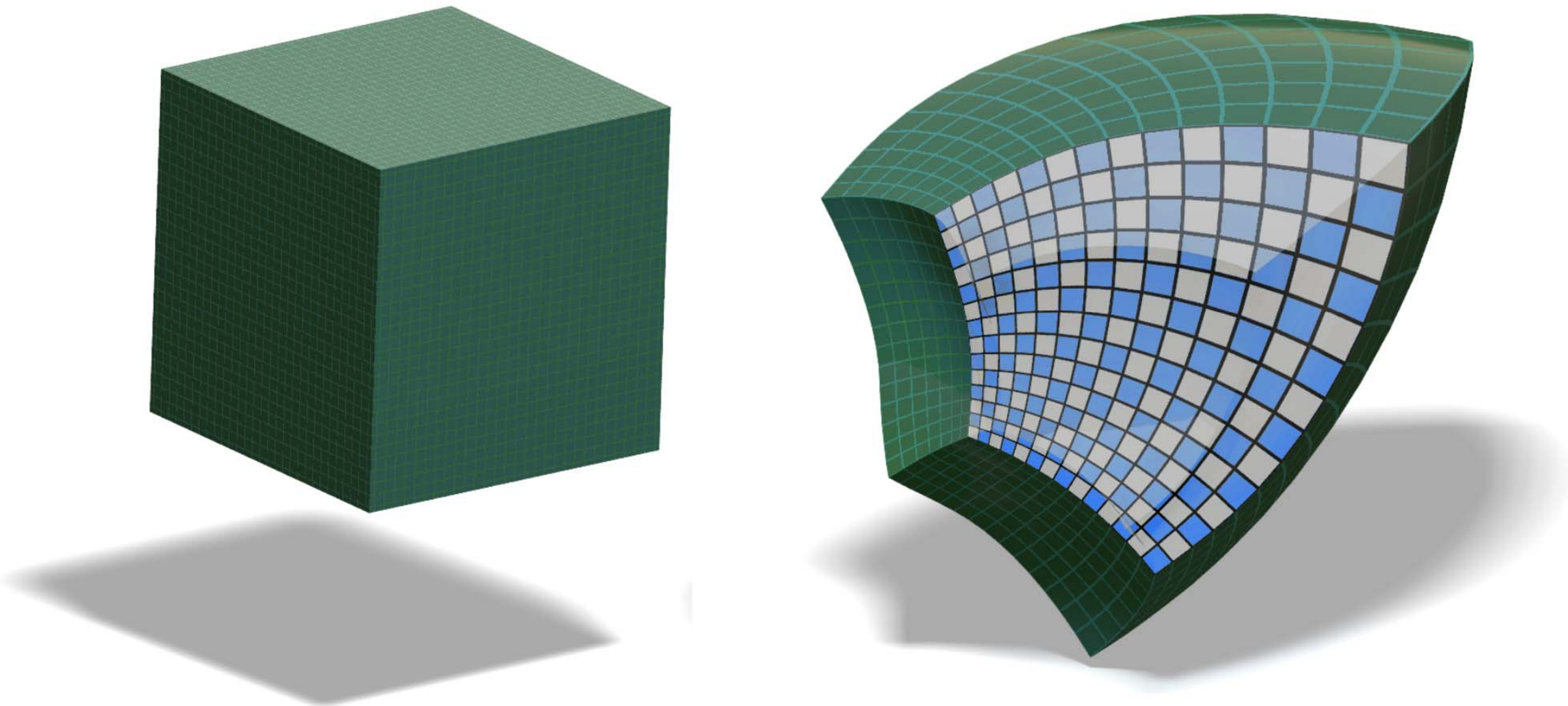


$f : \mathbb{M} \rightarrow \mathbb{S}^2$



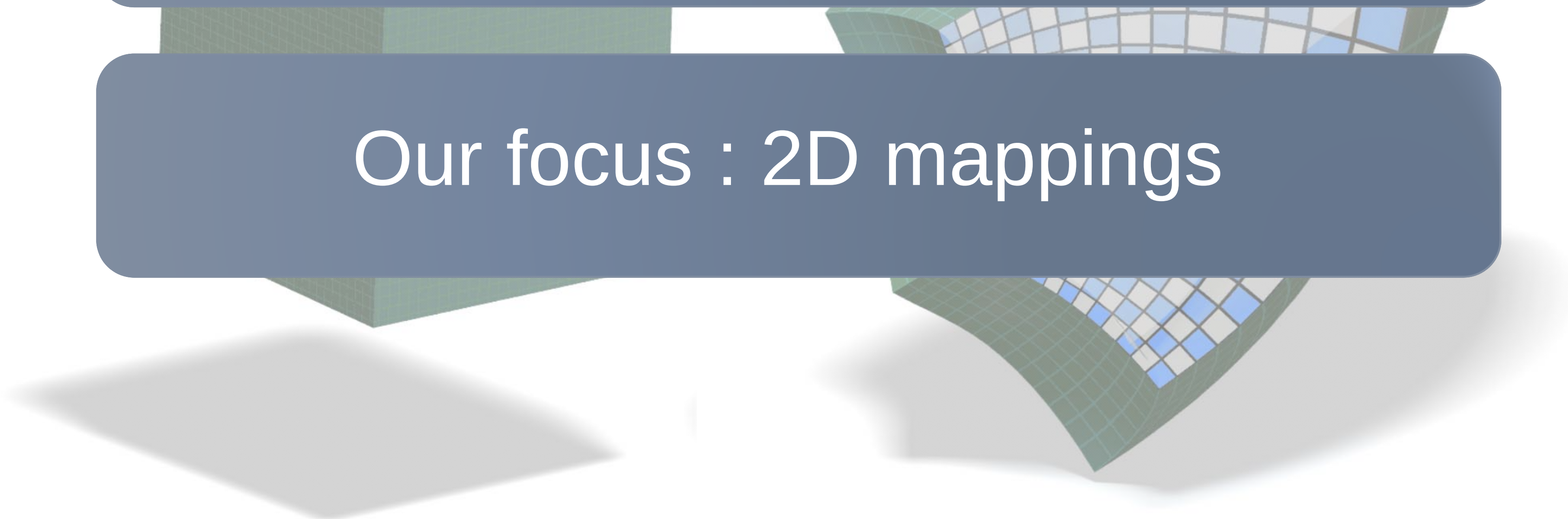
$f: \mathbb{M} \rightarrow \mathbb{M}'$

$$f: \mathbb{R}^3 \to \mathbb{R}^3$$



$f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

All cases are based on the same concepts and use similar techniques



Our focus : 2D mappings

Outline

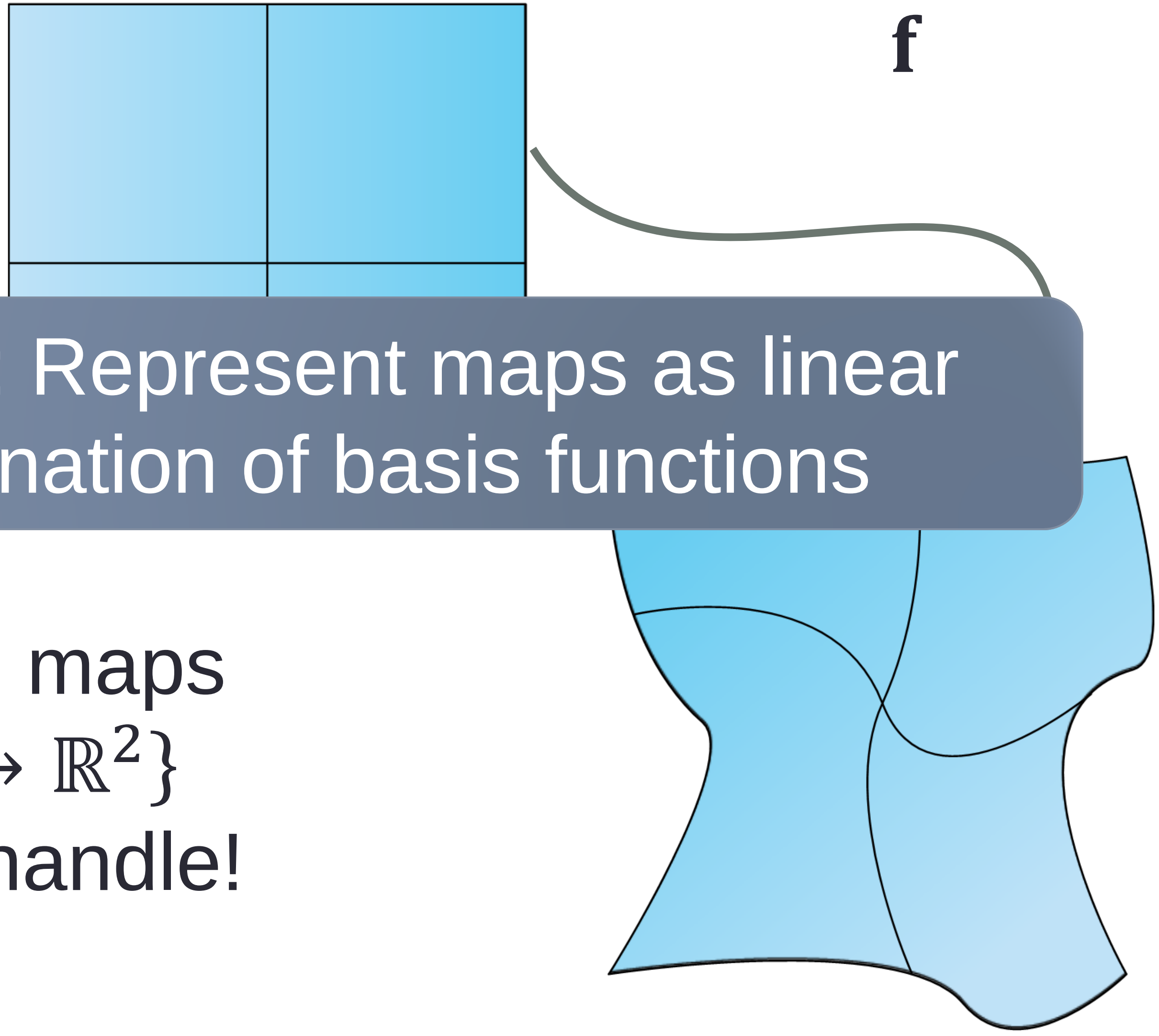
- Roi
  - Representation
  - Distortion of mappings
- Shahar
  - Optimization of mappings

2D Maps : Representation

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} u(\mathbf{x}) \\ v(\mathbf{x}) \end{pmatrix}$$

2D Maps : Representation

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} u(\mathbf{x}) \\ v(\mathbf{x}) \end{pmatrix}$$



Solution: Represent maps as linear combination of basis functions

The set of all maps

$$\{f(x): \mathbb{R}^2 \to \mathbb{R}^2\}$$

is too big to handle!

2D Maps : Representation

$$f_1, f_2, f_3, \dots, f_n$$

2D Maps : Representation

$$f_1, f_2, f_3, \dots, f_n$$

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} u(\mathbf{x}) \\ v(\mathbf{x}) \end{pmatrix}$$

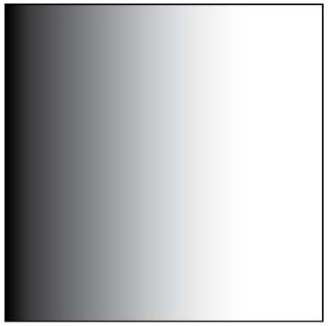
2D Maps : Representation

$$f_1, f_2, f_3, \dots, f_n$$

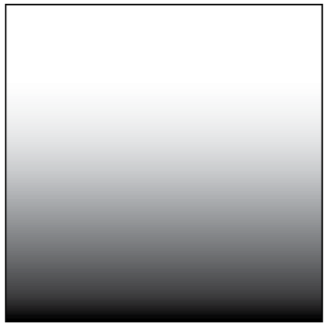
$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} u(\mathbf{x}) \\ v(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \sum a_i f_i(\mathbf{x}) \\ \sum b_i f_i(\mathbf{x}) \end{pmatrix}$$

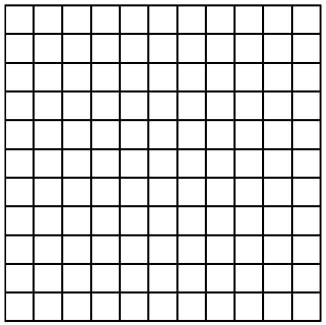
2D Maps : Representation

$f_1(x, y) = x$

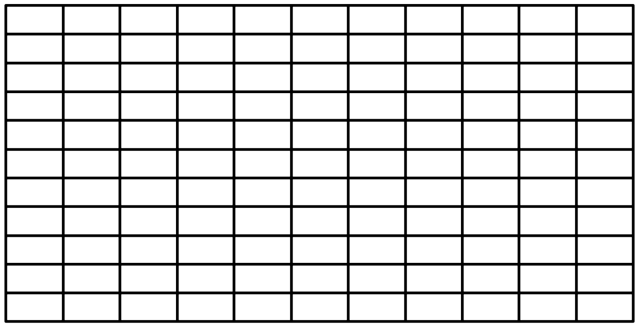


$f_2(x, y) = y$



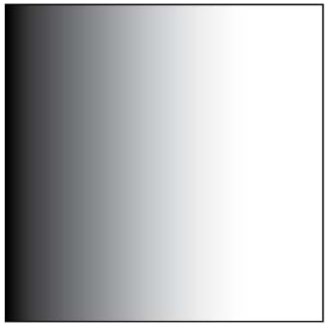


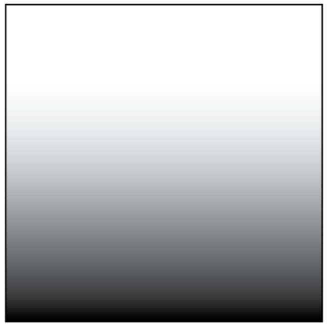
$(x, y) \Rightarrow (2f_1, f_2)$

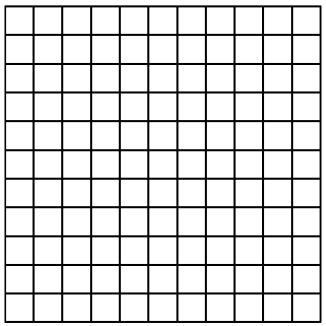


2D Maps : Representation

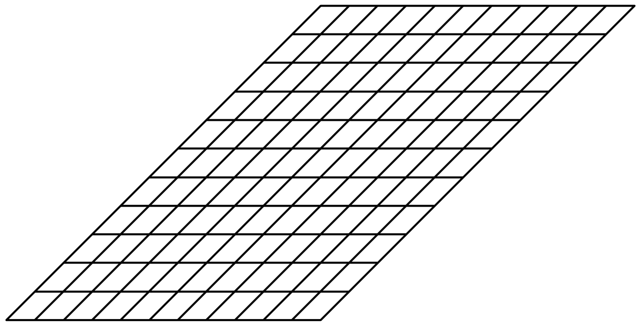
$f_1(x, y) = x$ $\quad$ $f_2(x, y) = y$





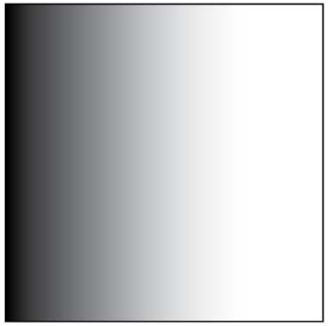


$(x, y) \Rightarrow (f_1 + f_2, f_2)$

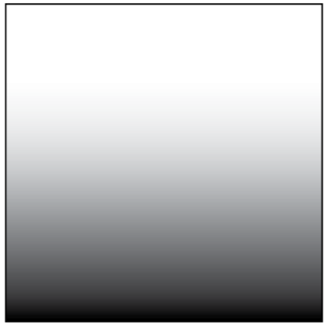


2D Maps : Representation

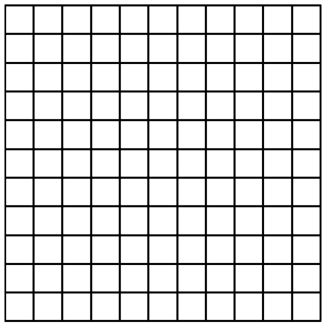
$f_1(x, y) = x$



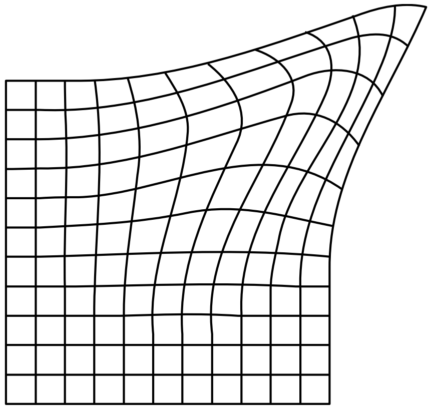
$f_2(x, y) = y$

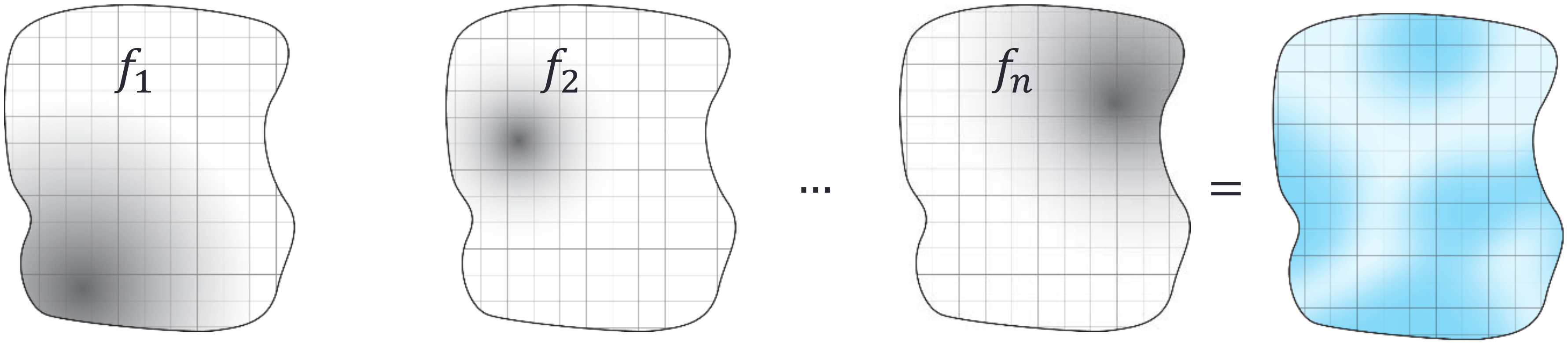


$f_3(x, y)$

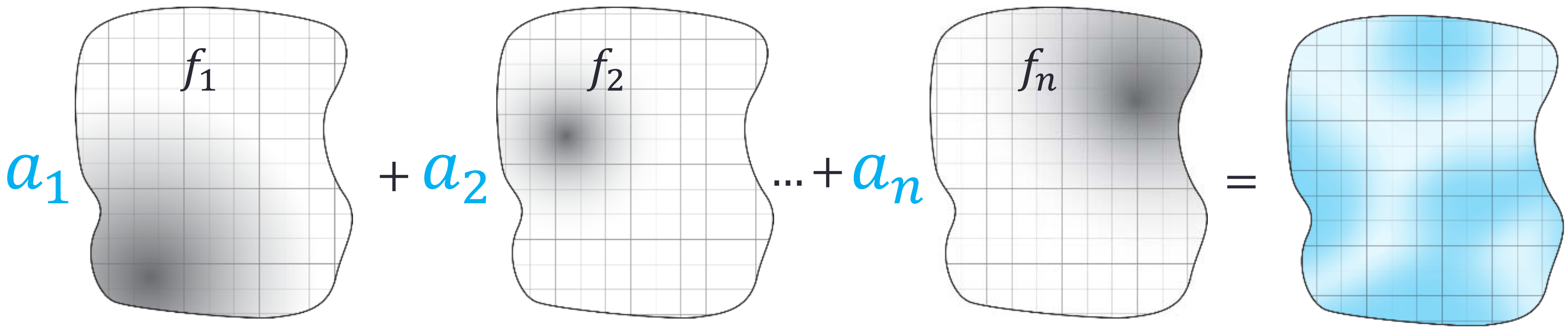


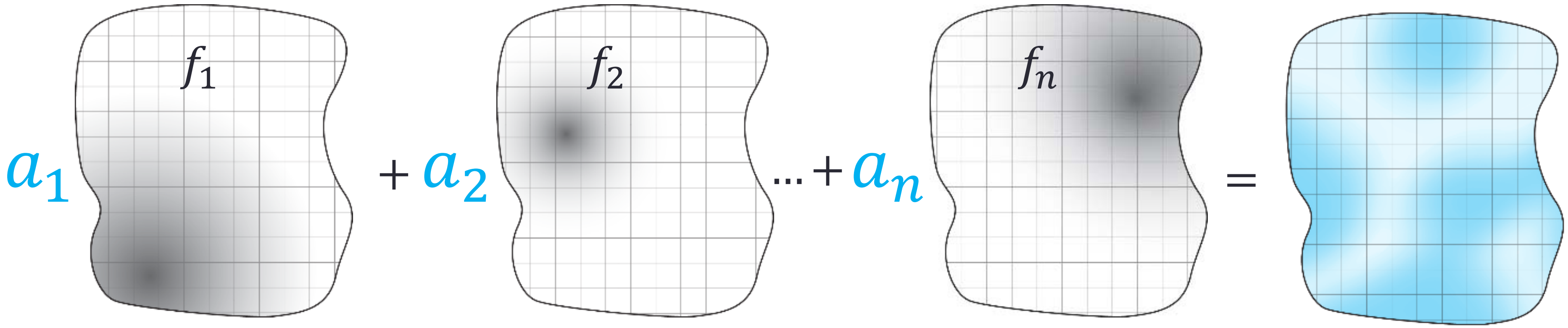
$(x, y) \Rightarrow (f_1 + f_3, f_2 + f_3)$



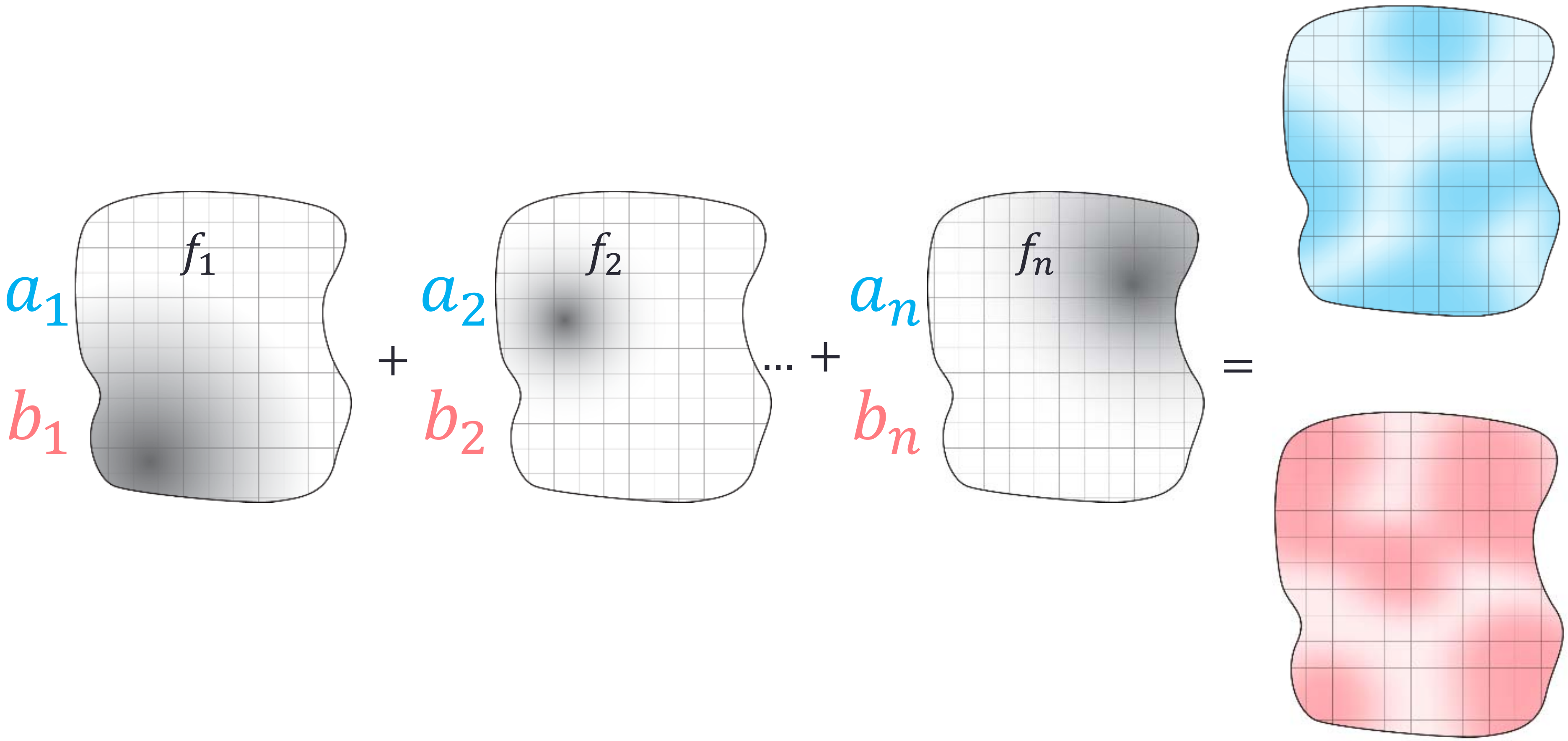


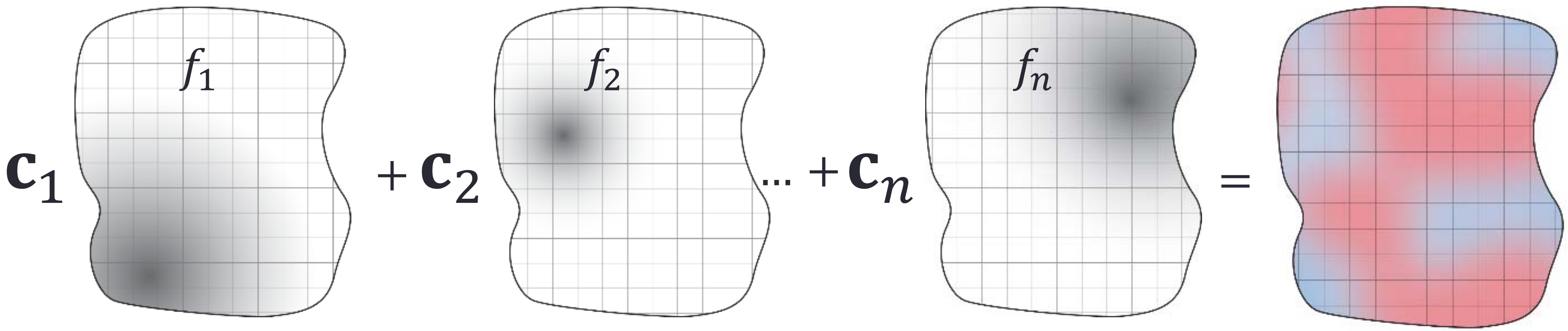
f_1
f_2
...
f_n
=


a_1
f_1
$+ a_2$
f_2
$\dots + a_n$
f_n
$=$



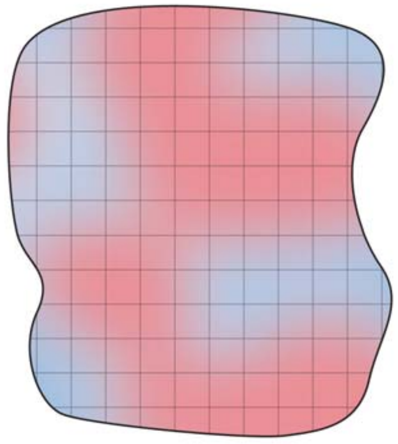
a_1
f_1
$+ a_2$
f_2
$\dots + a_n$
f_n
$=$


a_1
b_1
f_1
+
a_2
b_2
f_2
… +
a_n
b_n
f_n
=

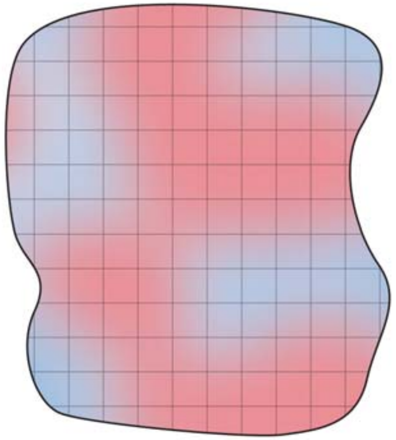


c1
f1
+ c2
f2
… + cn
fn
=

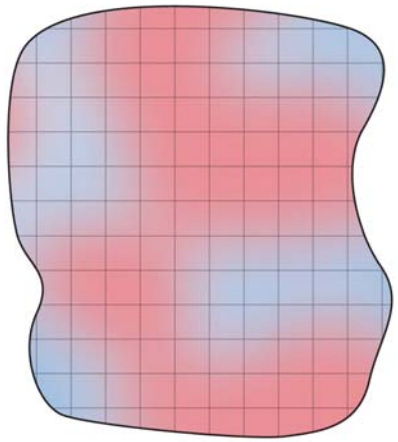
$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \sum a_i f_i(\mathbf{x}) \\ = \sum \mathbf{c}_i f_i(\mathbf{x}) \\ \sum b_i f_i(\mathbf{x}) \end{pmatrix} =$$

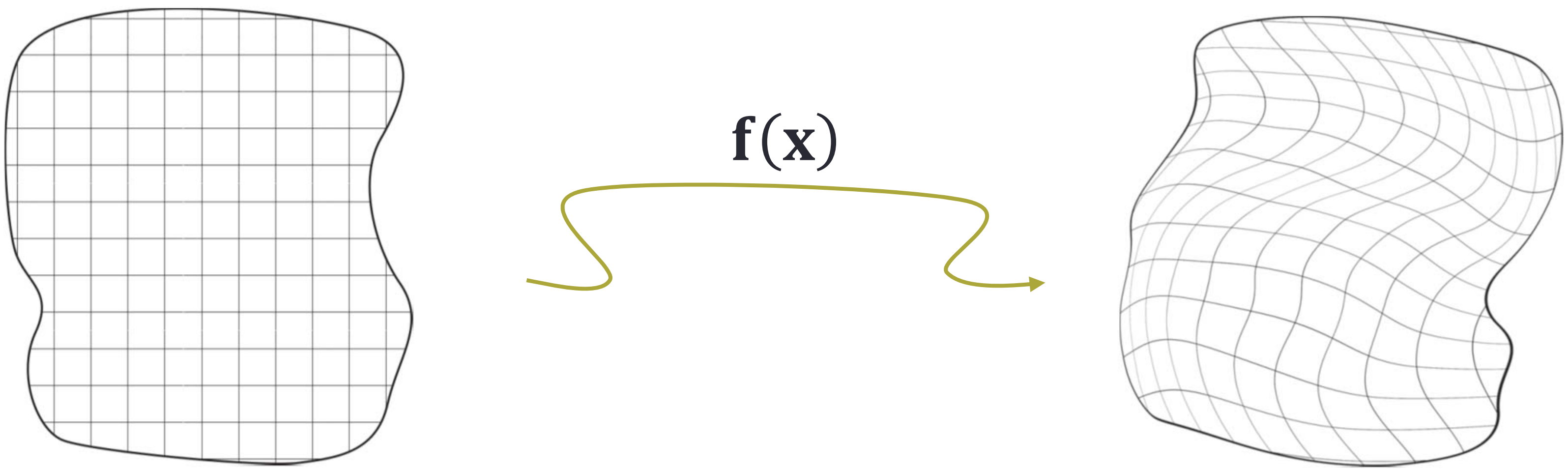


$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \sum a_i f_i(\mathbf{x}) \\ \sum b_i f_i(\mathbf{x}) \end{pmatrix} = \sum \mathbf{c}_i f_i(\mathbf{x}) =$$



$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \sum a_i f_i(\mathbf{x}) \\ \sum b_i f_i(\mathbf{x}) \end{pmatrix} = \sum \mathbf{c}_i f_i(\mathbf{x}) =$$



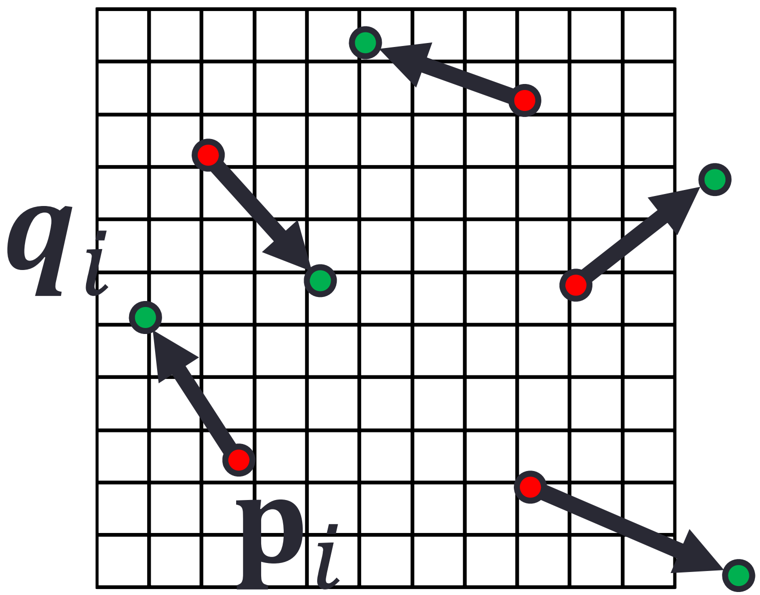


Mappings for deformations

Mappings for deformations

$\mathbf{q}_i$

Deformation as an interpolation problem



$$\mathbf{f}(\mathbf{p}_i) = \sum \mathbf{c}_i f_i(\mathbf{x})$$

$$\mathbf{c}_i = ?$$

$$\mathbf{f}(\mathbf{p}_i) = \mathbf{q}_i, \forall i$$

$$\sum \mathbf{c}_i f_i(\mathbf{p}_i) = \mathbf{q}_i, \forall i$$

Example: Thin Plate Spline

Solve the problem

$$\min E_{\mathrm{TPS}}(\mathbf{f}) = \iint \left[\left(\frac{\partial^2 \mathbf{f}}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 \mathbf{f}}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 \mathbf{f}}{\partial y^2}\right)^2 \right]$$

Bending energy

$$\text{s.t. } \mathbf{f}(\mathbf{p}_i) = \mathbf{q}_i, \forall i$$

General solution

$$\mathbf{f}(\mathbf{p}_i) = \mathbf{c}_0 + \mathbf{c}_x \boldsymbol{x} + \mathbf{c}_y \boldsymbol{y} + \sum \mathbf{c}_i \phi(\|\mathbf{x} - \mathbf{p}_i\|)$$

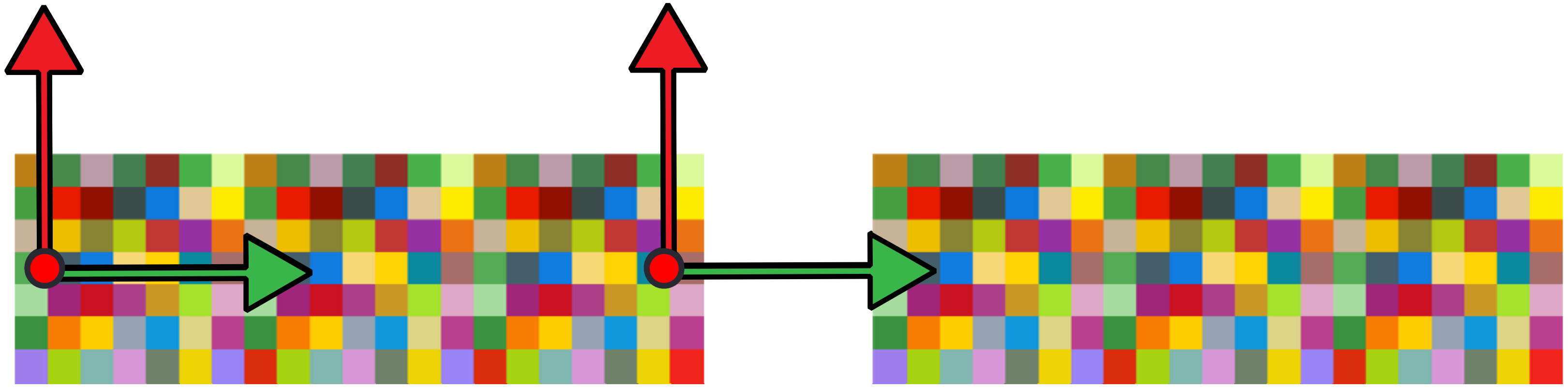
$$\phi(r) = r^2 \log r$$

Hermite interpolation

Interpolate derivatives

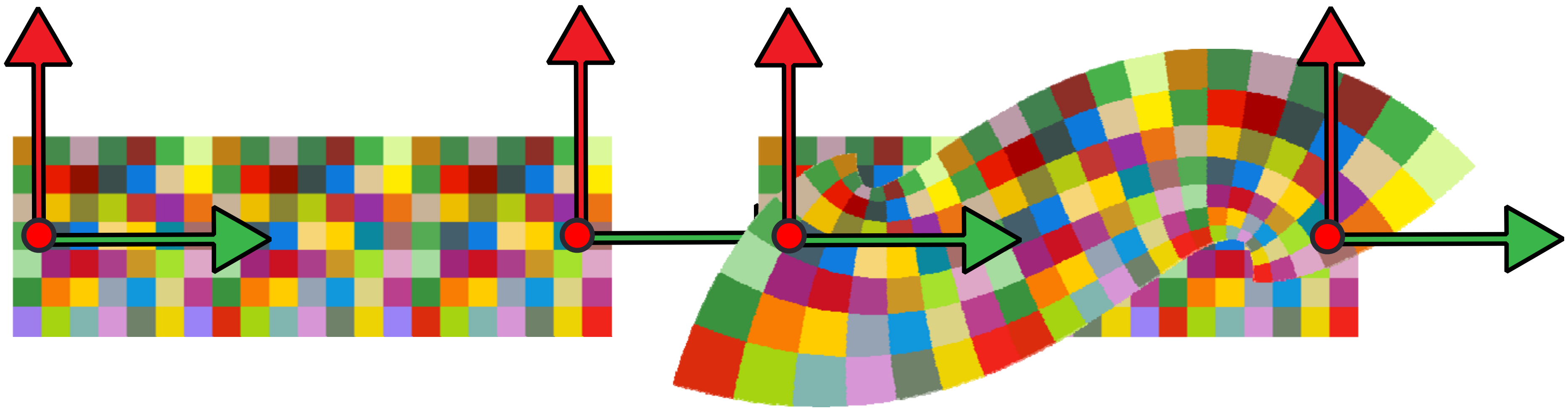
Hermite interpolation

Interpolate derivatives



Hermite interpolation

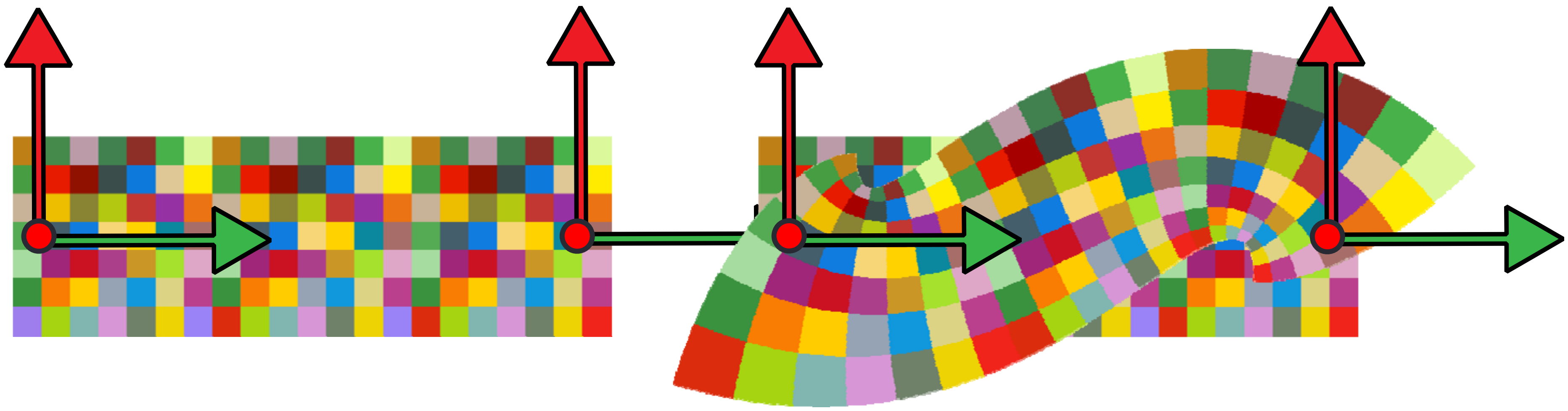
Interpolate derivatives



$$\mathbf{f}(\mathbf{p}_i) = \mathbf{q}_i \qquad \mathbf{Df}(\mathbf{p}_i) = \mathbf{D}_i$$

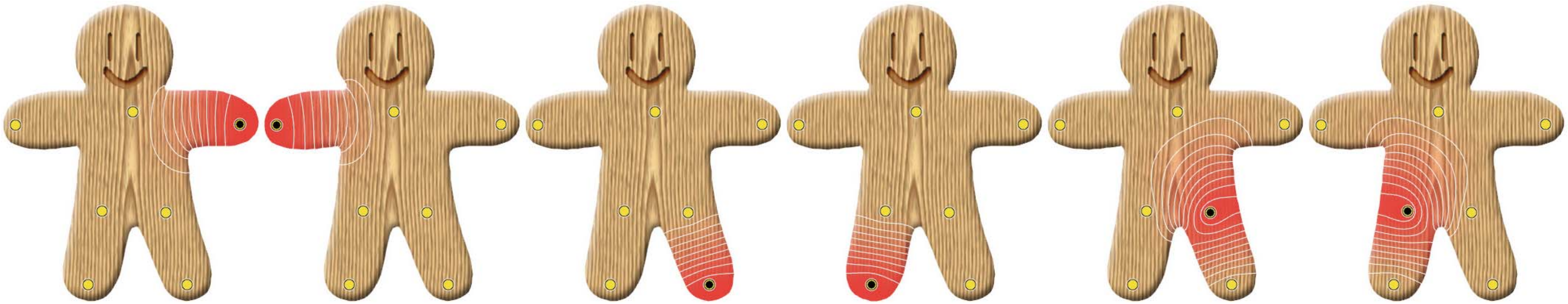
Hermite interpolation

Interpolate derivatives



$$\sum \mathbf{c}_i f_i(\mathbf{p}_i) = \mathbf{q}_i \qquad \sum \mathbf{c}_i \nabla f_i(\mathbf{p}_i) = \mathbf{D}_i$$

Example: Linear Blend Skinning



$$\mathbf{f}(\mathbf{x}) = \sum w_i(\mathbf{x})\underbrace{(\mathbf{T_i}x + \mathbf{q_i})}$$

Weights

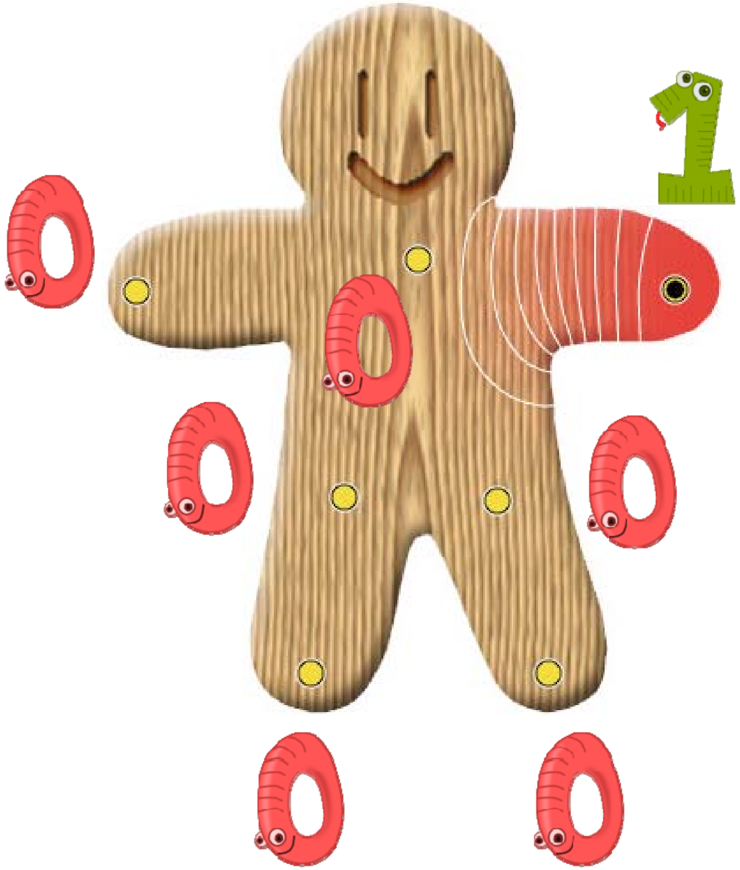
Affine transformations

Example: Linear Blend Skinning

$$\mathbf{f}(\mathbf{x}) = \sum w_i(\mathbf{x})(\mathbf{T}_i x + \mathbf{q}_i)$$

Example: Linear Blend Skinning

$$\mathbf{f}(\mathbf{x}) = \sum w_i(\mathbf{x})(\mathbf{T}_i \boldsymbol{x} + \mathbf{q}_i)$$



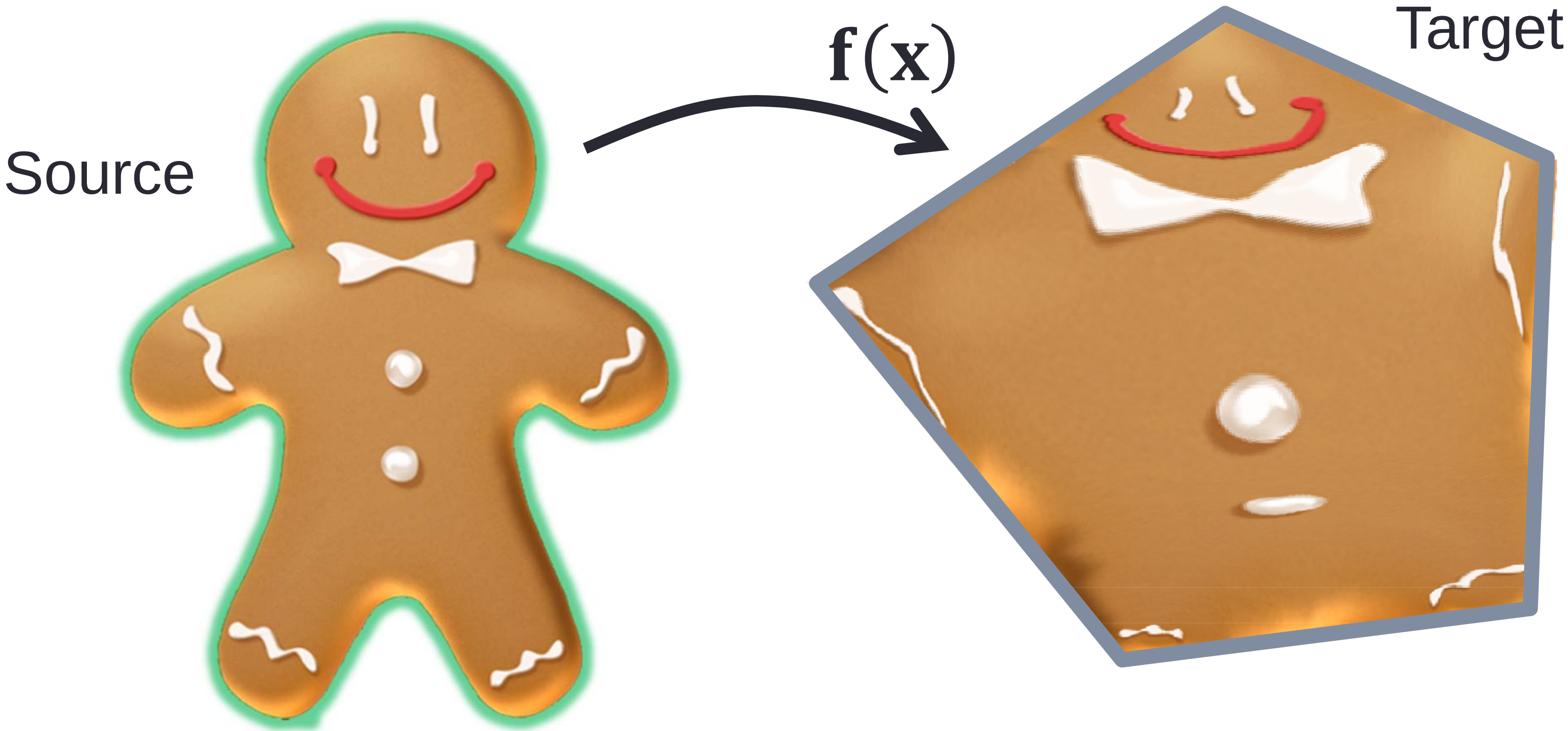
Lagrange property

$$w_i(\mathbf{p}_j) = \delta_{ij}$$

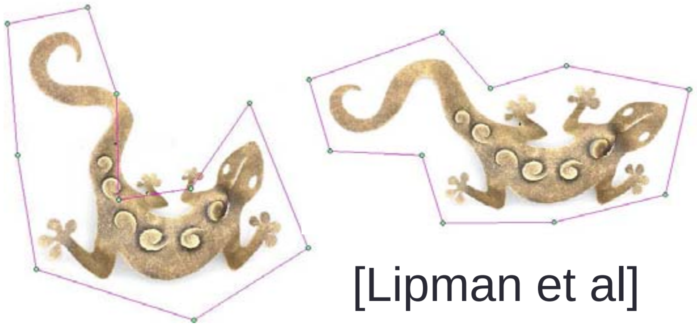
Hermite (derivative) property

$$\nabla w_i(\mathbf{p}_j) = \mathbf{0}$$

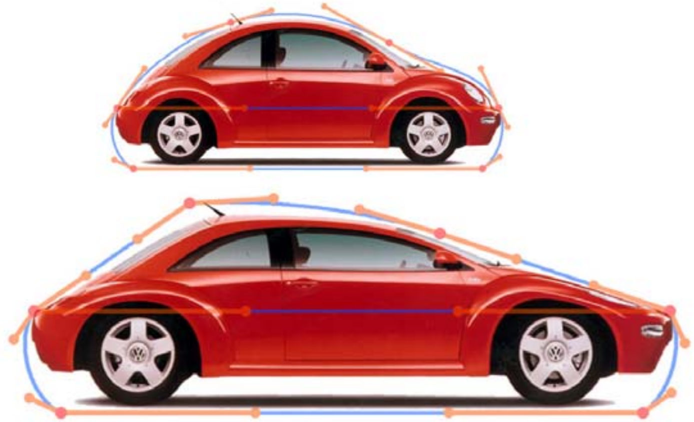
Deformation as an boundary interpolation



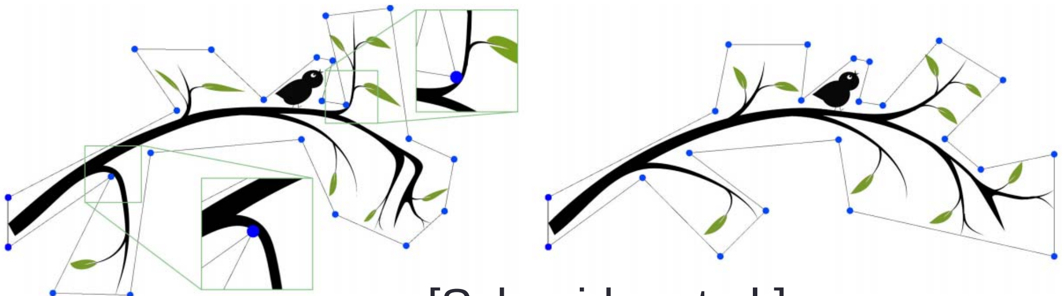
Example: Barycentric Coordinates



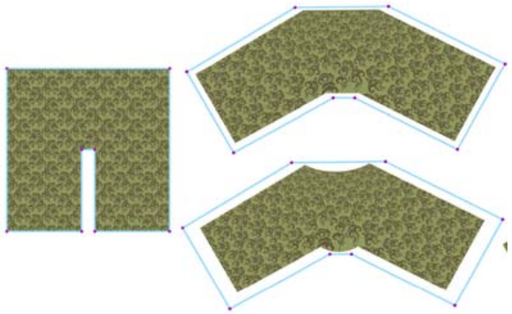
[Lipman et al]



[Li et al.]

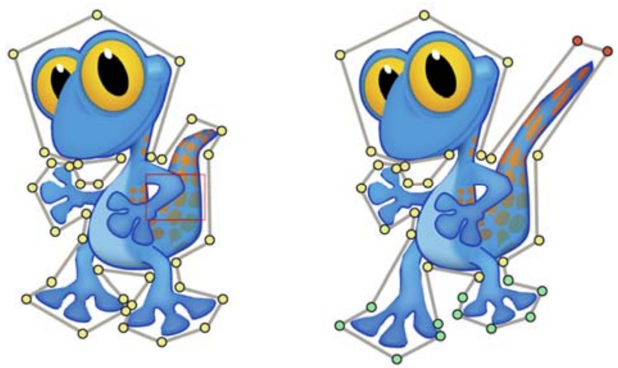


[Schneider et al.]



[Weber et al.]

[Weber et. al]

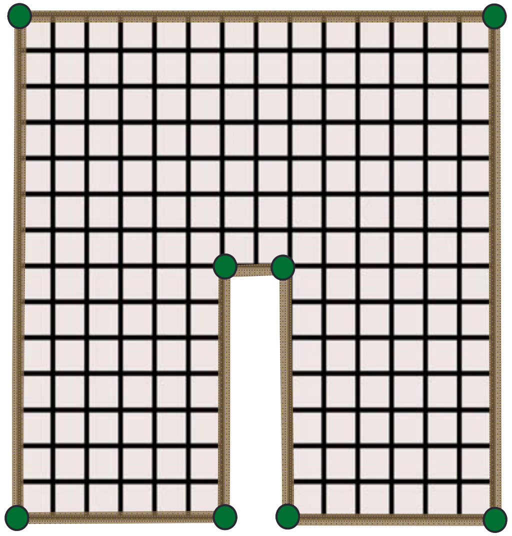


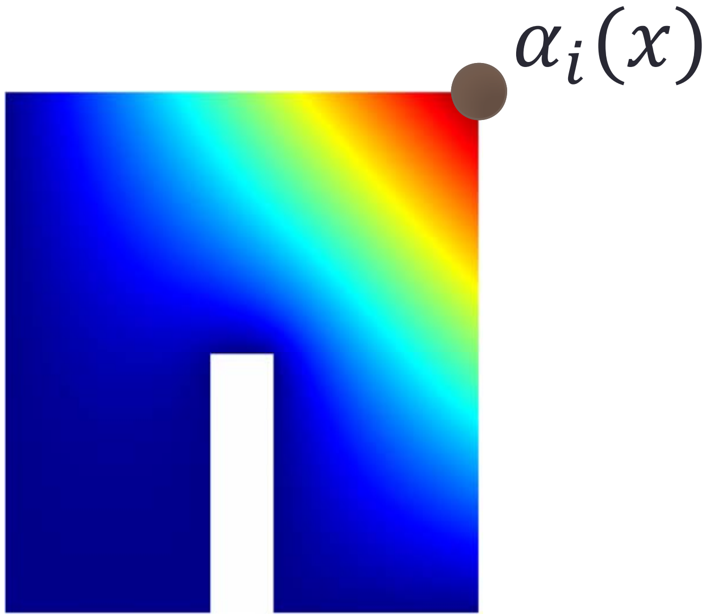
[Li et al.]

Example: Barycentric Coordinates

Stages:

- Source shape
- Polygonal cage
- Coordinates



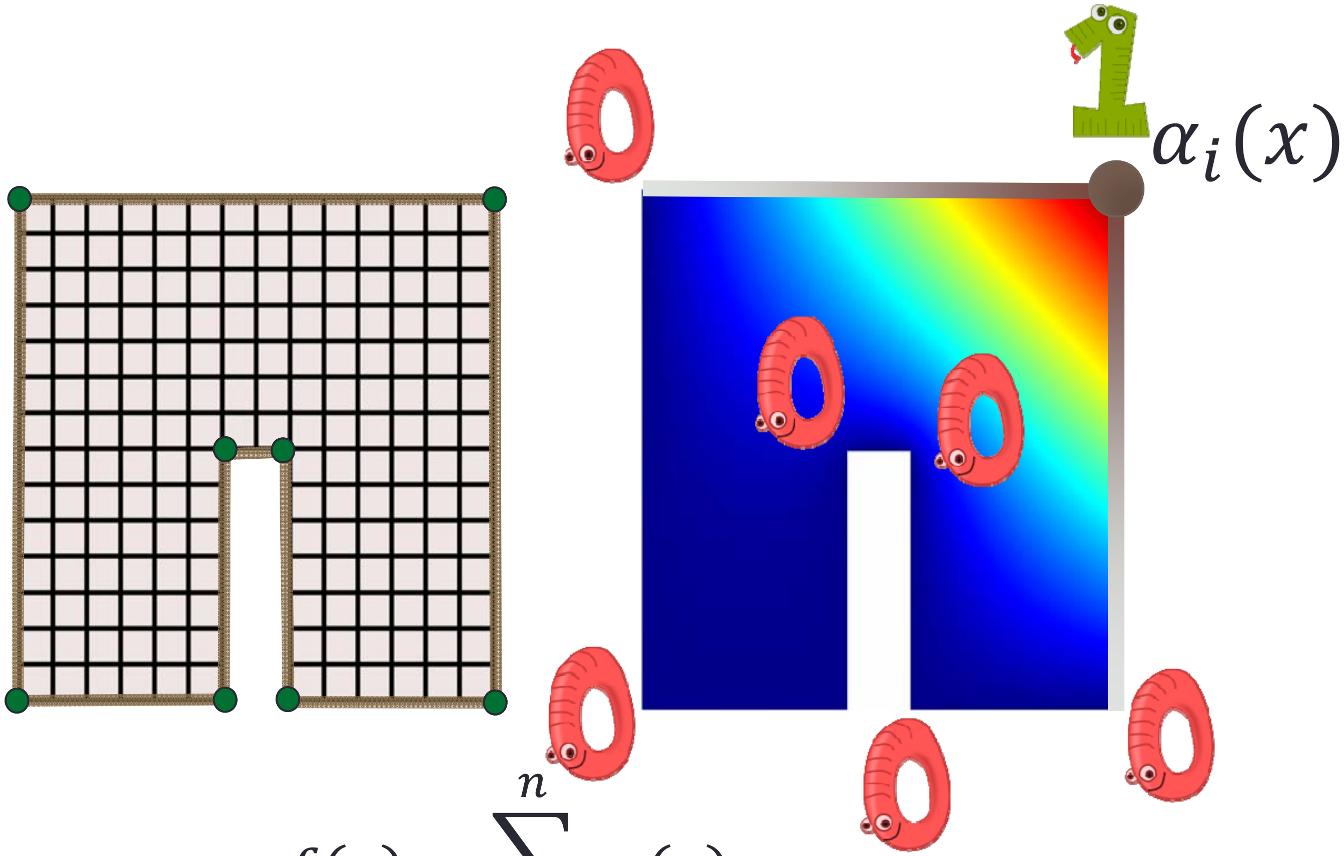


$$f(\mathbf{x}) = \sum_{i=1}^{n} \alpha_i(\mathbf{x})\mathbf{q}_i$$

Example: Barycentric Coordinates

Stages:

- Source shape
- Polygonal cage
- Coordinates

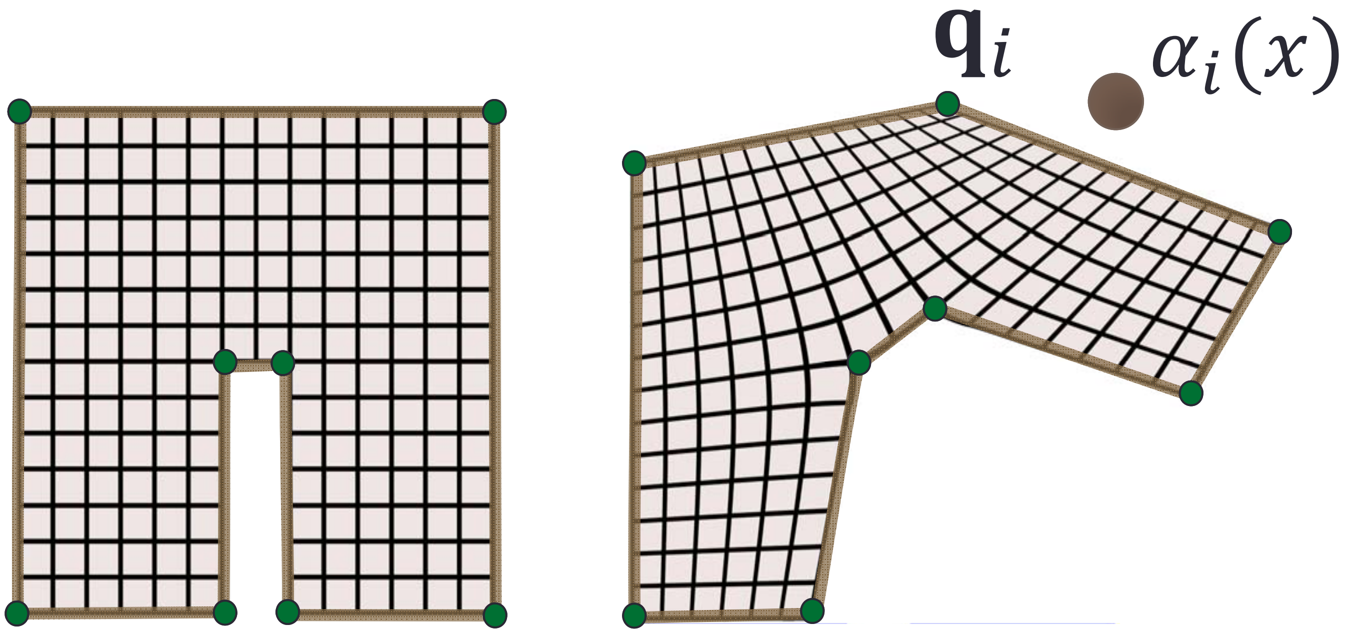


$$f(\mathbf{x}) = \sum_{i=1}^{n} \alpha_i(\mathbf{x})\mathbf{q}_i$$

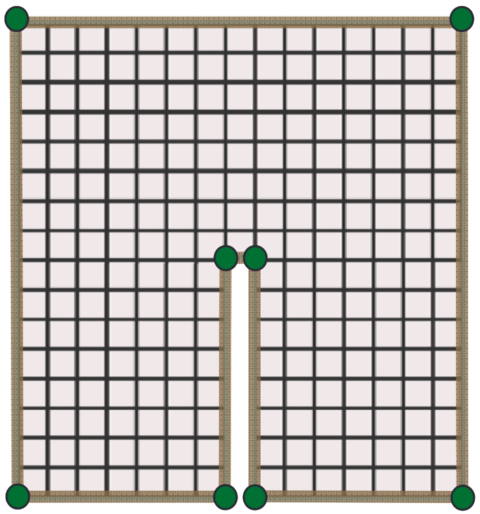
Example: Barycentric Coordinates

Stages:

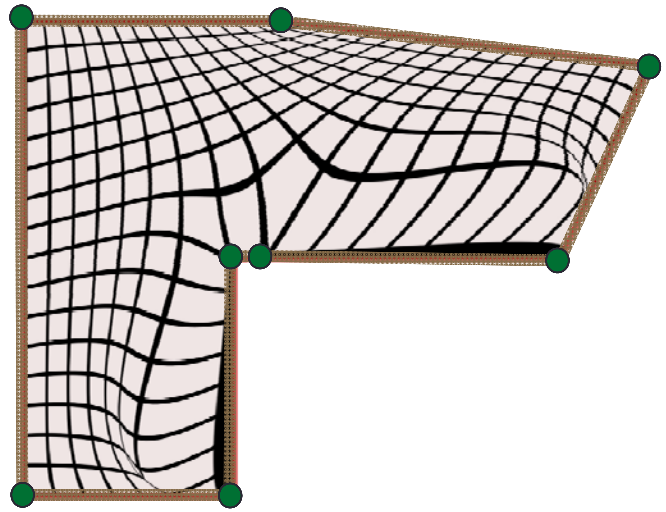
- Source shape
- Polygonal cage
- Coordinates
- Manipulate cage
- Apply deformation



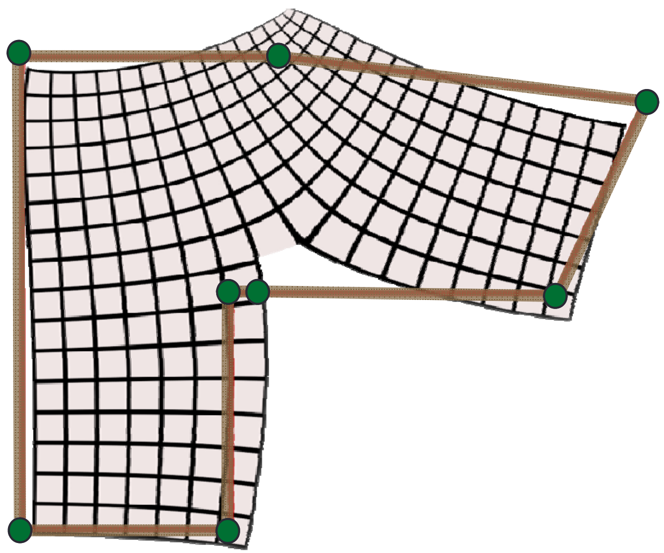
$$f(\mathbf{x}) = \sum_{i=1}^{n} \alpha_i(\mathbf{x})\mathbf{q}_i$$



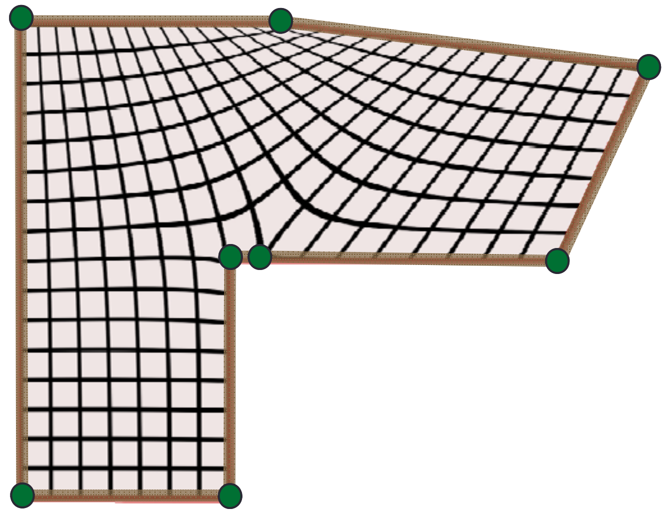
Mean-value coordinates



Cauchy coordinates

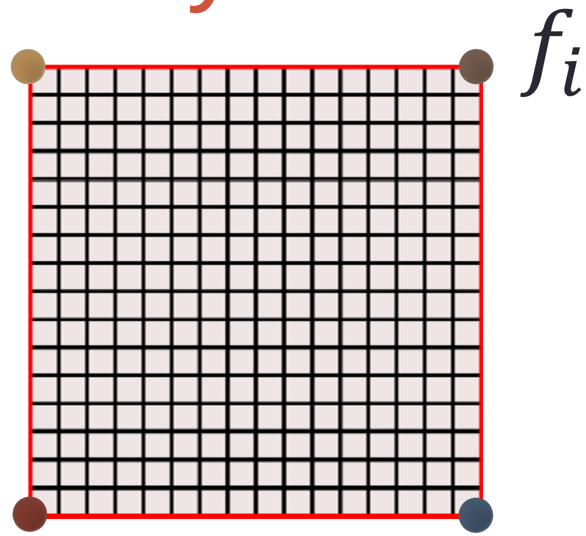


Harmonic coordinates



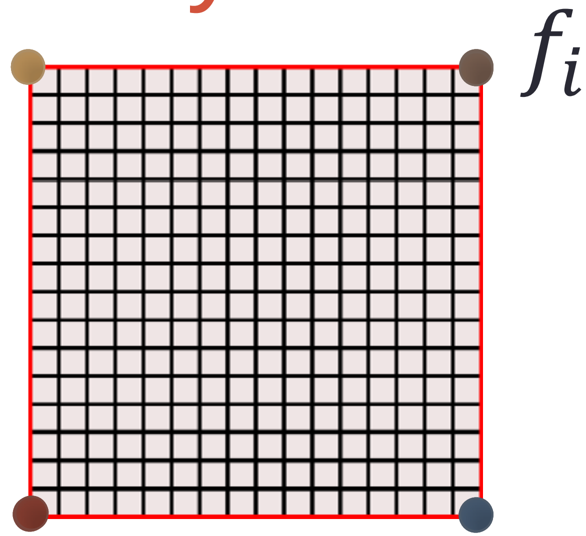
Example: Hermite Bary' Coordinates

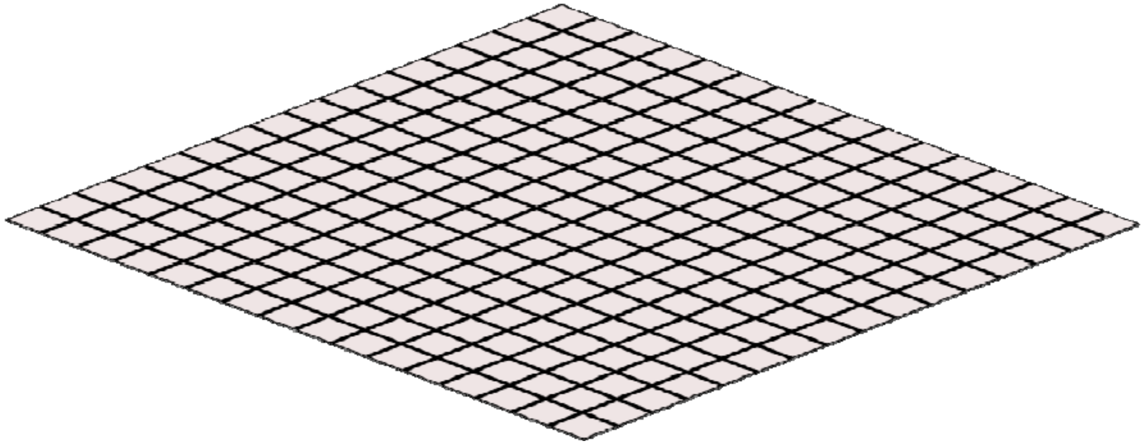
$$f(x) = \sum_{i=1}^{n} \alpha_i(x) f_i$$



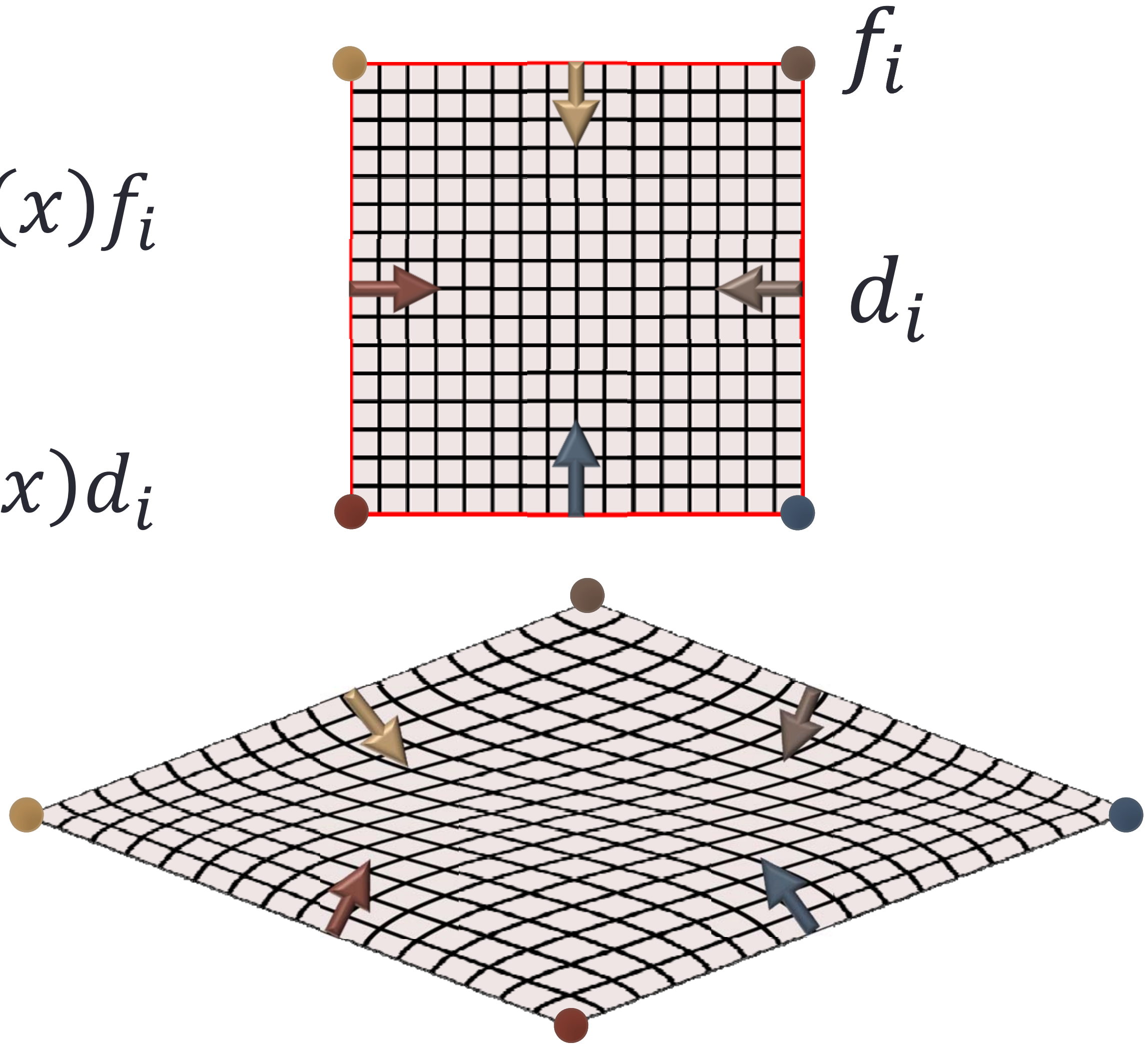
Example: Hermite Bary' Coordinates

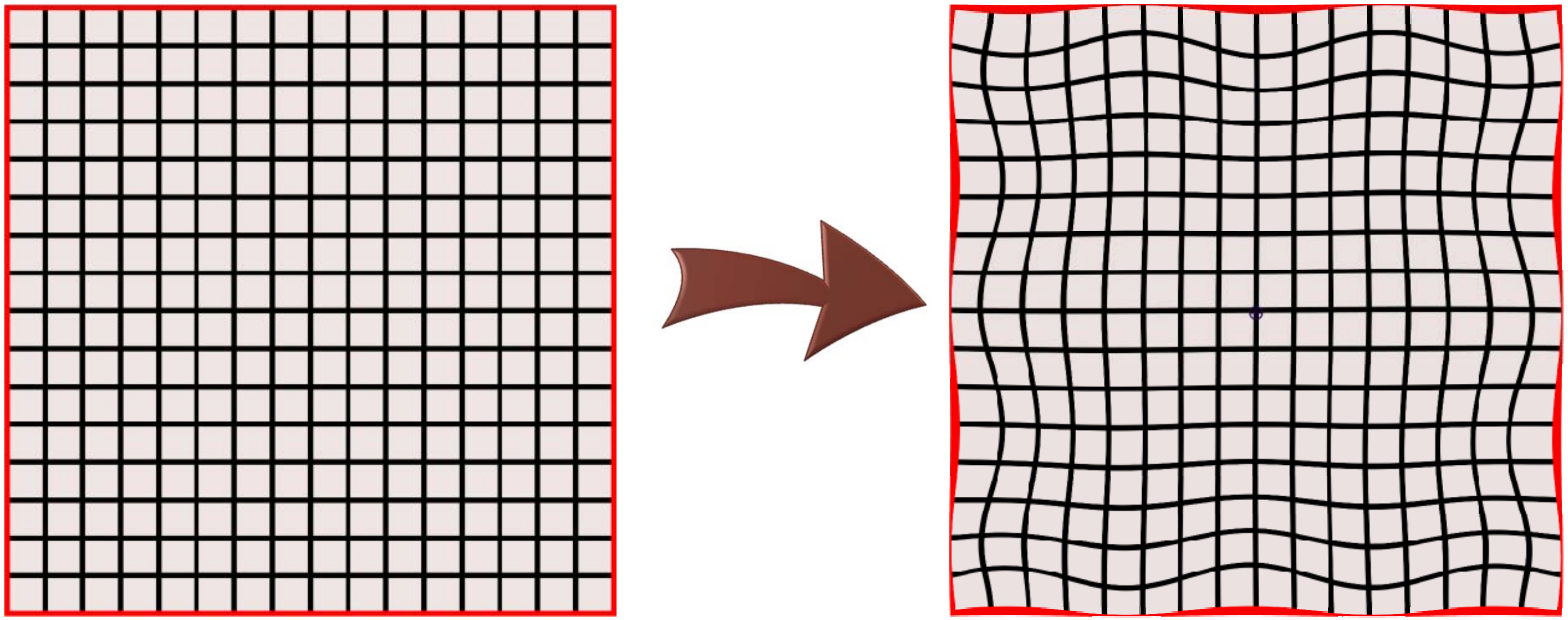
$$f(x) = \sum_{i=1}^{n} \alpha_i(x) f_i$$

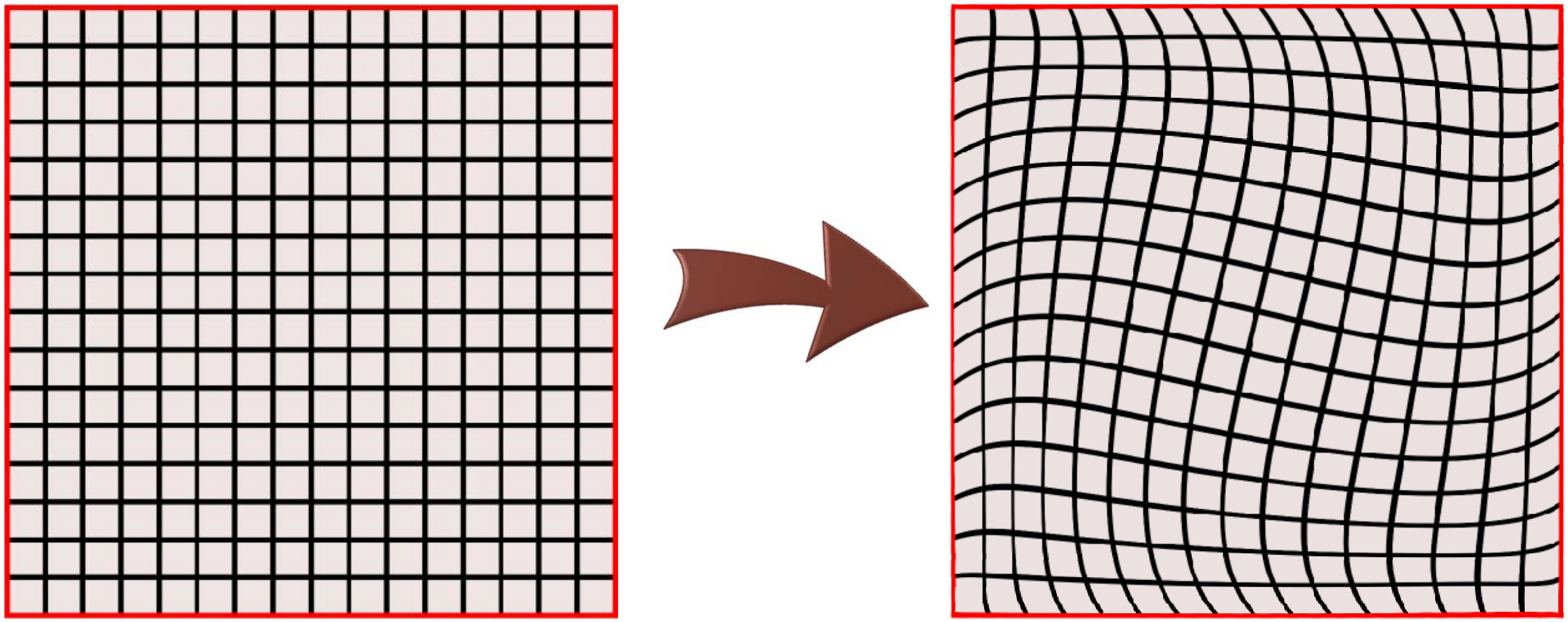




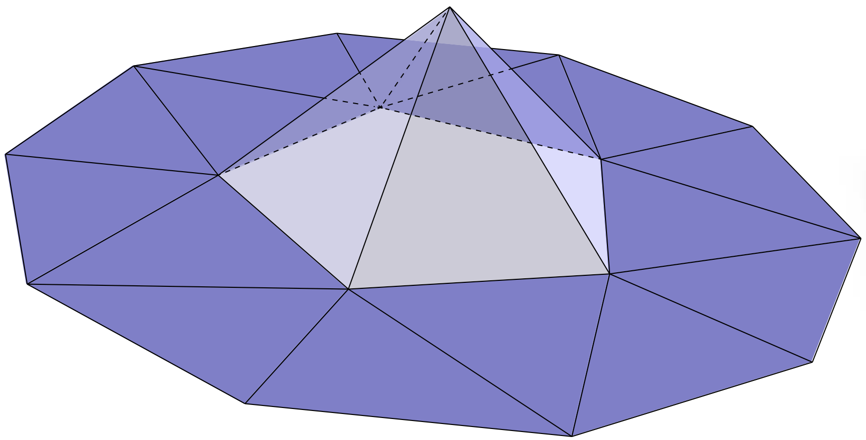
$$f(x) = \sum_{i=1}^{n} \alpha_i(x) f_i + \sum_{i=1}^{n} \beta_i(x) d_i$$

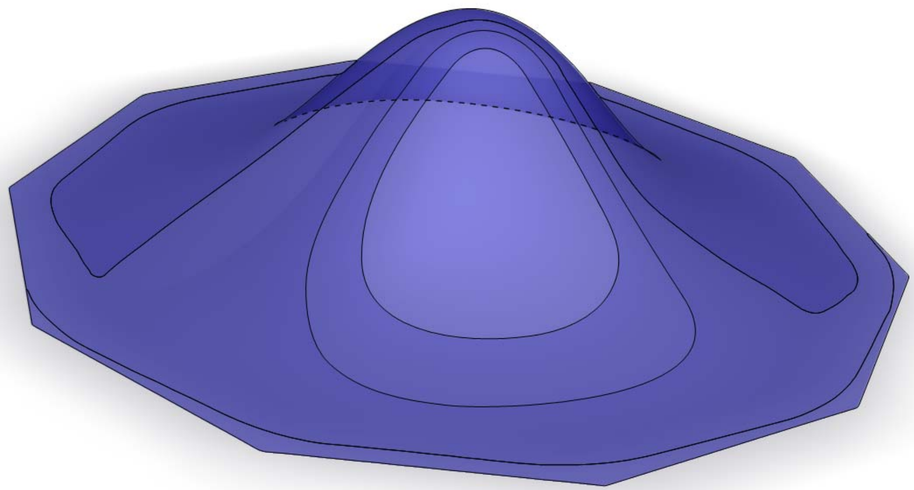






Basis functions



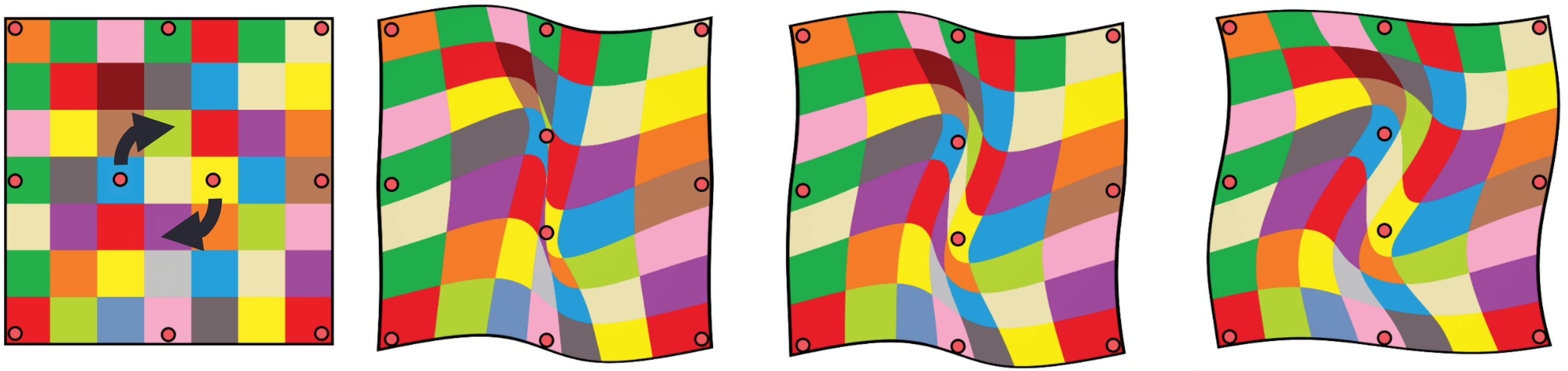


What are good maps?

Local

Bijectivity

Low distortion



Not Bijective

Bijective

Lower distortion

Globally Bijective VS. Locally Bijective

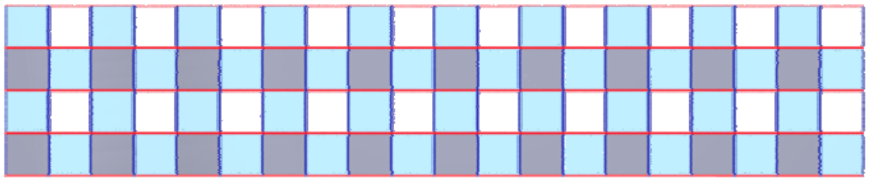
Globally Bijective → Locally Bijective

$\mathbf{f}$ is bijective

$\mathbf{f}: \mathbf{U} \rightarrow \mathbf{f}(\mathbf{U})$ is bijective

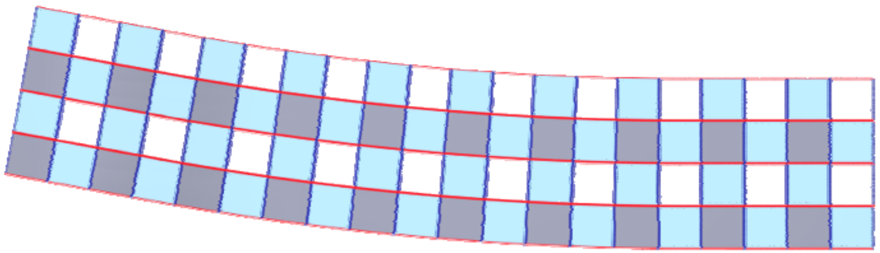
Globally Bijective VS. Locally Bijective

Globally Bijective	Locally Bijective
$\mathbf{f}$ is bijective	$\mathbf{f}: \mathbf{U} \rightarrow \mathbf{f}(\mathbf{U})$ is bijective



Globally Bijective VS. Locally Bijective

Globally Bijective	Locally Bijective
$\mathbf{f}$ is bijective	$\mathbf{f}: \mathbf{U} \rightarrow \mathbf{f}(\mathbf{U})$ is bijective



Globally Bijective VS. Locally Bijective

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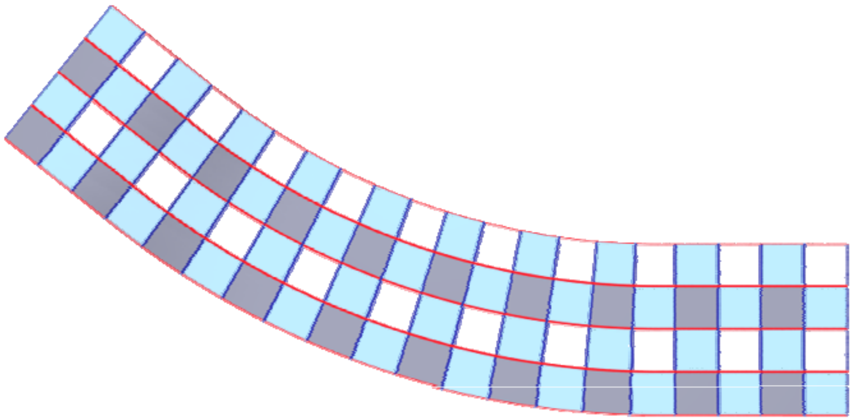

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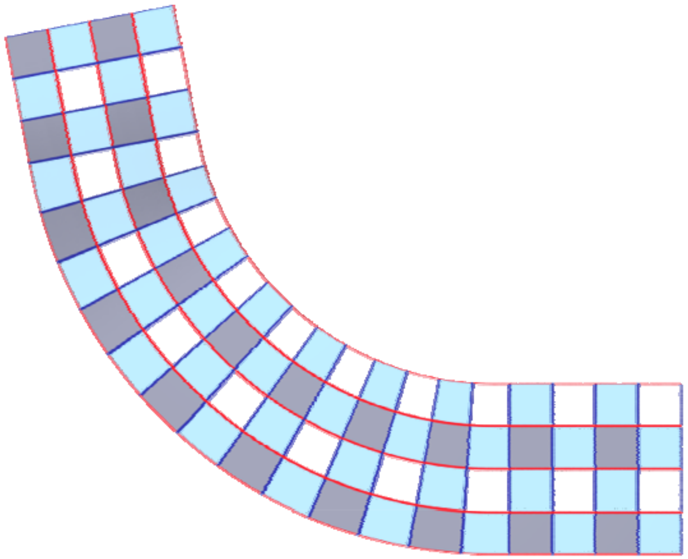

Globally Bijective VS. Locally Bijective

Globally Bijective

$\mathbf{f}$ is bijective

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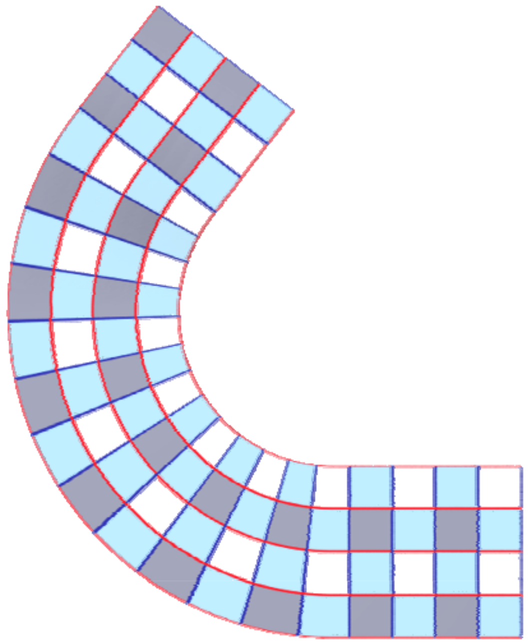
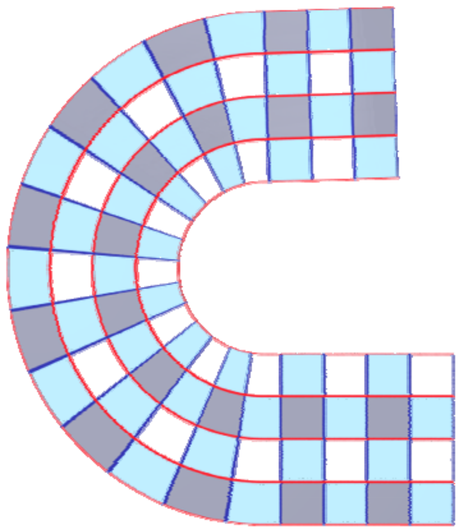
$\mathbf{f}: \mathbf{U} \to \mathbf{f}(\mathbf{U})$ is bijective



Globally Bijective VS. Locally Bijective

Globally Bijective	Locally Bijective
$\mathbf{f}$ is bijective	$\mathbf{f}: \mathbf{U} \to \mathbf{f}(\mathbf{U})$ is bijective

Still Bijective!



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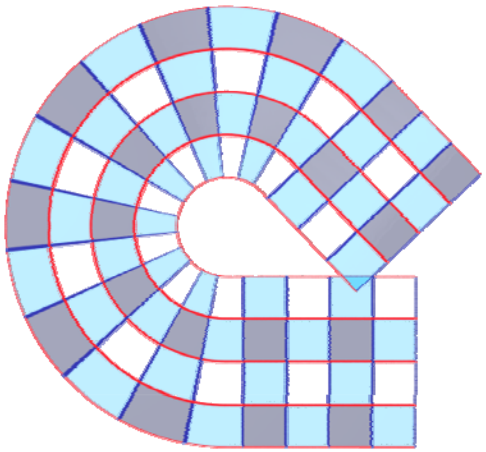


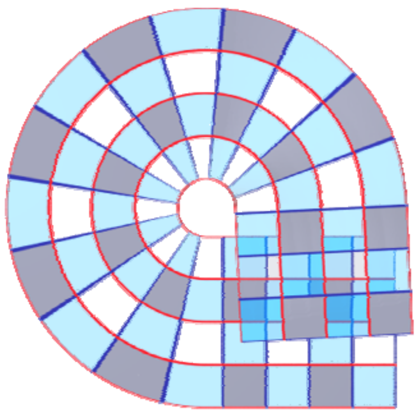
Globally Bijective VS. Locally Bijective

Globally Bijective

$\mathbf{f}$ is bijective

Locally Bijective

$\mathbf{f}: \mathbf{U} \rightarrow \mathbf{f}(\mathbf{U})$ is bijective



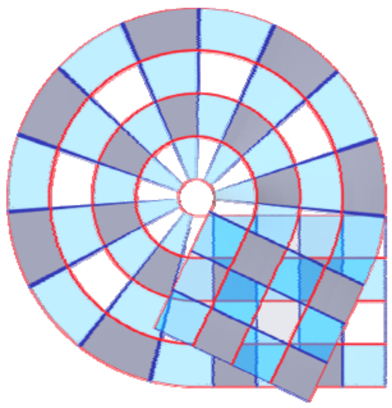
Globally Bijective VS. Locally Bijective

Globally Bijective

$\mathbf{f}$ is bijective

Locally Bijective

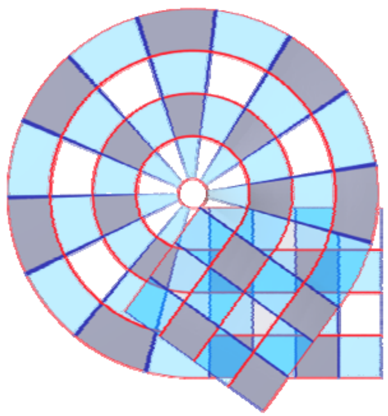
$\mathbf{f}: \mathbf{U} \rightarrow \mathbf{f}(\mathbf{U})$ is bijective



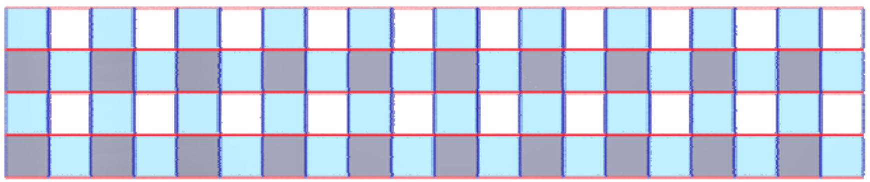
Globally Bijective VS. Locally Bijective

Globally Bijective	Locally Bijective
$\mathbf{f}$ is bijective	$\mathbf{f}: \mathbf{U} \rightarrow \mathbf{f}(\mathbf{U})$ is bijective

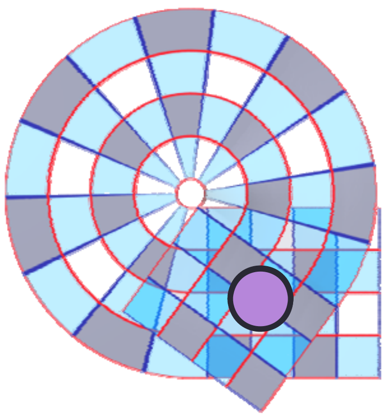
Not Bijective!



Globally Bijective VS. Locally Bijective

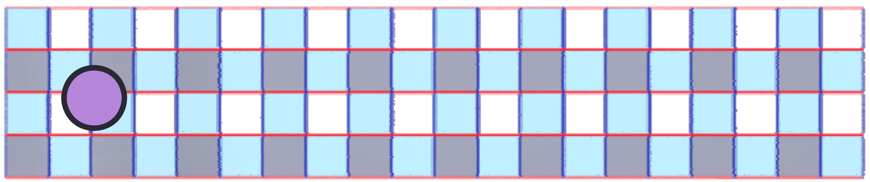


Not Bijective!

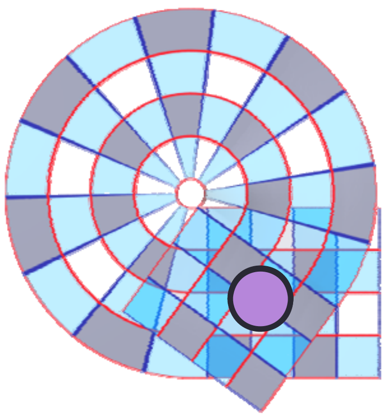


Two Pre-images

Globally Bijective VS. Locally Bijective

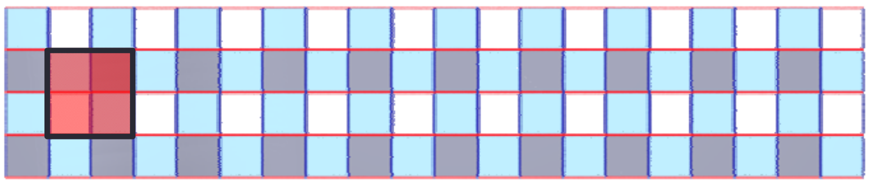


Not Bijective!

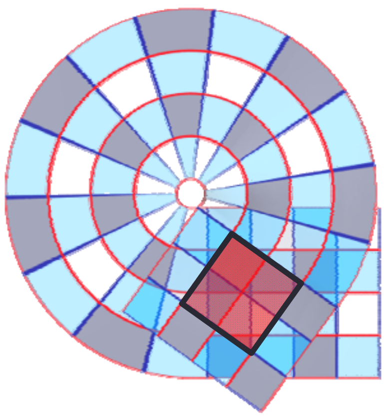


Globally Bijective VS. Locally Bijective

$\mathbf{U}$

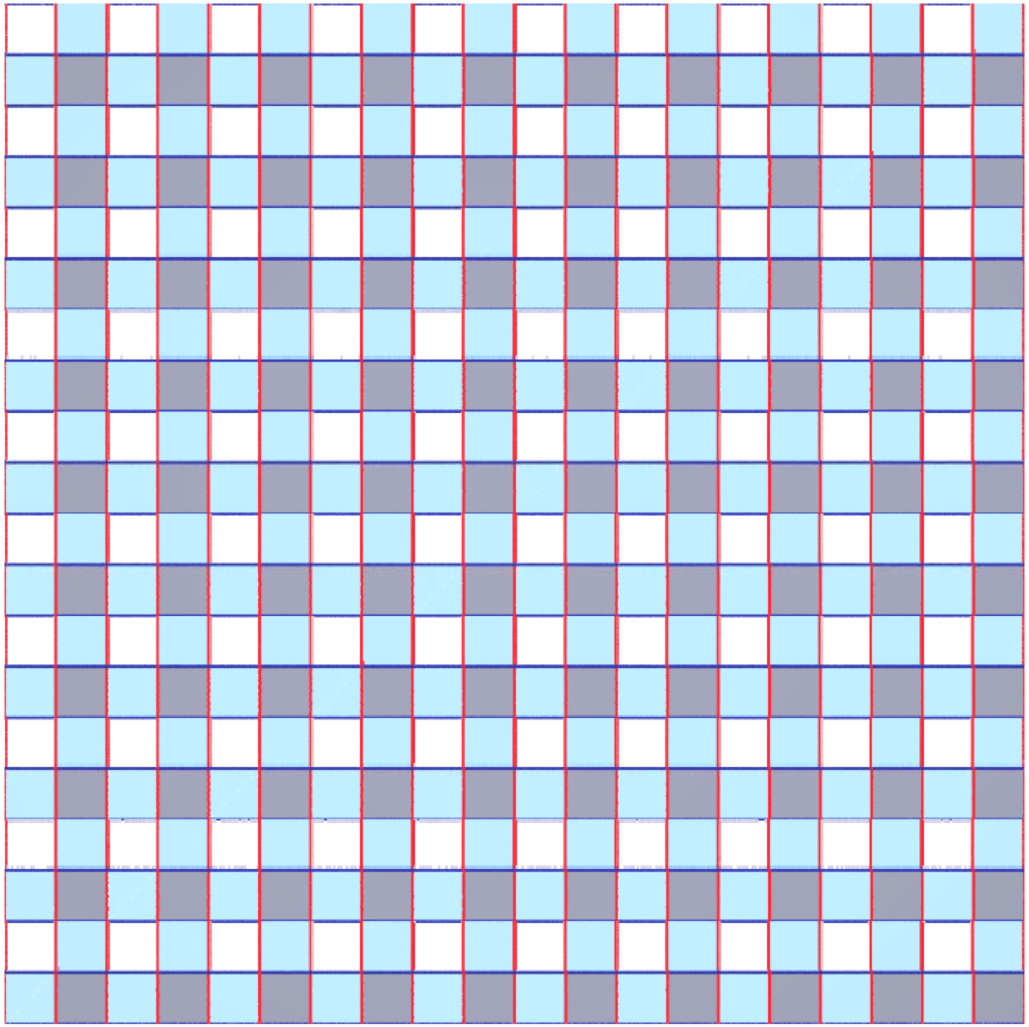


Only Locally Bijective!

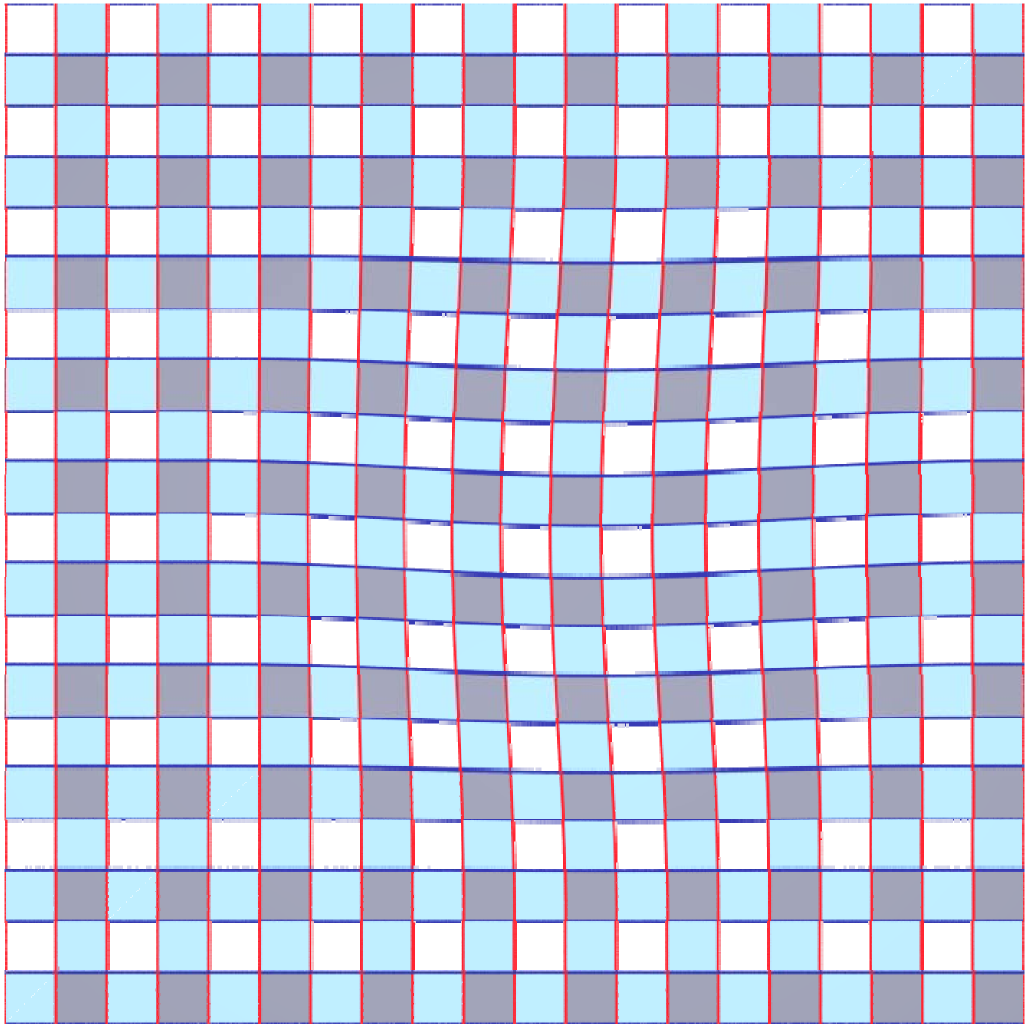


$\mathbf{f(U)}$

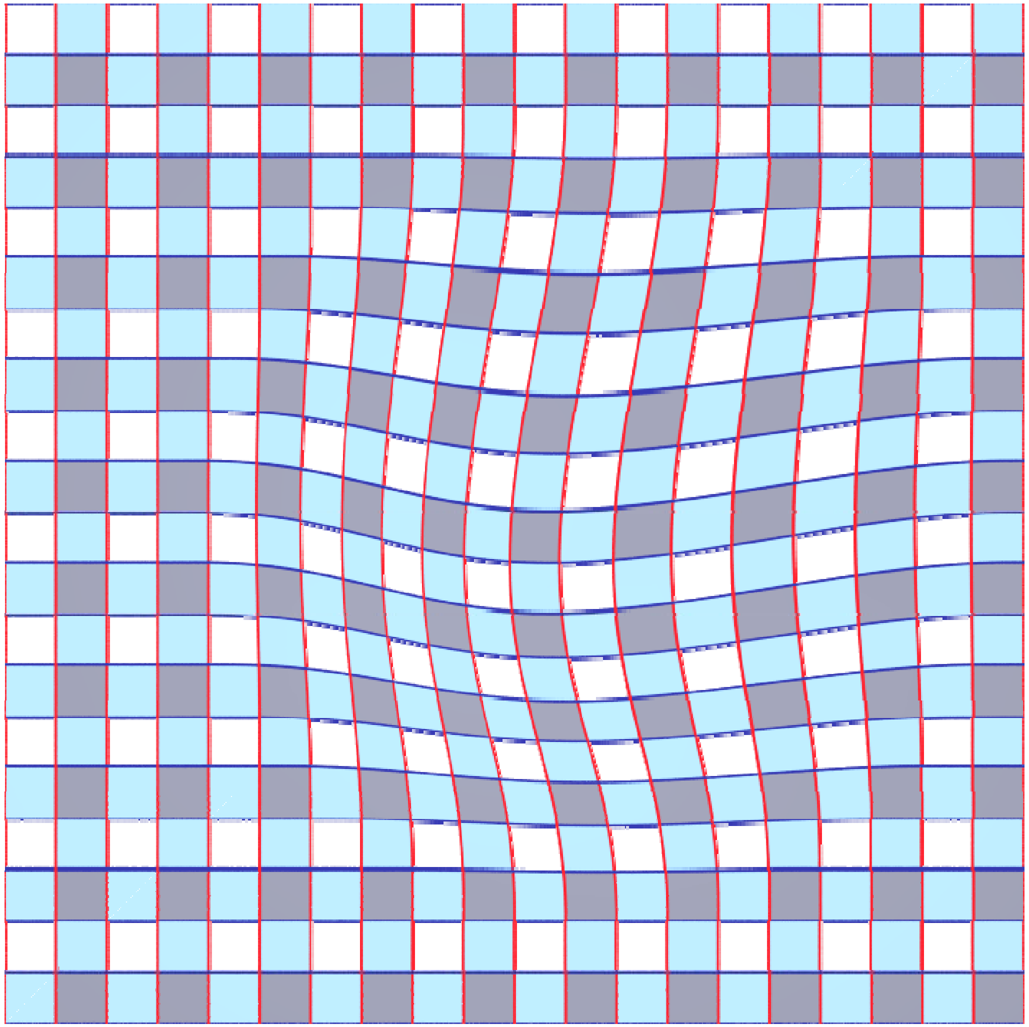
Locally Bijection – Non-example



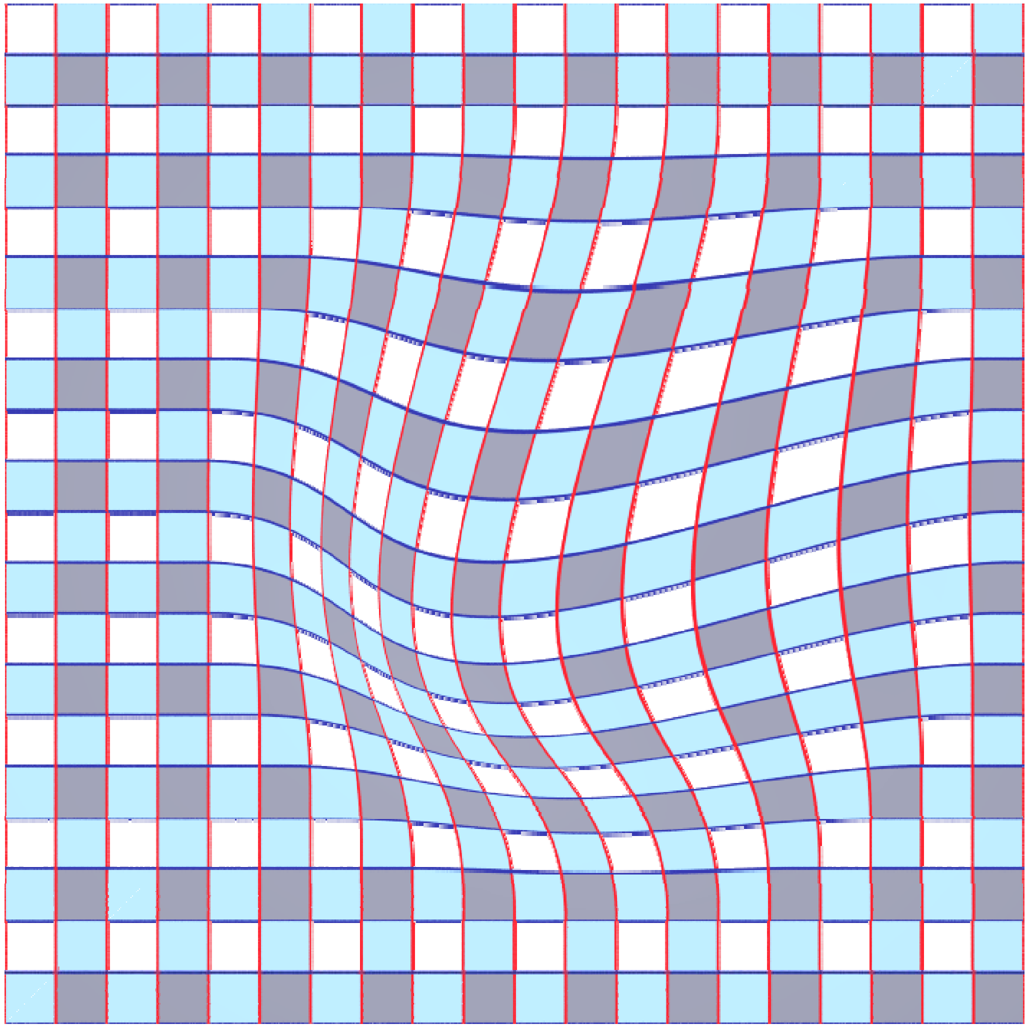
Locally Bijection – Non-example



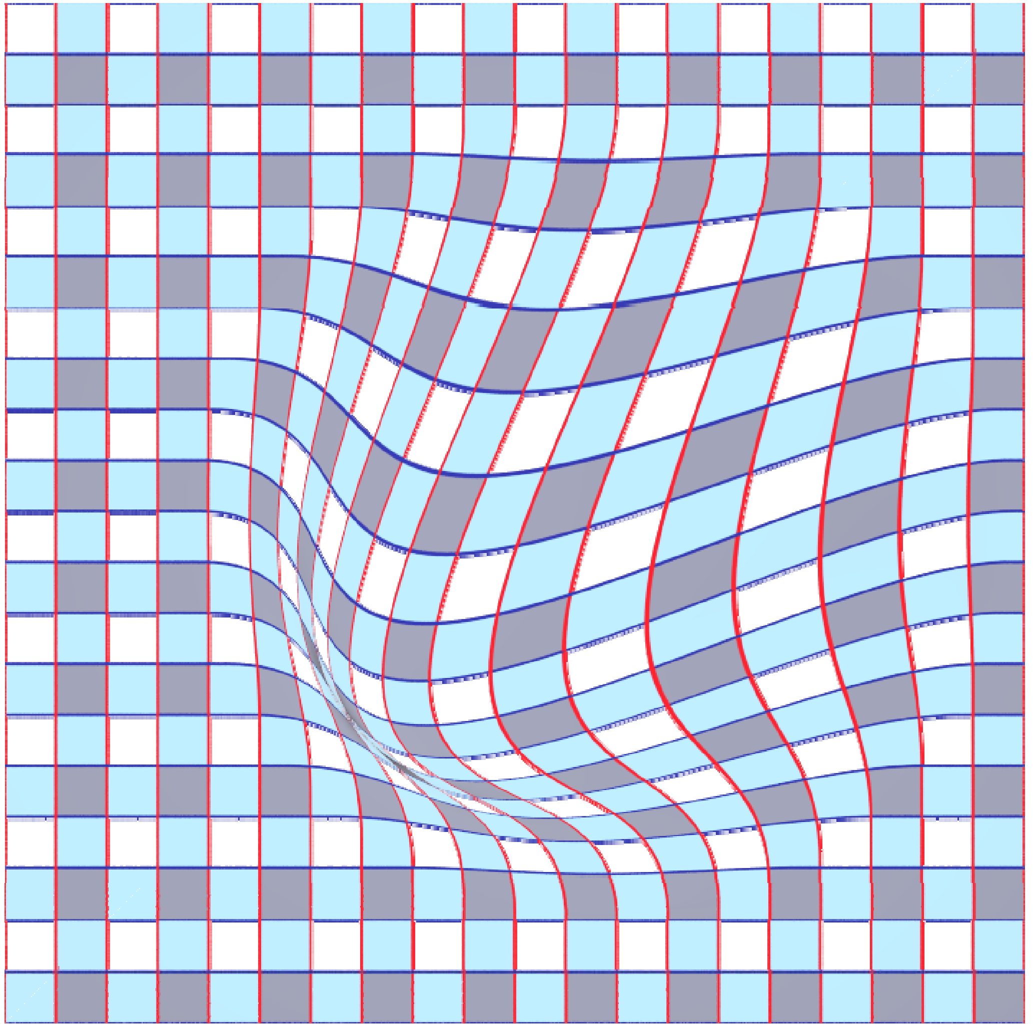
Locally Bijection – Non-example



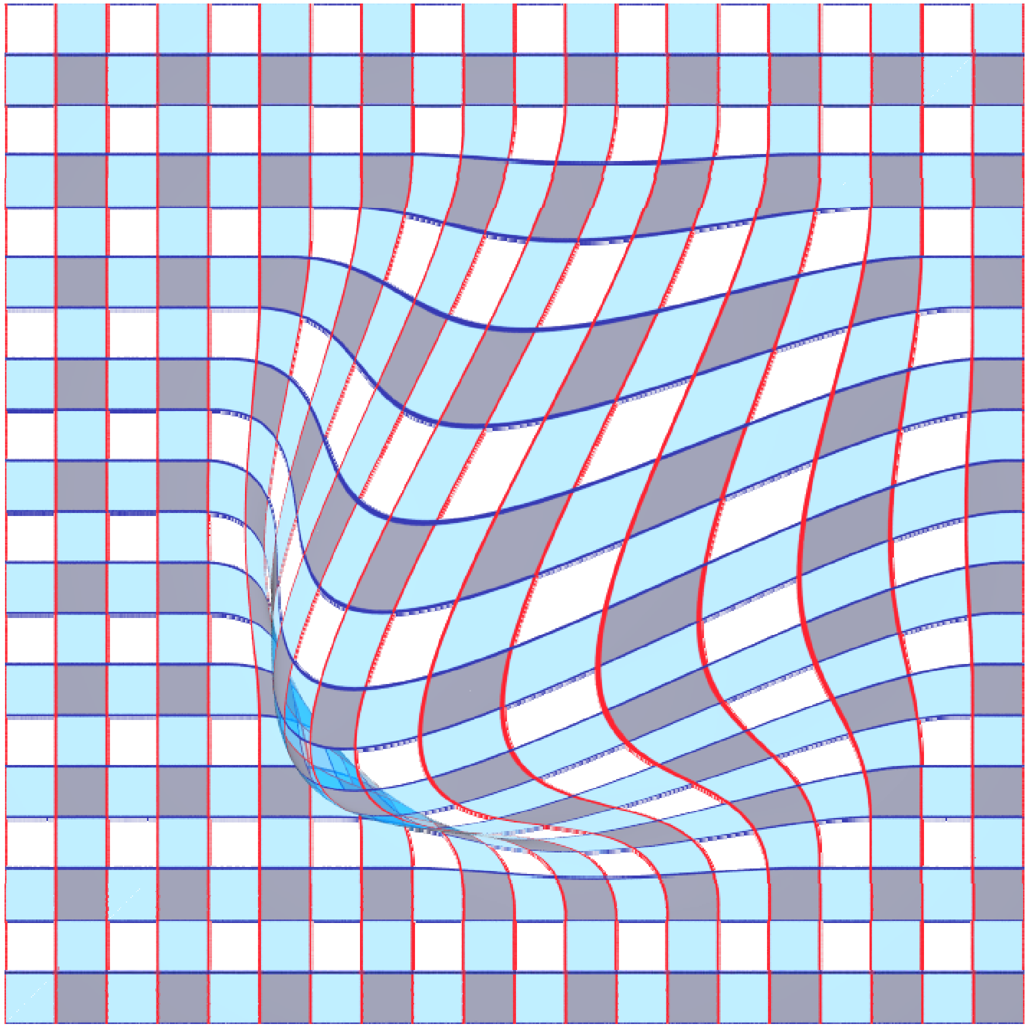
Locally Bijection – Non-example



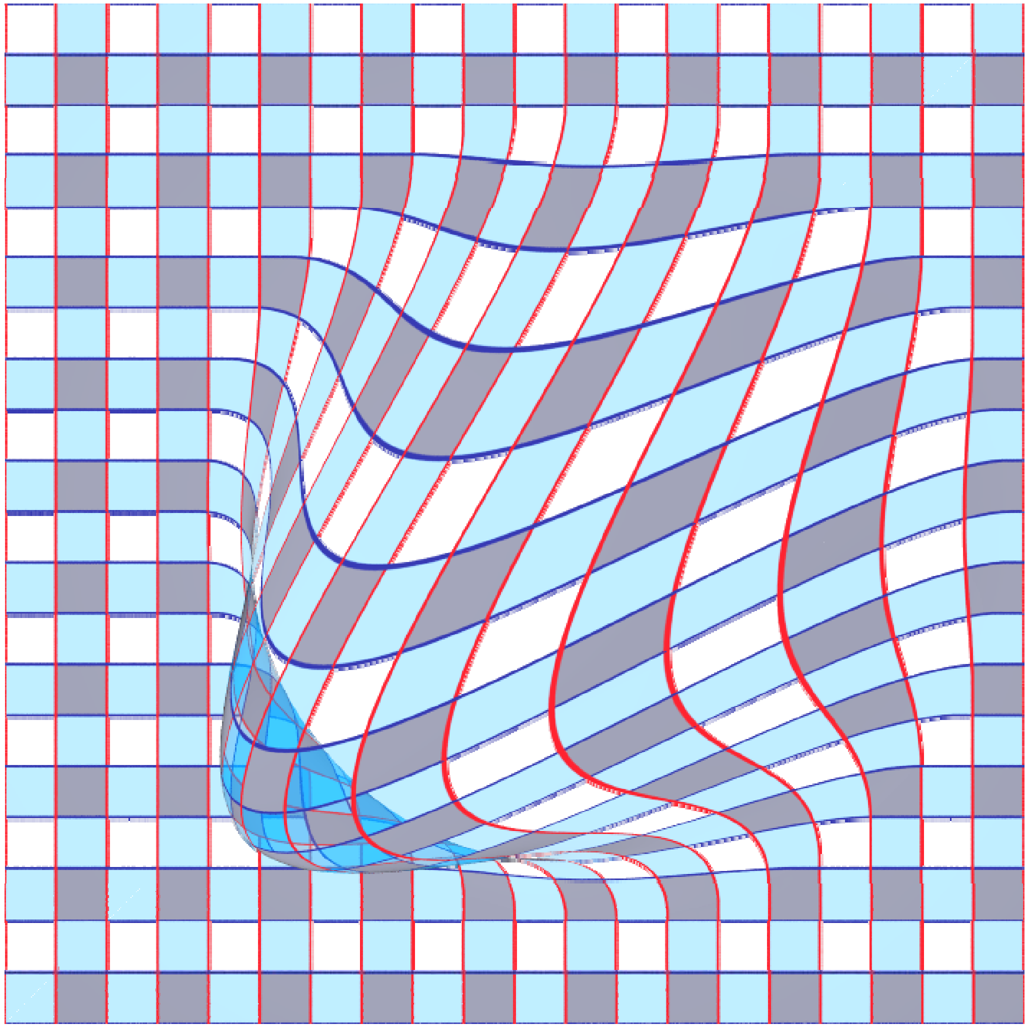
Locally Bijection – Non-example



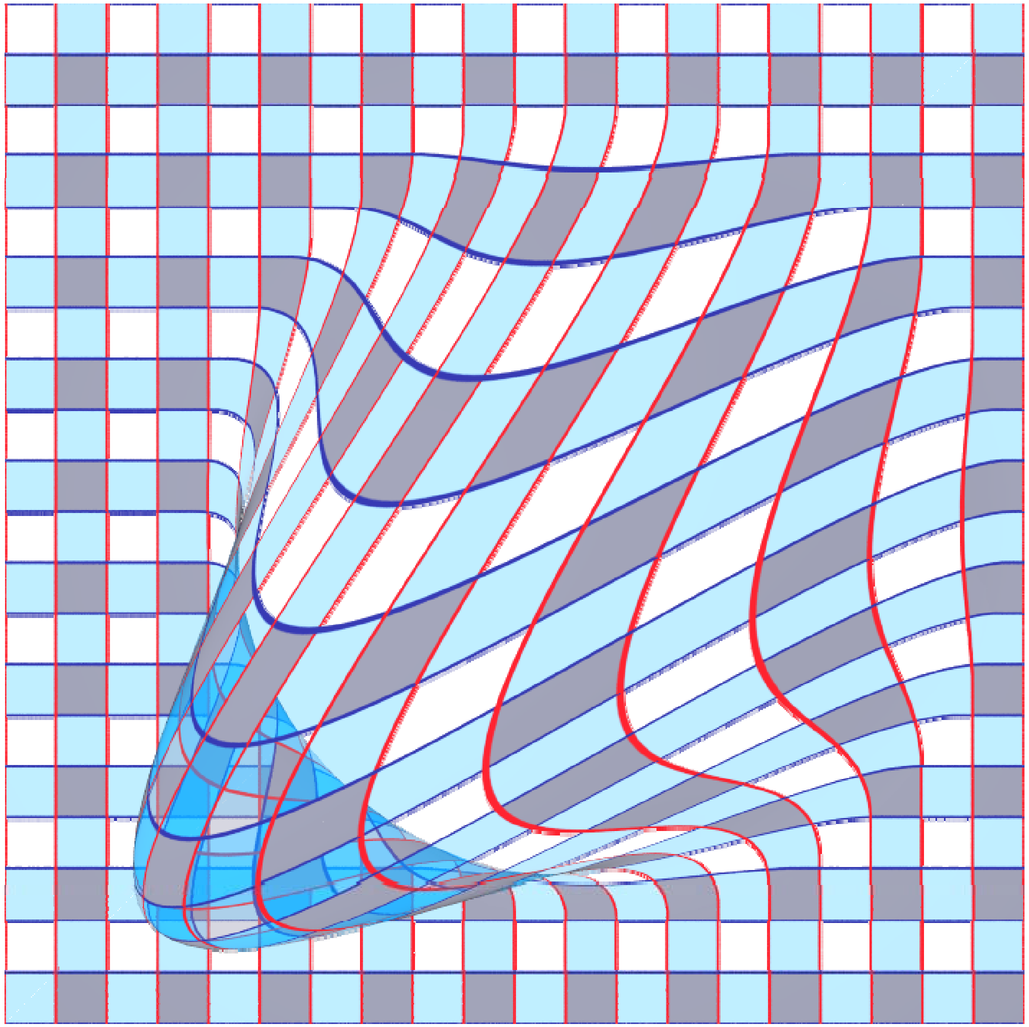
Locally Bijection – Non-example



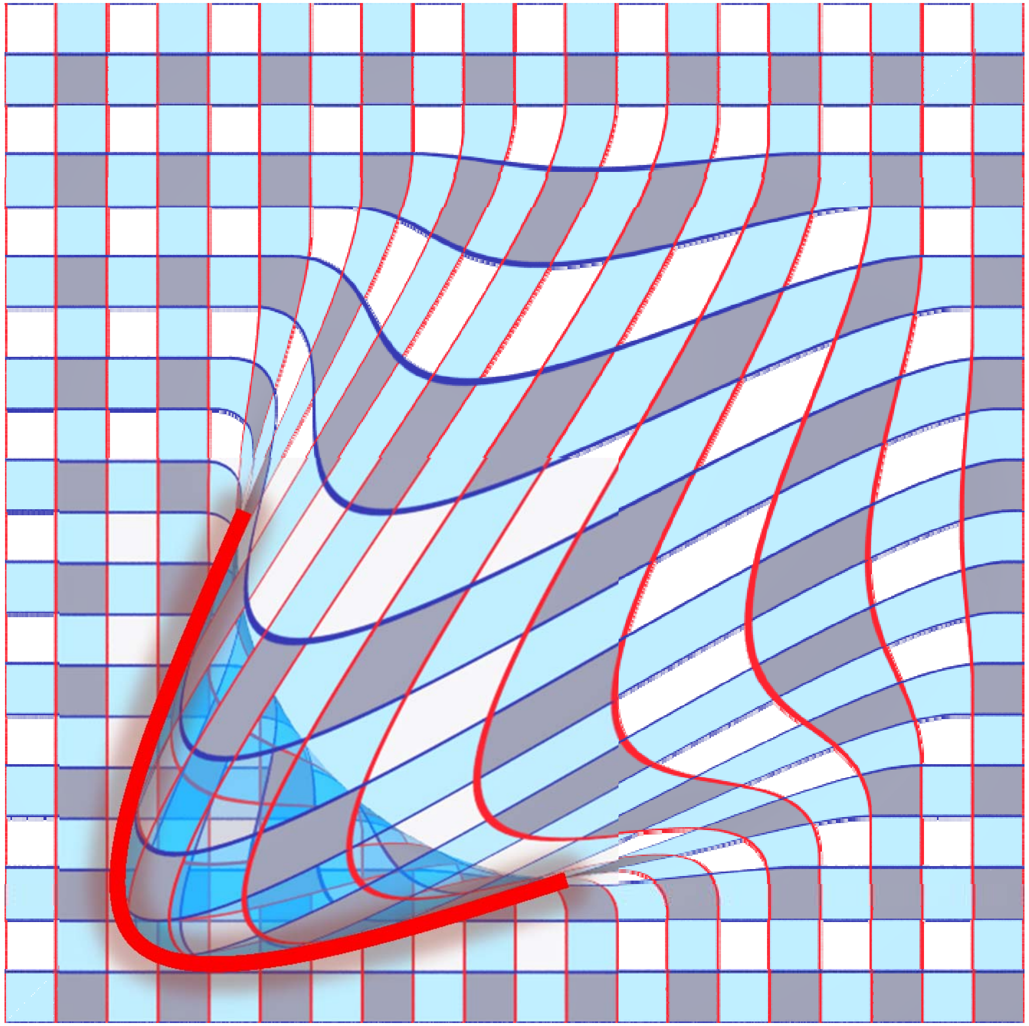
Locally Bijection – Non-example



Locally Bijection – Non-example

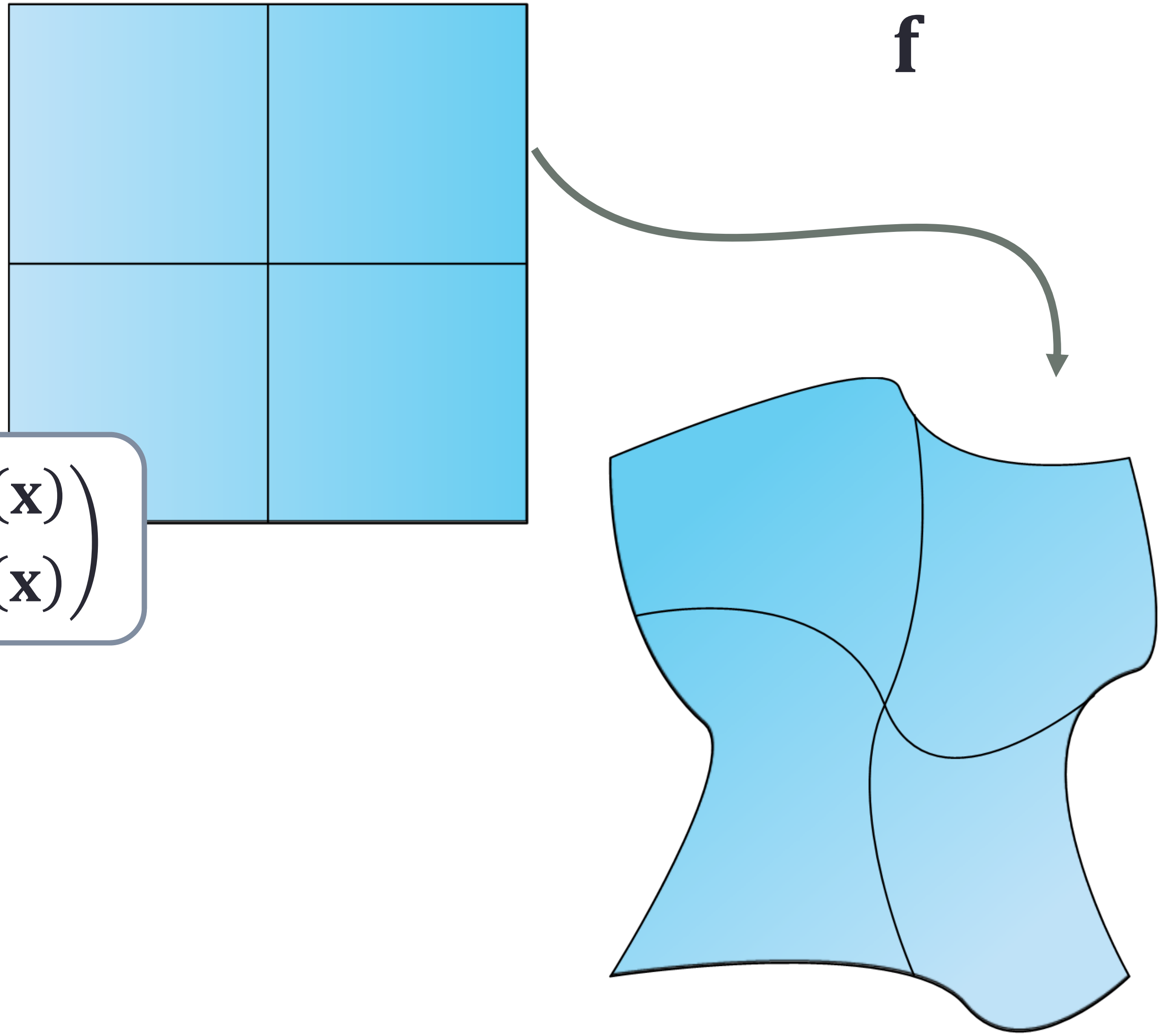


Locally Bijection – Non-example



Locally Bijection – Sufficient condition

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} u(\mathbf{x}) \\ v(\mathbf{x}) \end{pmatrix}$$

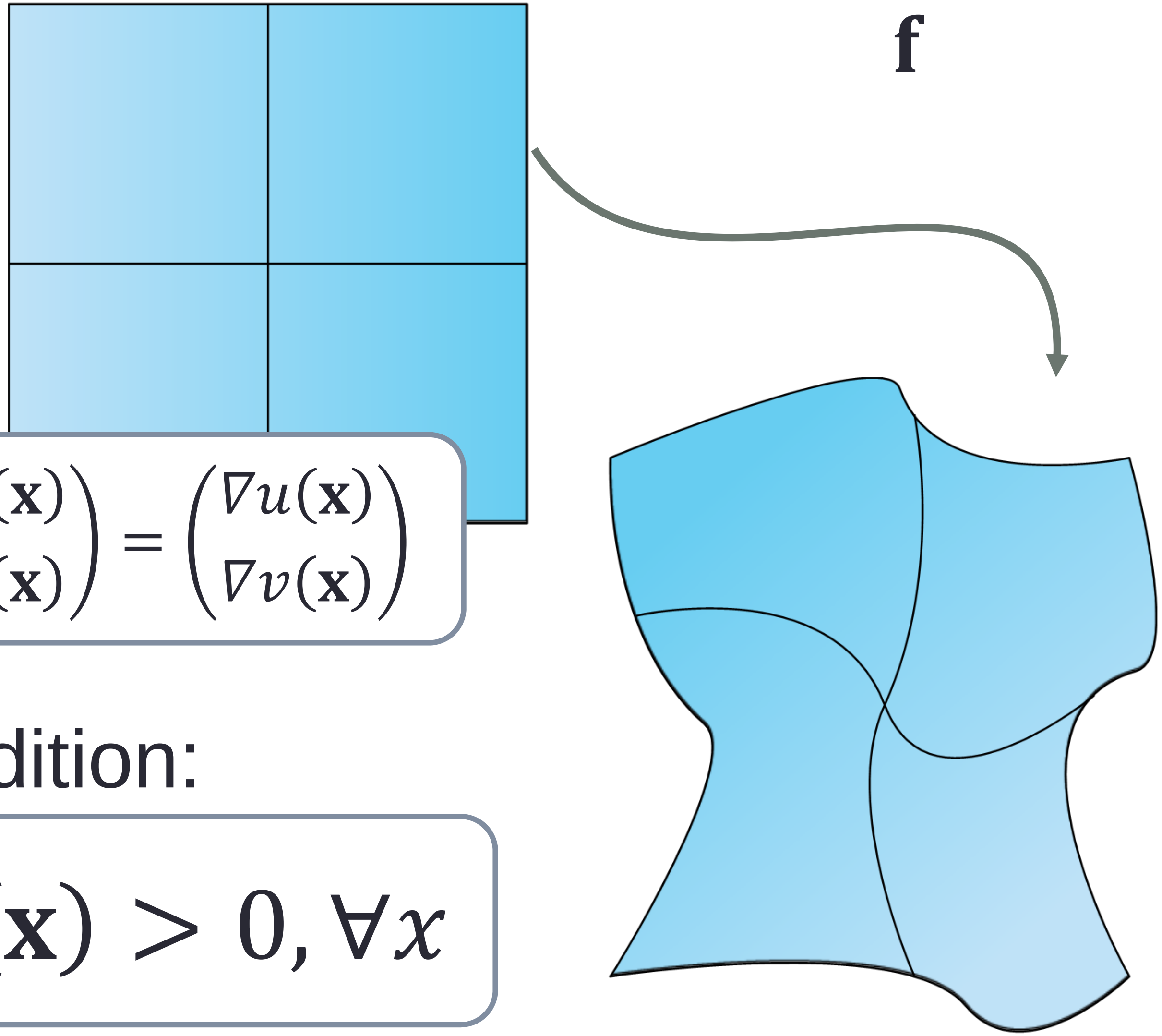


The Jacobian:

$$\mathcal{J}\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \partial_x u(\mathbf{x}) & \partial_y u(\mathbf{x}) \\ \partial_x v(\mathbf{x}) & \partial_y v(\mathbf{x}) \end{pmatrix}$$

Locally Bijection – Sufficient condition

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} u(\mathbf{x}) \\ v(\mathbf{x}) \end{pmatrix}$$



The Jacobian:

$$\mathcal{J}\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \partial_x u(\mathbf{x}) & \partial_y u(\mathbf{x}) \\ \partial_x v(\mathbf{x}) & \partial_y v(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \nabla u(\mathbf{x}) \\ \nabla v(\mathbf{x}) \end{pmatrix}$$

The Condition:

$$\det \mathcal{J}\mathbf{f}(\mathbf{x}) > 0, \forall x$$

Globally Bijective VS. Locally Bijective

Globally Bijective → Locally Bijective

$\mathbf{f}$ is bijective

$\mathbf{f}: \mathbf{U} \rightarrow \mathbf{f}(\mathbf{U})$ is bijective

Globally Bijective VS. Locally Bijective

Globally Bijective ⟵ Locally Bijective

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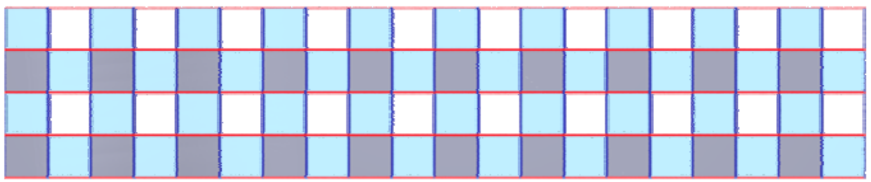

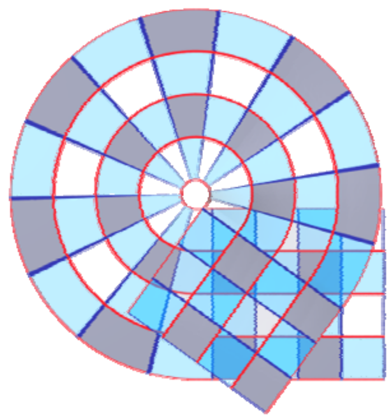

Globally Bijective VS. Locally Bijective

Globally Bijective ⟵ Locally Bijective

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Google: "Global inversion theorems"

What are good maps?

Local

Bijectivity

Low distortion

Not Bijective

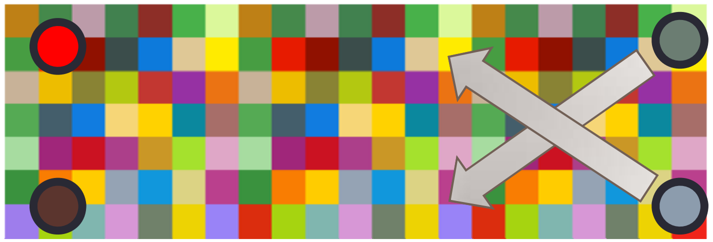
Bijective

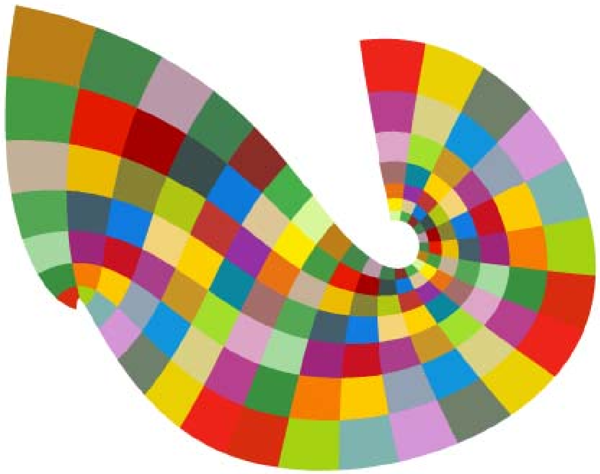
Lower distortion

Distortion - Types

Conformal distortion

Isometric distortion

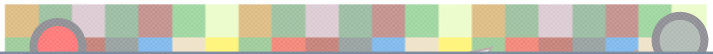




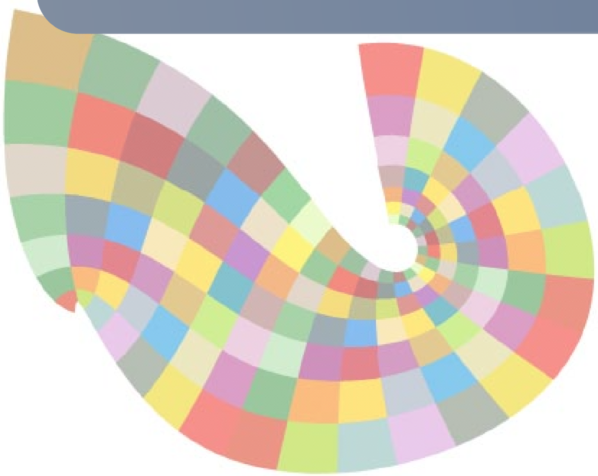
Distortion - Types

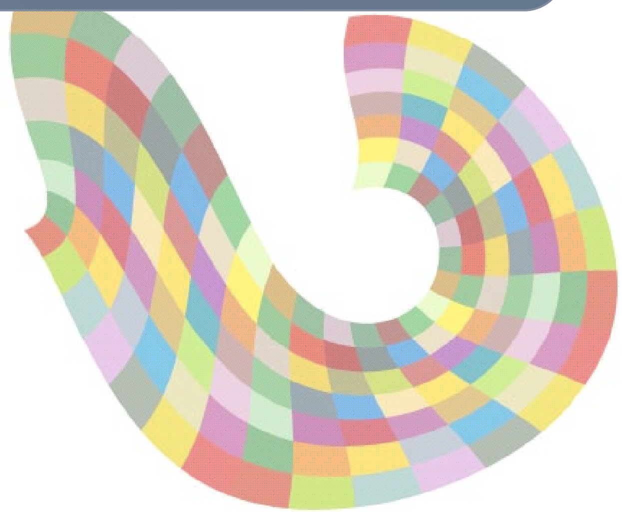
Conformal distortion

Isometric distortion



The distortion is a function of the Jacobian at a point





[Lévy et al. 2002]

Distortion - LSCM

LSCM – **L**east **S**quares **C**onformal **M**ap

We want the Jacobian

$$\begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}$$

to be a similarity matrix

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

$$\partial_x u = \partial_y v$$
$$\partial_y u = -\partial_x v$$

Cauchy-Riemann Equations

Distortion - LSCM

LSCM – **L**east **S**quares **C**onformal **M**ap

We want the Jacobian

$$\begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}$$

to be a similarity matrix

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

$$\mathcal{D}_{\mathrm{LSCM}} = (\partial_x u - \partial_y v)^2 + (\partial_y u + \partial_x v)^2$$

Quick Notation Change

$$\mathcal{J}\mathbf{f} \Longrightarrow \mathbf{A}$$

$$\begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} \Longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathcal{D}_{\mathrm{LSCM}} = (\partial_x u - \partial_y v)^2 + (\partial_y u + \partial_x v)^2$$

$$(a - d)^2 + (b + c)^2$$

[Liu et al. 2008]

Distortion – ASAP

ASAP– **A**s **S**imilar **A**s **P**ossible

$$\mathcal{D}_{\mathrm{ASAP}} = \|\mathbf{A} - \mathcal{S}_{\mathbf{A}}\|_F^2$$

Jacobian

Closest Similarity

How to compute closest similarity?

Distortion – ASAP

ASAP– **A**s **S**imilar **A**s **P**ossible

How to compute closest similarity?

Distortion – ASAP

ASAP– **A**s **S**imilar **A**s **P**ossible

How to compute closest similarity?

In 2D:

$$\min_{\mathcal{S}} \|\mathbf{A} - \mathcal{S}\|_F^2$$
$$\text{s.t.} \quad \mathcal{S} \text{ is similarity}$$

$$\min_{\alpha,\beta} \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \right\|_F^2$$

Distortion – ASAP

ASAP– **A**s **S**imilar **A**s **P**ossible

How to compute closest similarity?

In 2D:

$$\min_{\mathcal{S}} \|\mathbf{A} - \mathcal{S}\|_F^2$$
$$\text{s.t.} \quad \mathcal{S} \text{ is similarity}$$

$$\mathcal{S} = \frac{1}{2}\begin{pmatrix} a+d & c-b \\ b-c & a+d \end{pmatrix}$$

Distortion – ASAP

ASAP– **A**s **S**imilar **A**s **P**ossible

How to compute closest similarity?

In 2D:

$$\mathbf{A} = \frac{1}{2}\begin{pmatrix} a+d & c-b \\ b-c & a+d \end{pmatrix} + \frac{1}{2}\begin{pmatrix} a-d & c+b \\ b+c & d-a \end{pmatrix}$$

Distortion – ASAP

ASAP– **A**s **S**imilar **A**s **P**ossible

How to compute closest similarity?

In 2D:

$$\mathbf{A} = \underbrace{\frac{1}{2}\begin{pmatrix} a+d & c-b \\ b-c & a+d \end{pmatrix}}_{\mathcal{S}_A} + \underbrace{\frac{1}{2}\begin{pmatrix} a-d & c+b \\ b+c & d-a \end{pmatrix}}_{\mathcal{S}_A^{\perp}}$$

Similarity Anti-Similarity

Distortion – ASAP

ASAP– **A**s **S**imilar **A**s **P**ossible

$$\| \quad \mathcal{S}_A^{\perp} \quad \|_F^2$$

Measure of antisimilarity

Jacobian

Closest Similarity

$$\left\| \begin{pmatrix} a-d & c+b \\ b+c & d-a \end{pmatrix} \right\|_F^2$$

$$(a-d)^2 + (b+c)^2$$

Distortion – ASAP

ASAP– **A**s **S**imilar **A**s **P**ossible

$$\| \quad S_A^{\perp} \quad \|_F^2$$

Jac

LSCM = ASAP

st

Similarity

$$\left\| \begin{pmatrix} a-d & c+b \\ b+c & d-a \end{pmatrix} \right\|_F^2$$

Distortion - ARAP

ARAP– **A**s **R**igid **A**s **P**ossible

$$\mathcal{D}_{\mathrm{ARAP}} = \|\mathbf{A} - \mathcal{R}_{\mathbf{A}}\|_F^2$$

Jacobian

Closest Rotation

How to compute closest Rotation?

Singular Value Decomposition

Every Matrix M has a factorization of the form

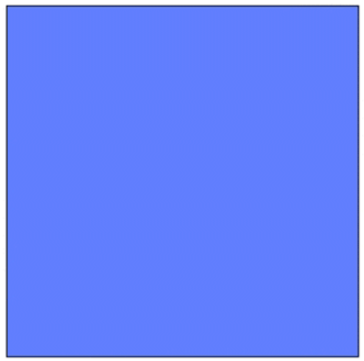
$$M \quad = \quad U \qquad\qquad S \qquad\qquad V^T$$

$$\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \qquad \sigma_1 > \sigma_2$$

Singular Value Decomposition

Every Matrix M has a factorization of the form

$$M \quad = \quad U \qquad\qquad S \qquad\qquad V^T$$



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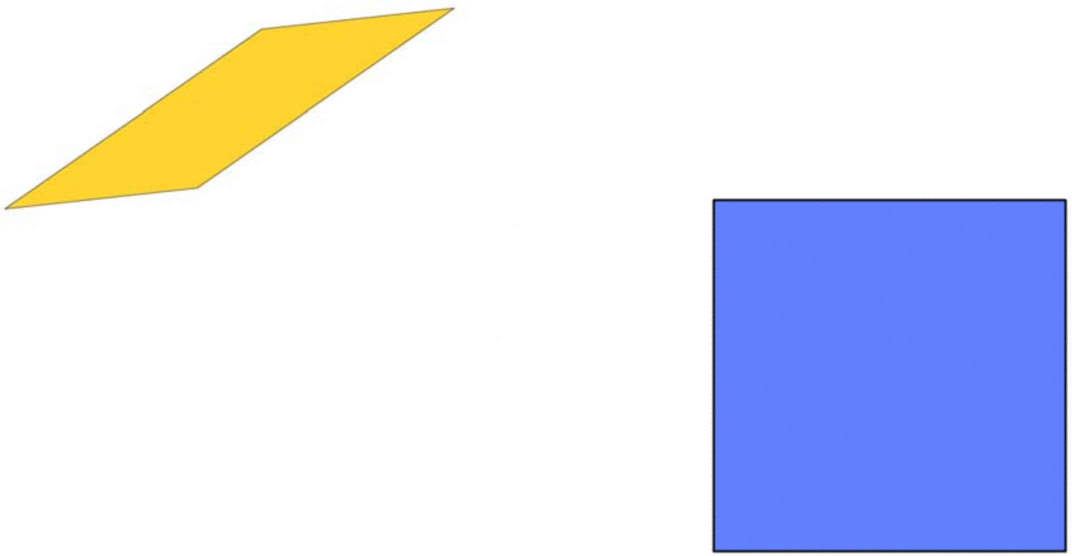

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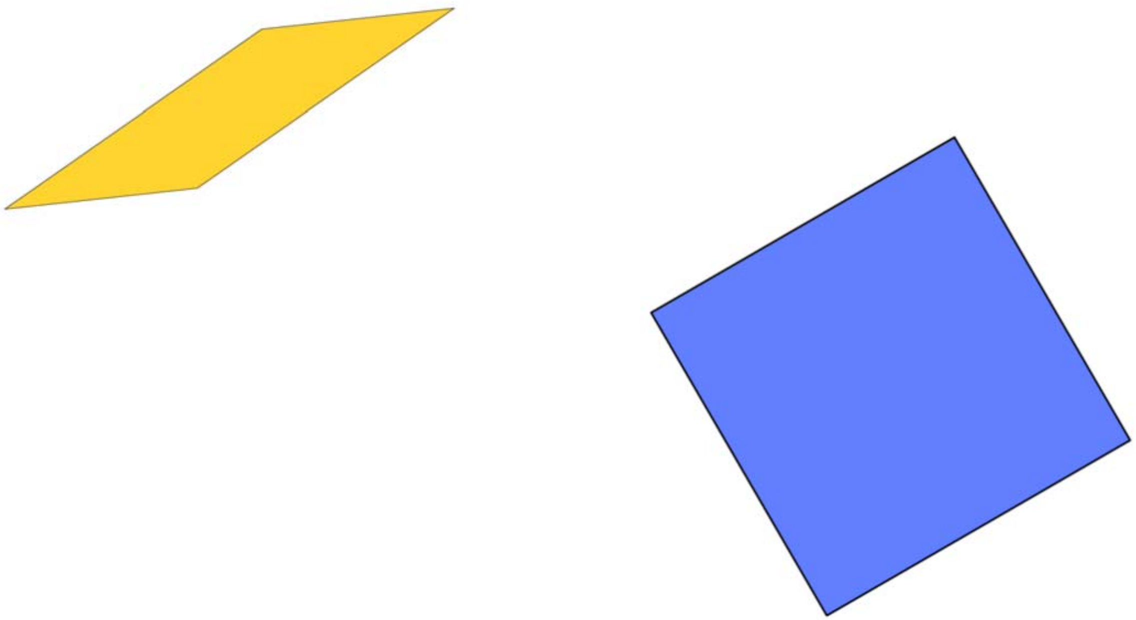

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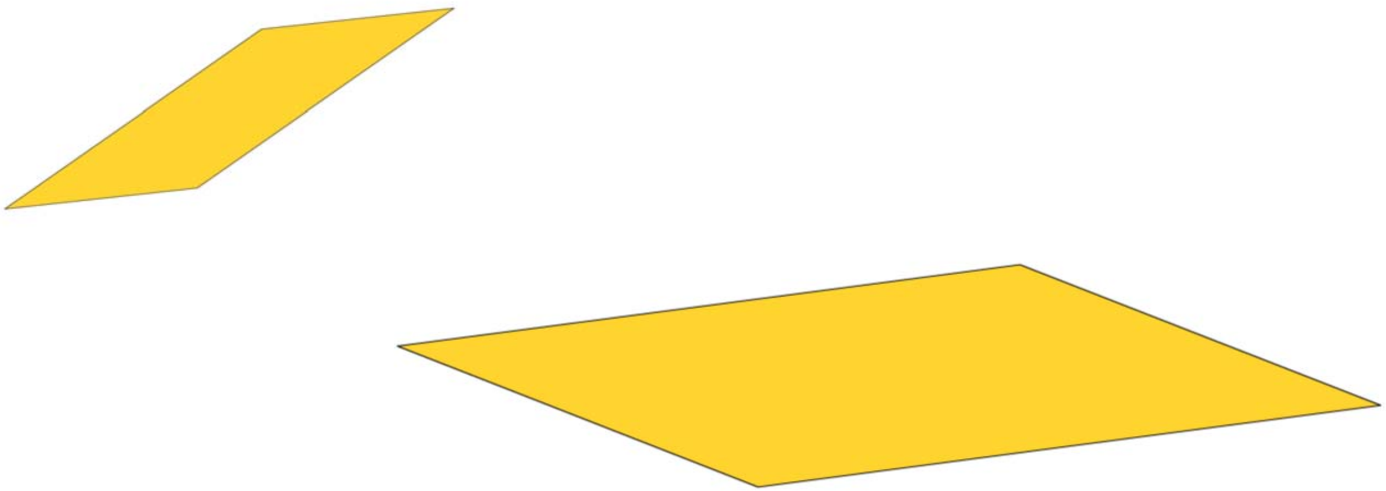


Singular Value Decomposition

Every Matrix M has a factorization of the form

$$M \quad = \quad U \qquad\qquad S \qquad\qquad V^T$$

U and V are not rotations!

Singular Value Decomposition

Every Matrix M has a factorization of the form

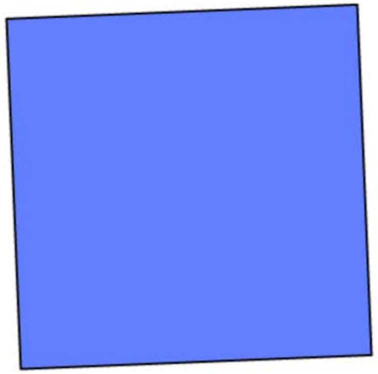
$$M = U \qquad S \qquad V^T$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Singular Value Decomposition

Every Matrix M has a factorization of the form

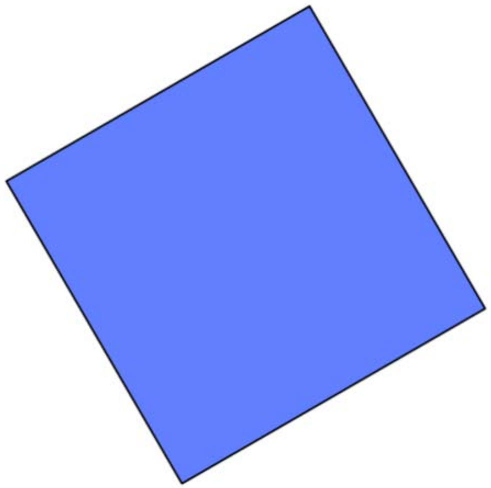
$$M \quad = \quad U \qquad\qquad S \qquad\qquad RV^T$$



Singular Value Decomposition

Every Matrix M has a factorization of the form

$$M \quad = \quad U \qquad\qquad S \qquad\qquad RV^T$$



Singular Value Decomposition

Every Matrix M has a factorization of the form

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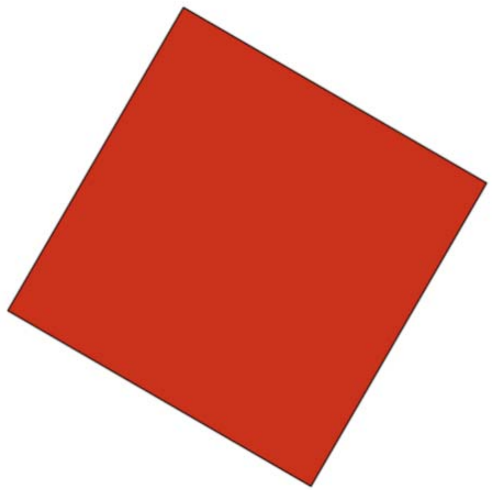
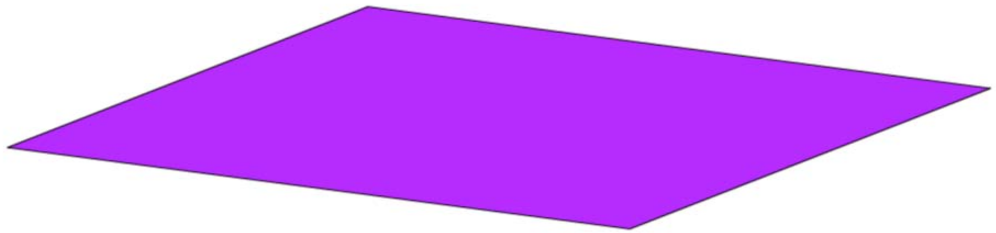


Singular Value Decomposition

Every Matrix M has a factorization of the form

$$M \quad = \quad U \qquad\qquad S \qquad\qquad RV^T$$



Signed Singular Value Decomposition

Every Matrix M has a factorization of the form

$$M \quad = \quad U \qquad\qquad S \qquad\quad RV^T$$

Signed Singular Value Decomposition

Every Matrix M has a factorization of the form

$$M \quad = \quad U \qquad\qquad SR \qquad\qquad V^T$$

$$\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_2 \end{pmatrix} \sigma_1 > \sigma_2$$

Now U and V are rotations!

Signed Singular Value Decomposition

Every Matrix M has a factorization of the form

$$M \quad = \quad U \qquad\qquad SR \qquad\qquad V^T$$

$$\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_2 \end{pmatrix} \sigma_1 > \sigma_2$$

Now U and V are rotations!

What if U and V both had reflections?

Signed Singular Value Decomposition

Every Matrix M has a factorization of the form

$$M \quad = \quad U \qquad\qquad SR \qquad\qquad V^T$$

$$\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_2 \end{pmatrix} \sigma_1 > \sigma_2$$

Now U and V are rotations!

What if U and V both had reflections?

$$\operatorname{sign} \det M = \operatorname{sign}(\sigma_2)$$

Singular values in 2D

Closed form expression

$$\mathcal{J}\mathbf{f} = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} = \begin{pmatrix} \nabla u \\ \nabla v \end{pmatrix}$$

$$\alpha = \frac{\nabla u + \triangleleft v}{2} \qquad \beta = \frac{\nabla u + \triangleright v}{2}$$

$$\sigma_1 = \|\alpha\| + \|\beta\| \qquad \sigma_2 = \|\alpha\| - \|\beta\|$$

Distortion - ARAP

ARAP– **A**s **R**igid **A**s **P**ossible

$$\mathcal{D}_{\text{ARAP}} = \|\mathbf{A} - \mathcal{R}_{\mathbf{A}}\|_F^2$$

Jacobian

Closest Rotation

How to compute closest Rotation?

Distortion - ARAP

ARAP– **A**s **R**igid **A**s **P**ossible

$$\mathcal{D}_{\mathrm{ARAP}} = \|\mathbf{A} - \mathcal{R}_{\mathbf{A}}\|_F^2$$

Jacobian

Closest Rotation

$$\mathbf{A} = USV^T \qquad \mathcal{R}_{\mathbf{A}} = UV^T$$

Proof: Using Lagrange multipliers

Distortion – ASAP

ASAP– **A**s **S**imilar **A**s **P**ossible

$$\mathcal{D}_{\mathrm{ASAP}} = \|\mathbf{A} - \mathcal{S}_{\mathbf{A}}\|_F^2$$

Jacobian

Closest Similarity

$$\mathbf{A} = USV^T$$

$$\mathcal{S}_{\mathbf{A}} = \bar{\sigma}UV^T$$

$$\bar{\sigma} = \frac{\sigma_1 + \sigma_2}{2}$$

Distortion - ARAP

ARAP– **A**s **R**igid **A**s **P**ossible

$$\mathcal{D}_{\text{ARAP}} = \|\mathbf{A} - \mathcal{R}_{\mathbf{A}}\|_F^2$$

Jacobian

Closest Rotation

$$\mathbf{A} = USV^T \qquad \mathcal{R}_{\mathbf{A}} = UV^T$$

Proof: Using Lagrange multipliers

Distortion - ARAP

ARAP– **A**s **R**igid **A**s **P**ossible

$$\mathcal{D}_{\text{ARAP}} = \|\mathbf{A} - \mathcal{R}_{\mathbf{A}}\|_F^2 = \|\mathbf{A} - UV^T\|_F^2$$

$$= \|USV^T - UV^T\|_F^2$$

$$= \|U(S - I)V^T\|_F^2$$

$$= \|(S - I)\|_F^2$$

$$= (\sigma_1 - 1)^2 + (\sigma_2 - 1)^2$$

Distortion - ASAP

ASAP– **A**s **S**imilar **A**s **P**ossible

$$
\begin{aligned}
\mathcal{D}_{\text{ASAP}} &= \|\mathbf{A} - \mathcal{S}_{\mathbf{A}}\|_F^2 \\
&= \|USV^T - \bar{\sigma}UV^T\|_F^2 \\
&= \|U(S - \bar{\sigma}I)V^T\|_F^2 \\
&= (\sigma_1 - \bar{\sigma})^2 + (\sigma_2 - \bar{\sigma})^2 \\
&= (\sigma_1 - \sigma_2)^2
\end{aligned}
$$

Distortion bias

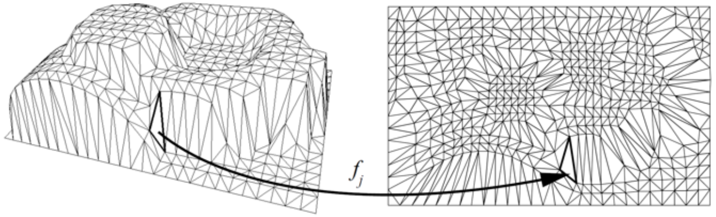
$$\mathcal{D}_{\mathrm{ASAP}}(A) \quad < \quad \mathcal{D}_{\mathrm{ASAP}}(2A)$$

$$(\sigma_1 - \sigma_2)^2 \qquad\qquad (2\sigma_1 - 2\sigma_2)^2$$

Conformal distortion

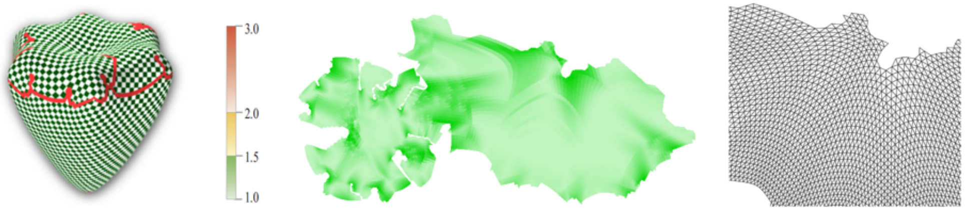
$$\frac{\sigma_1}{\sigma_2}$$

$$\frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}$$



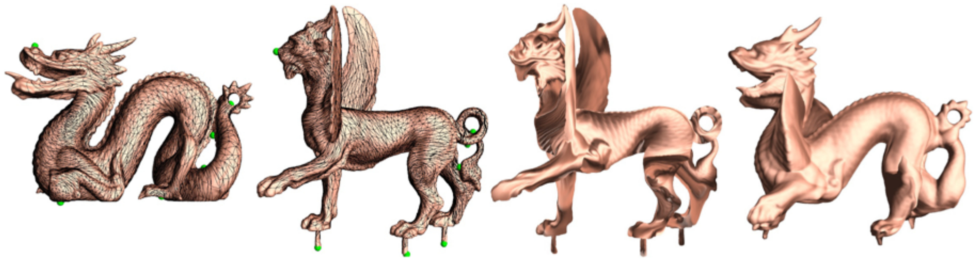
MIPS [Hormann & Greiner 2000]

$$\max\left\{\sigma_1, \frac{1}{\sigma_2}\right\}$$



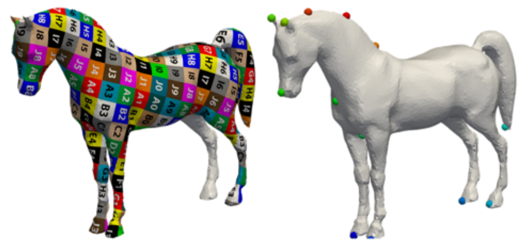
[Sorkine et al. 2000]

$$\sigma_1^2 + \sigma_2^2 + \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$$



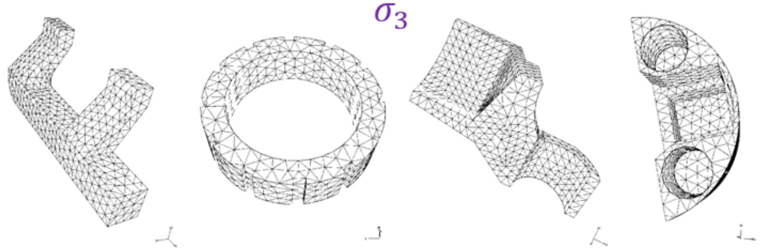
[Schreiner et al. 2014]

$$\sqrt{\sigma_1^2 + \frac{1}{\sigma_2^2}}$$



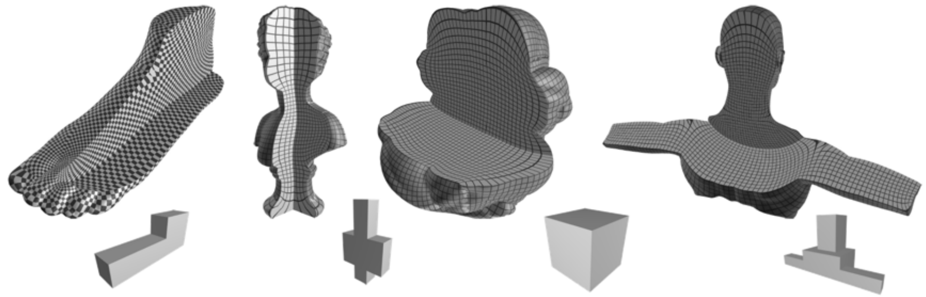
[Aigerman et al. 2014]

$$\frac{\sigma_1}{\sigma_3}$$



[Freitag & Knupp 2002]

$$\frac{\sigma_1}{\sigma_3} + \frac{\sigma_3}{\sigma_1}; \quad \sigma_1\sigma_2\sigma_3 + \frac{1}{\sigma_1\sigma_2\sigma_3}$$



[Paillé & Poulin 2012]