Mappings
and their computation

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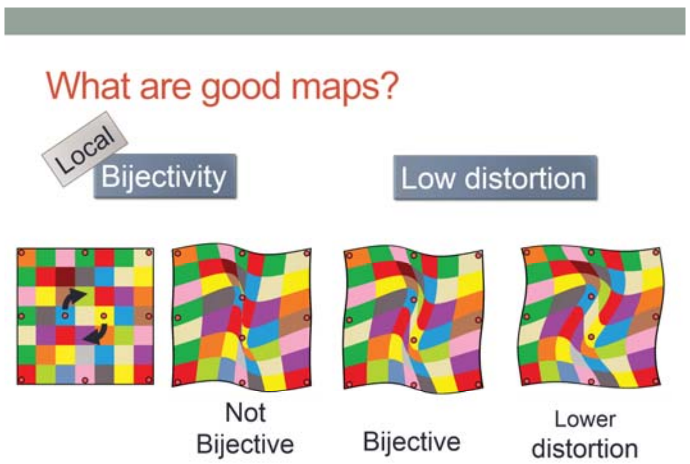
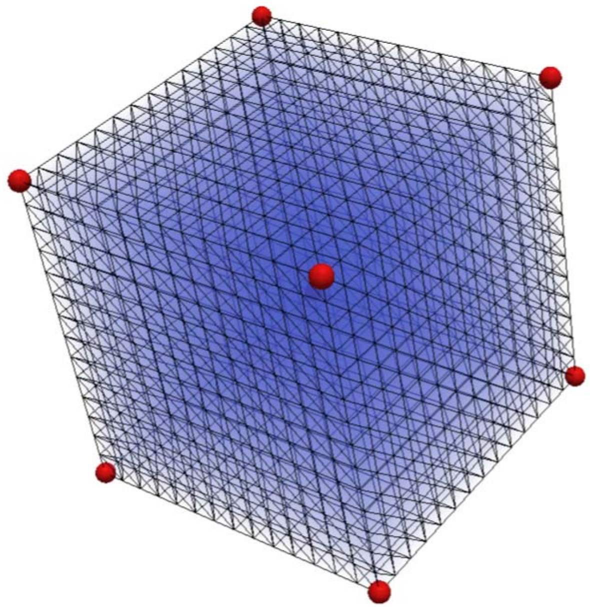
Recap

- Mappings
- Representations of mappings
- What are good maps?
  - Bijectivity
  - Distortion



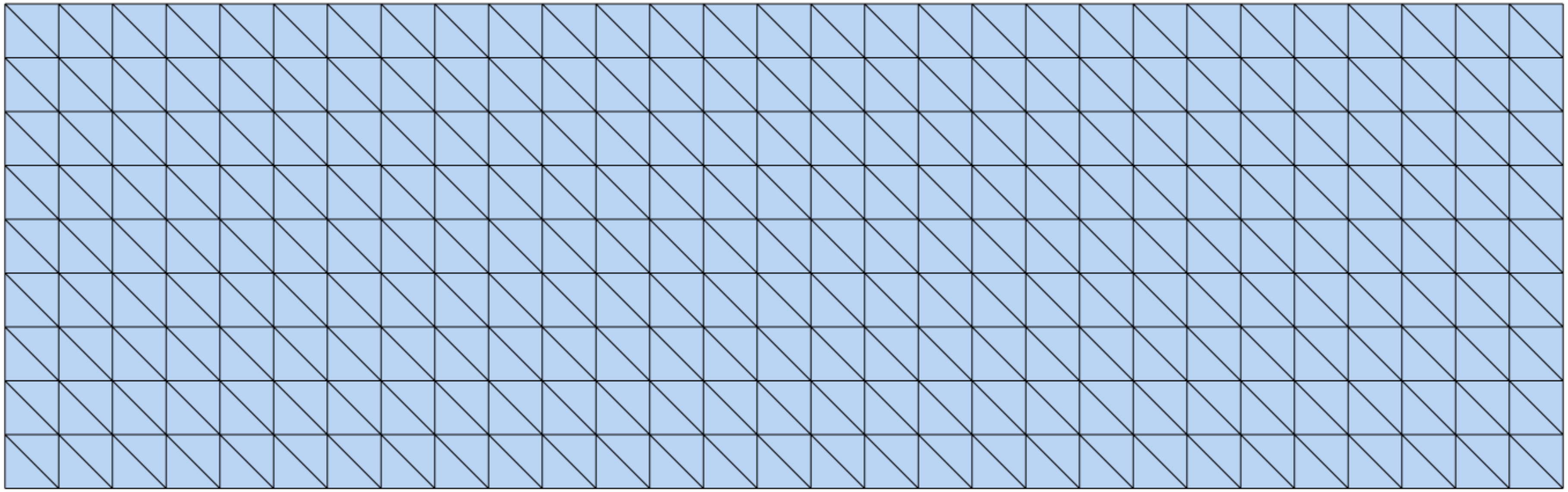
This part

- Computation of simplicial maps

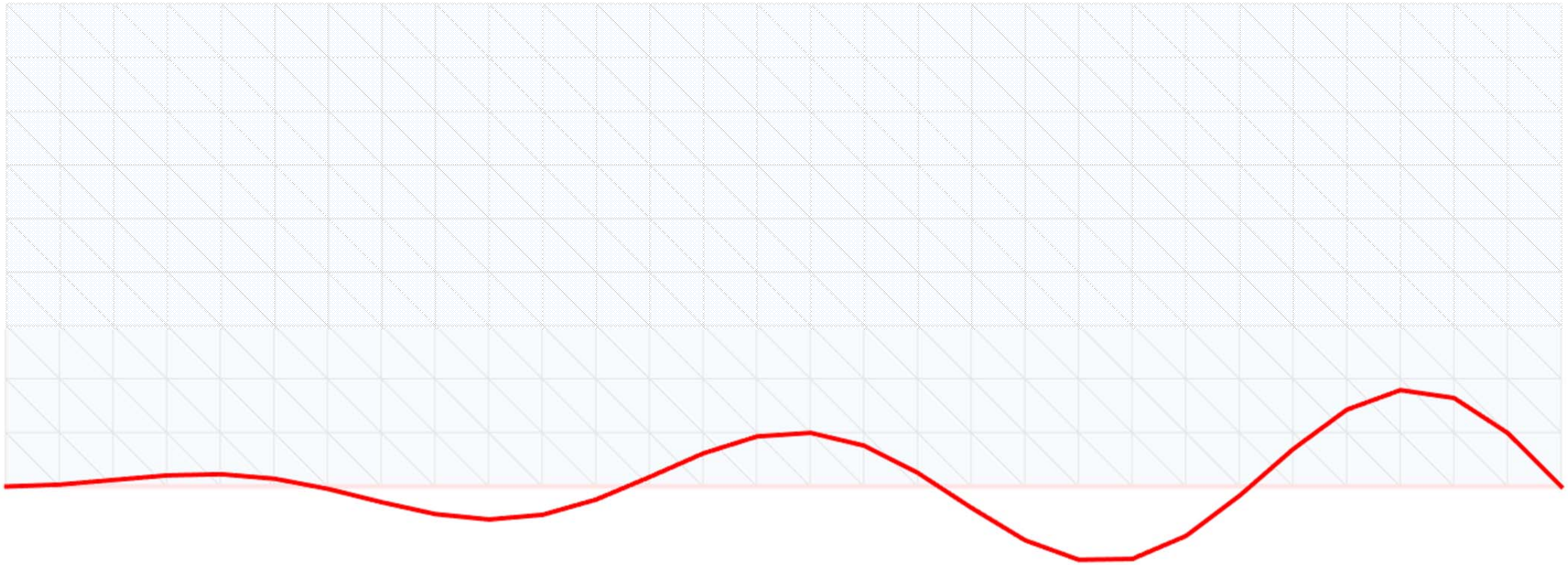


- Focus: linear algebra aspects

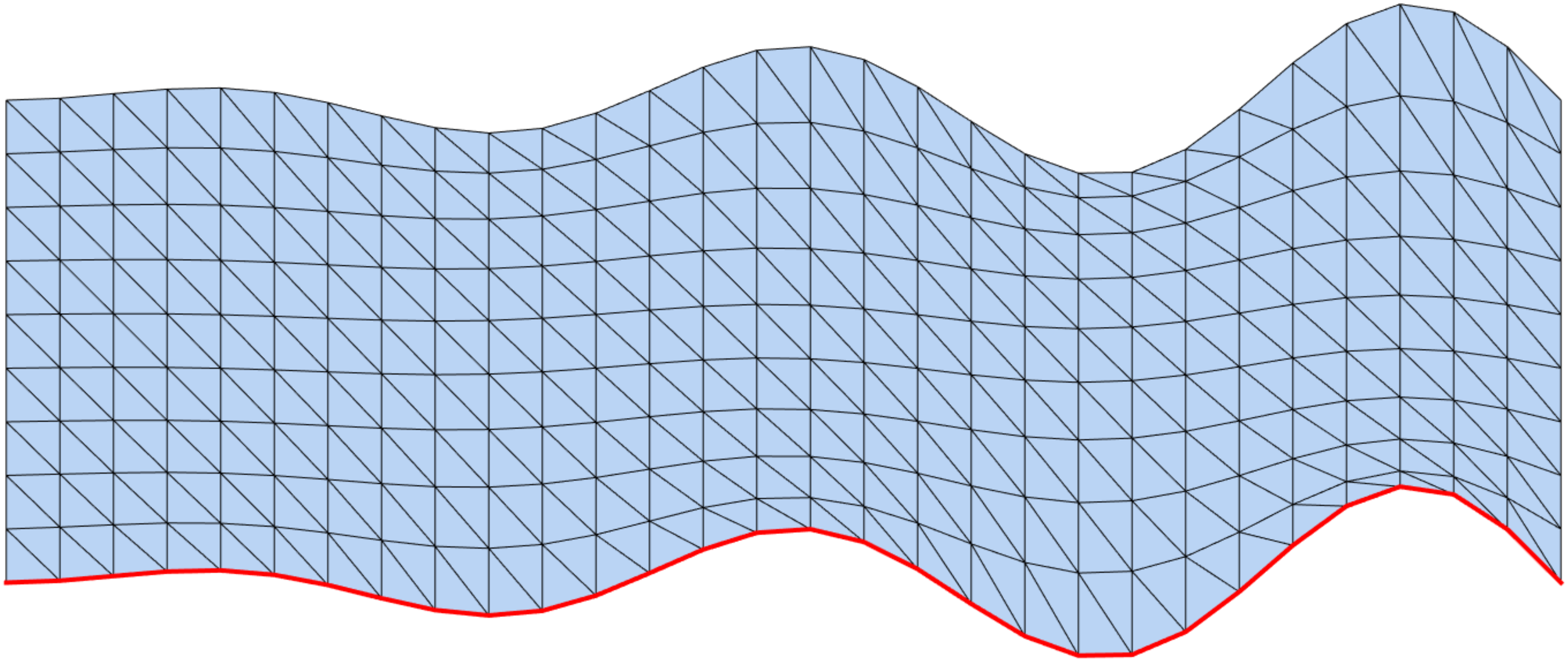
Computing maps



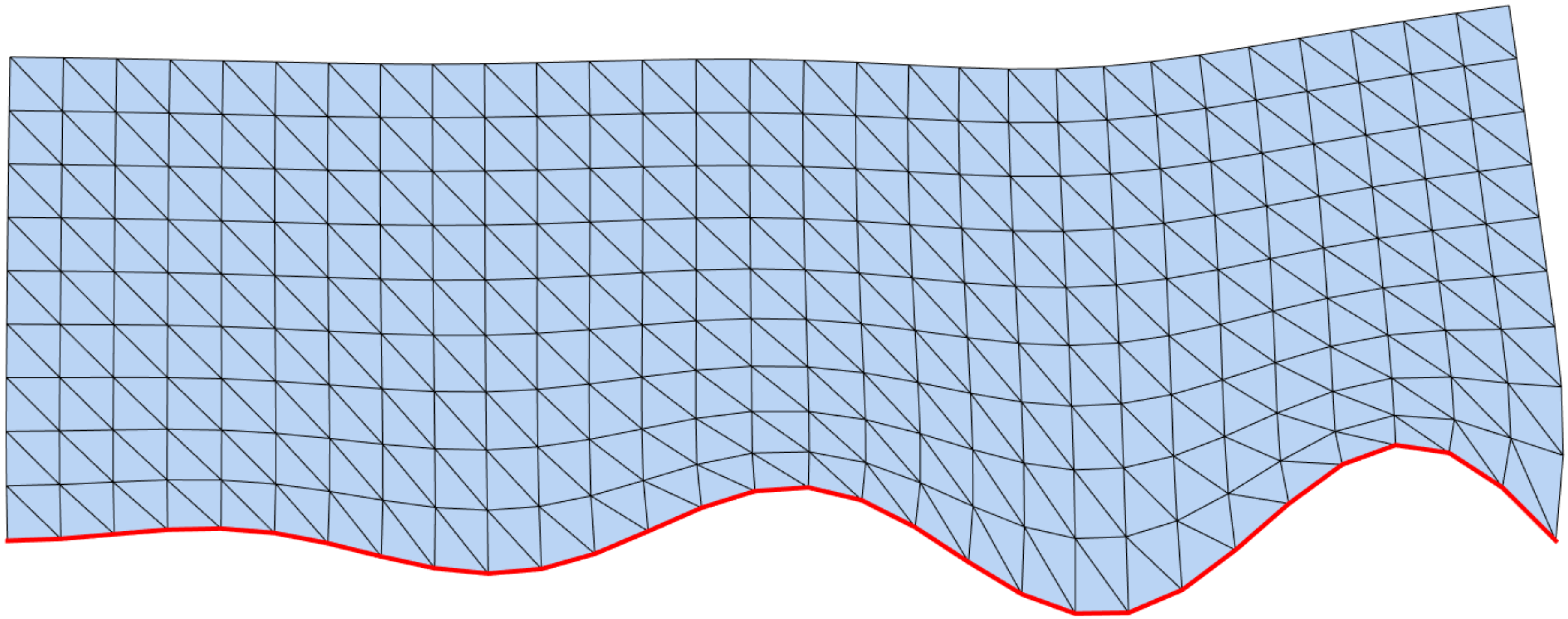
Computing maps



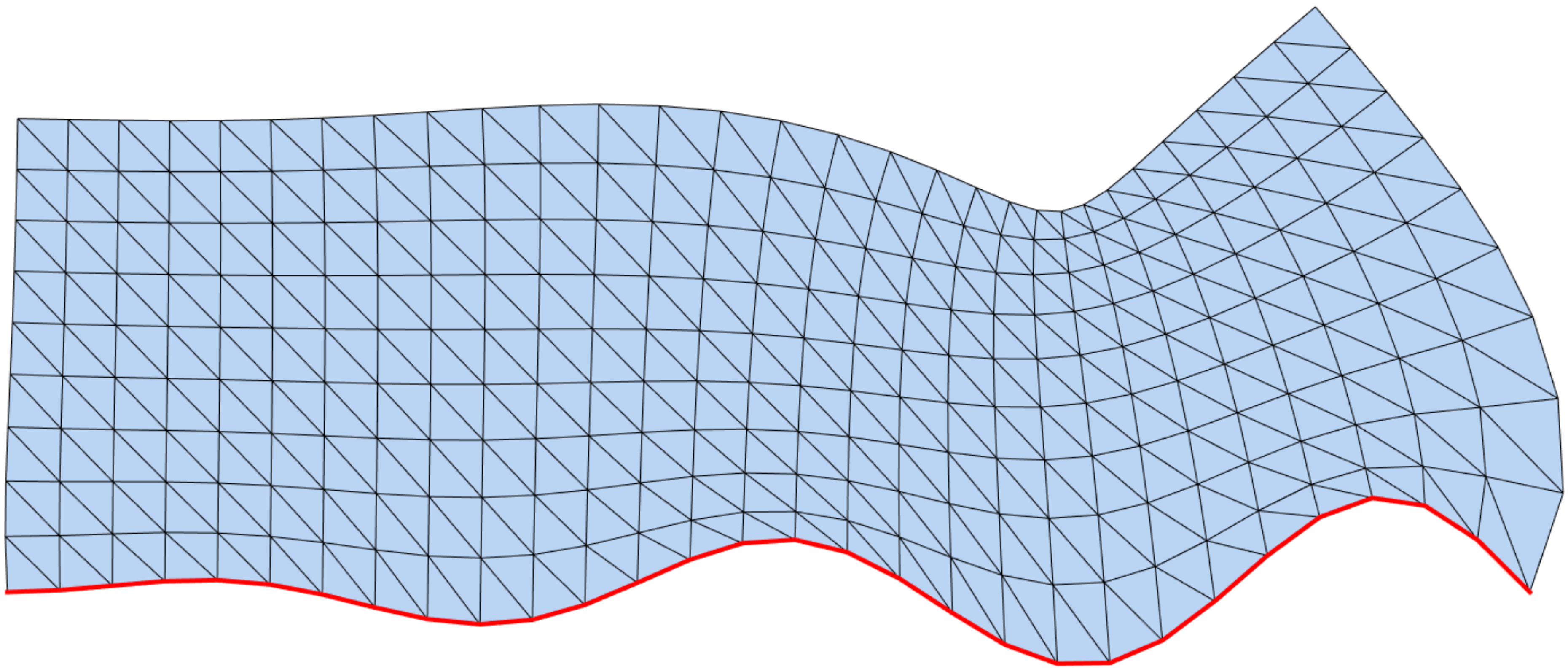
Computing maps



Computing maps



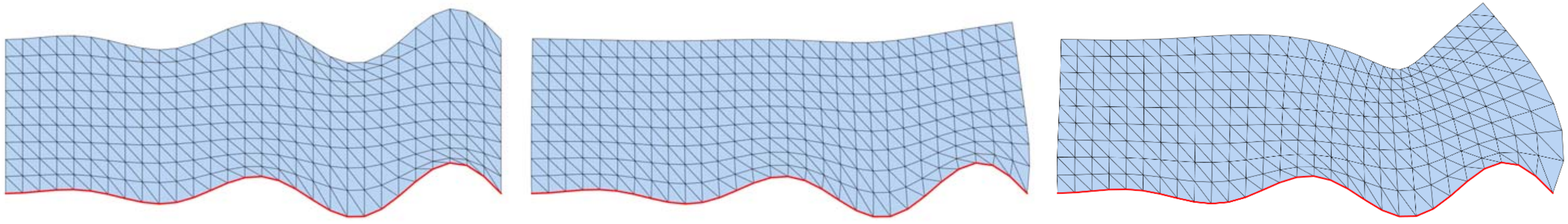
Computing maps



Computing maps

- Imposing constraints

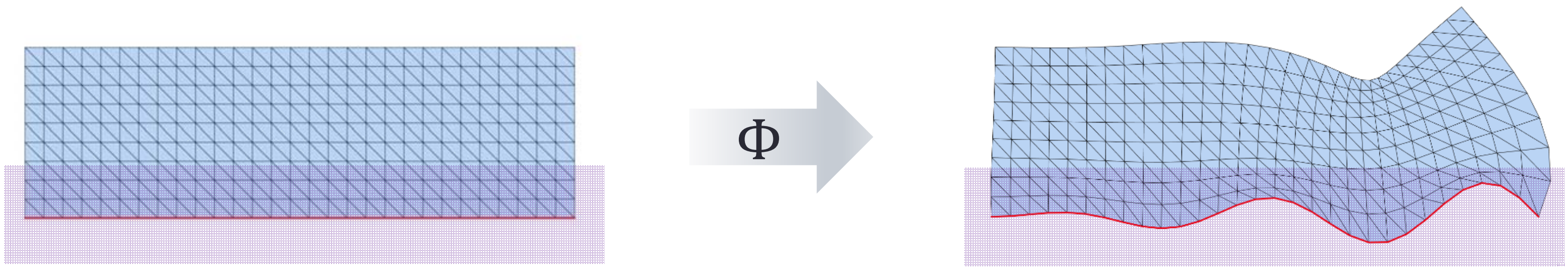
- Finding maps that are most…



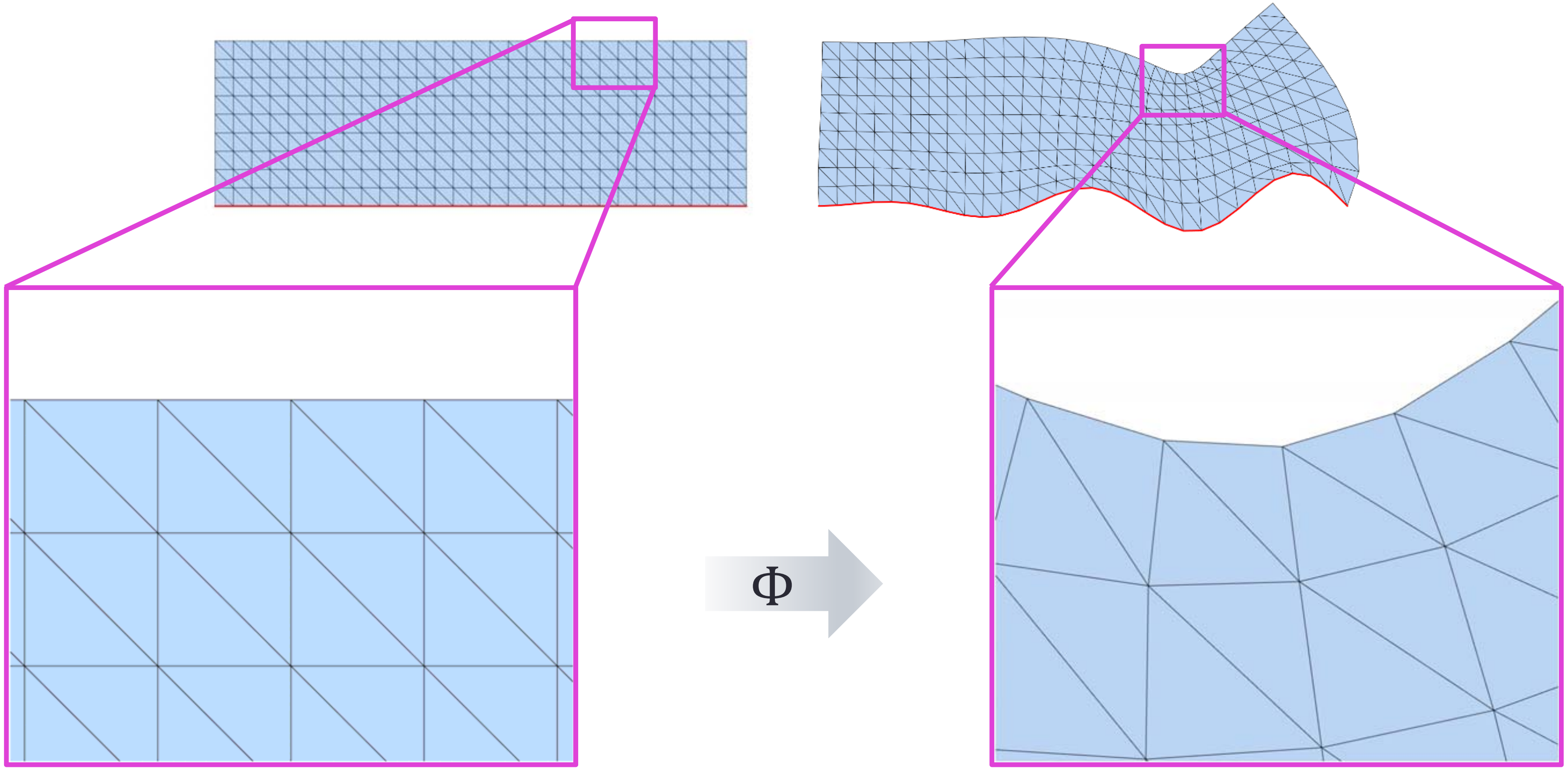
Constrained Optimization

$$\underset{\Phi}{\operatorname{argmin}} \; E(\Phi) \quad \text{Energy}$$

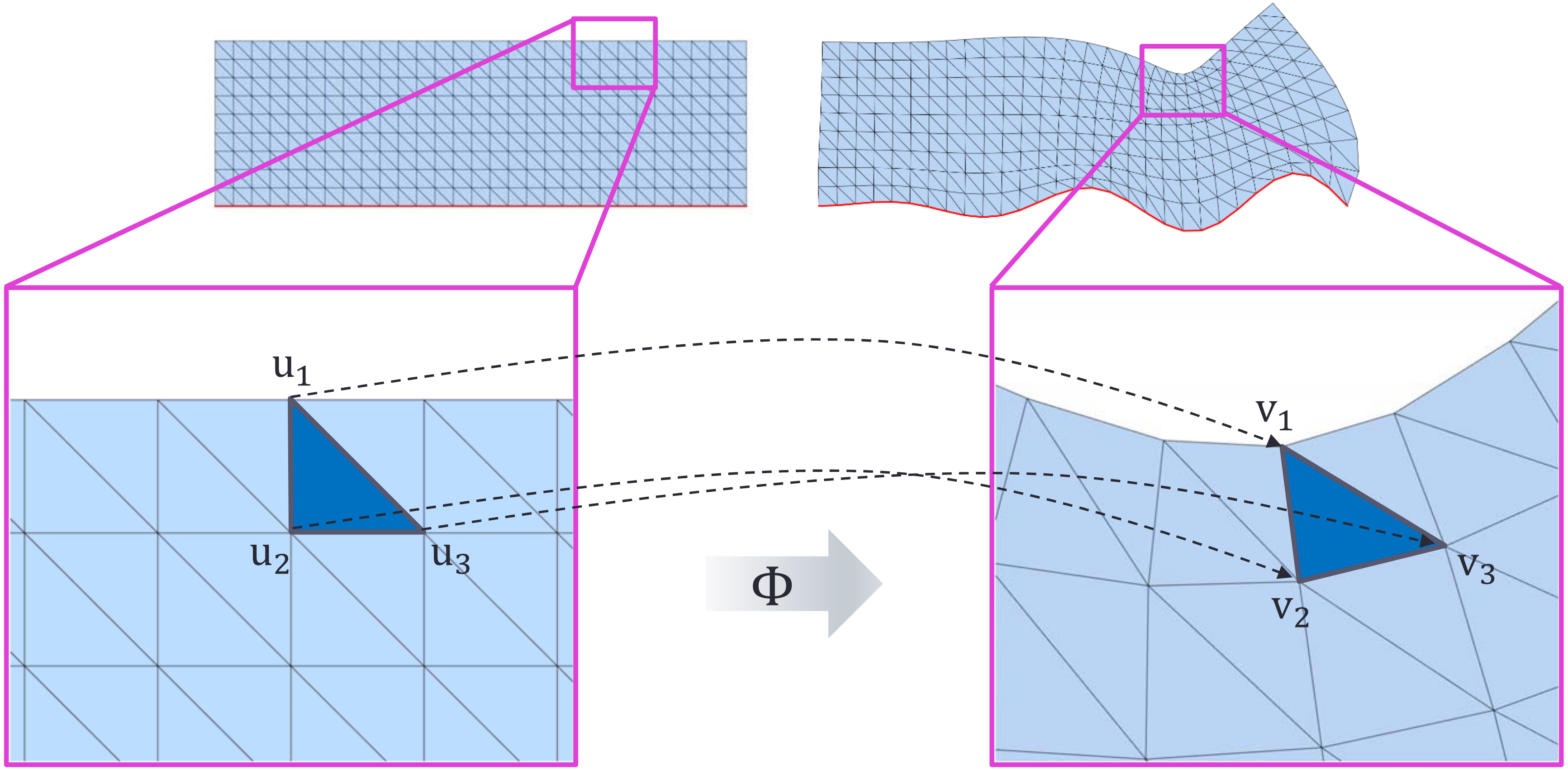
$$\text{s.t.} \quad \Phi \in K \quad \text{Constraints}$$



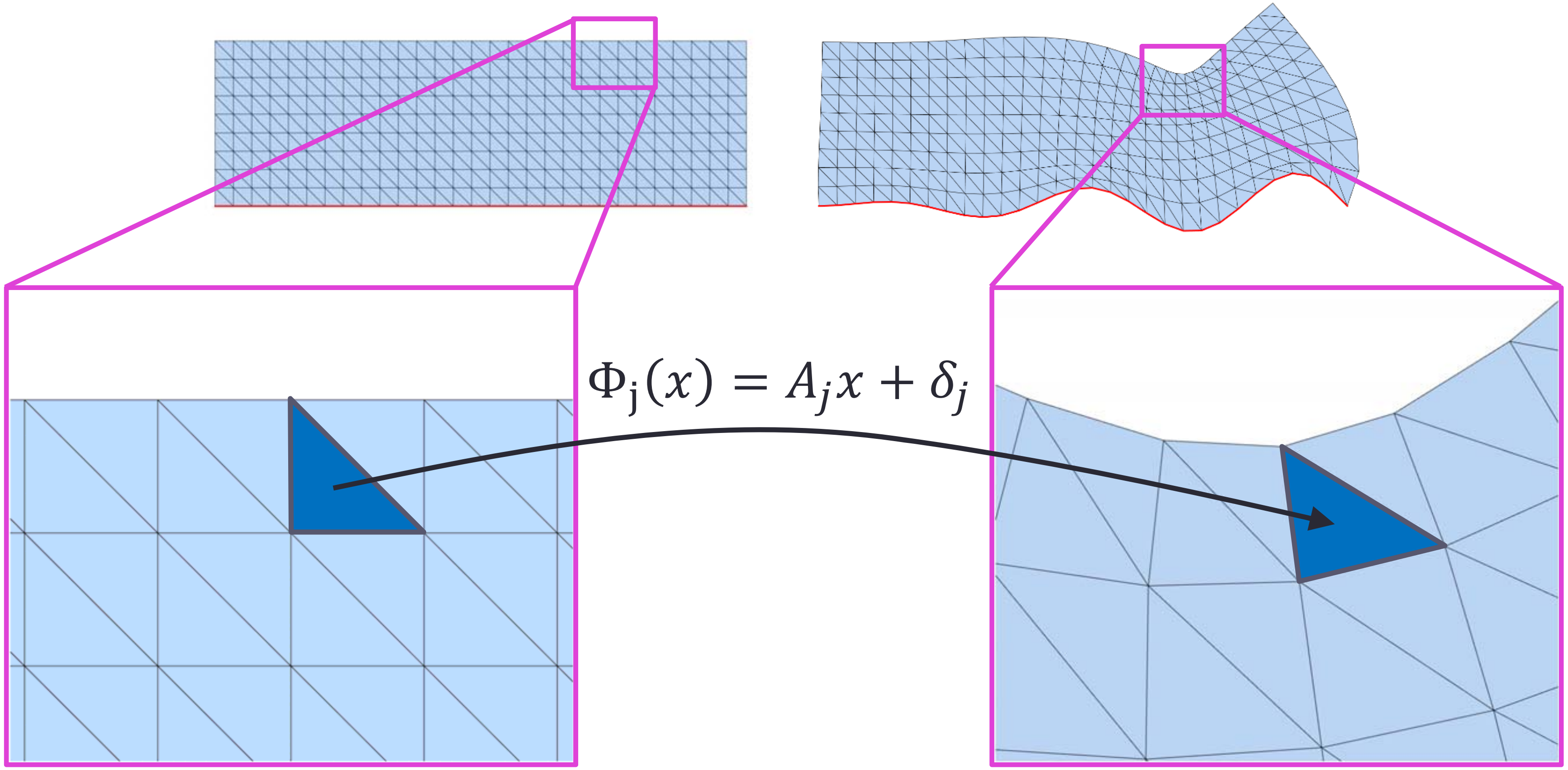
Energy



Energy



Energy

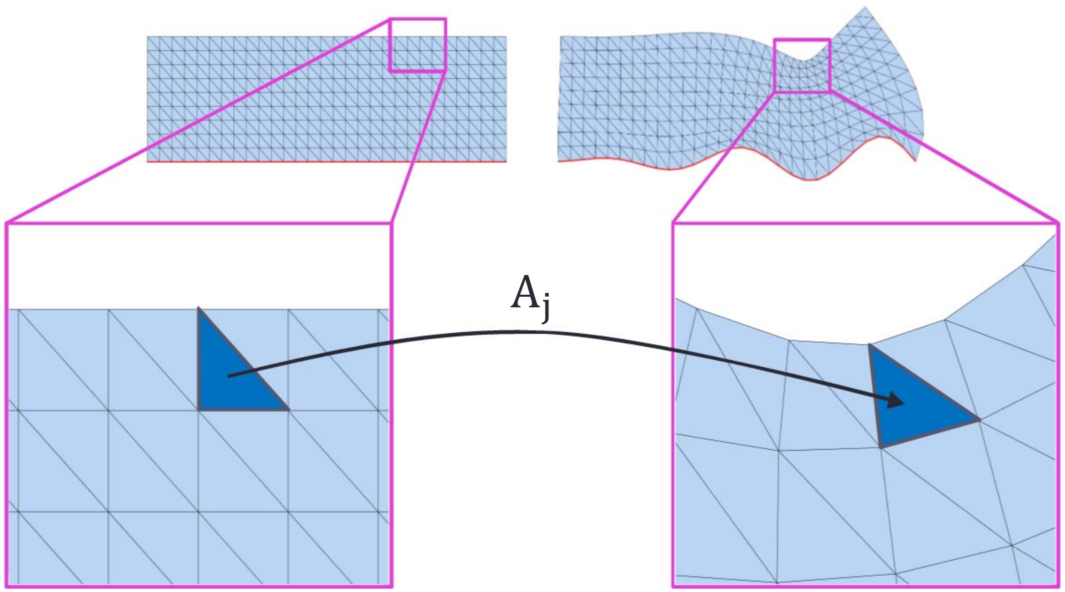


$$E(\Phi) = E(A_1, \dots, A_m)$$

Map optimization

- In terms of differentials:

$$\operatorname{argmin} E(\mathrm{A}_1, \ldots, \mathrm{A}_\mathrm{m})$$

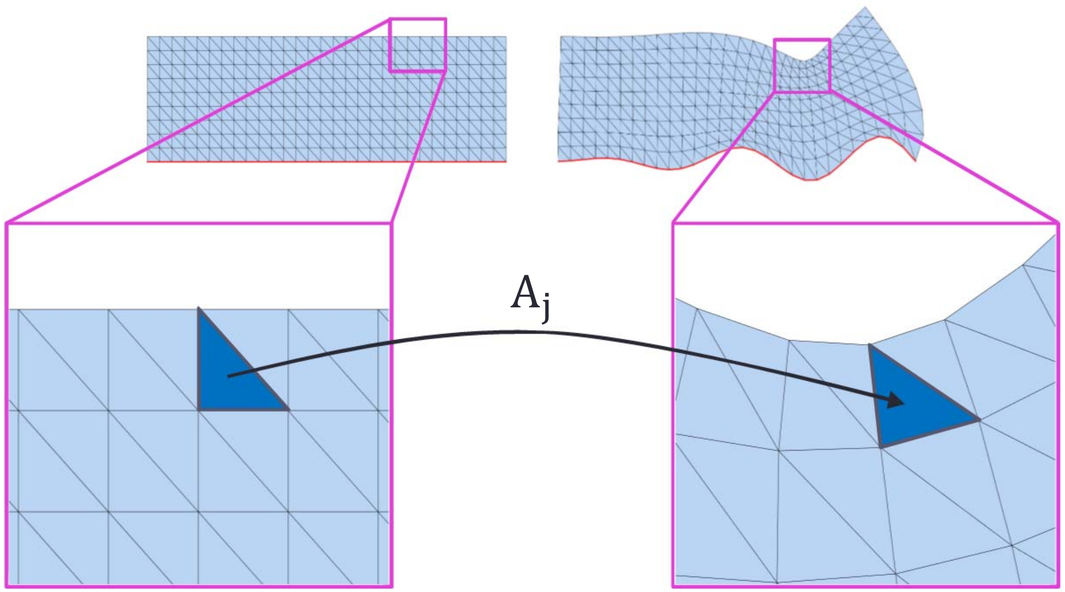


Map optimization

- In terms of differentials:

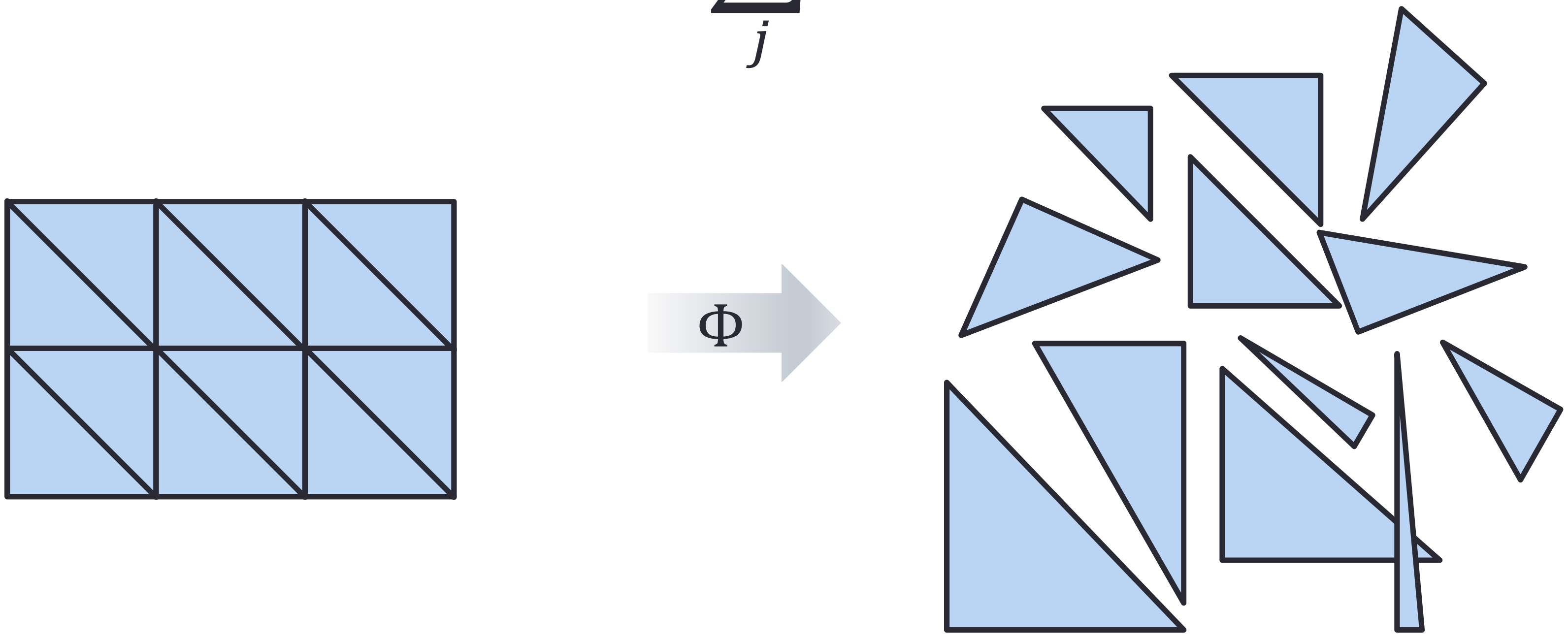
$$\text{argmin} \sum_j f(A_j)$$

Separable



Map optimization

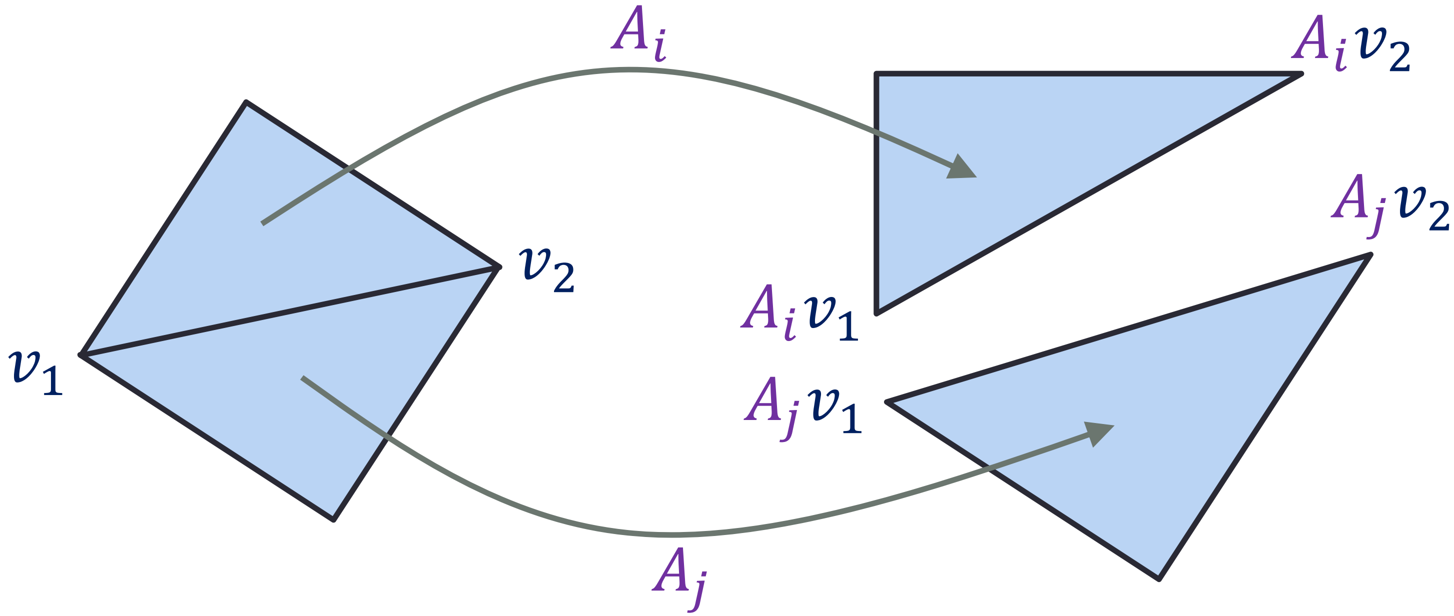
$$\operatorname{argmin} \sum_j f(A_j)$$



Must impose continuity!

Explicit continuity

- Optimization variables: $A_1, A_2, \ldots, A_m$
- Adjacent A_j's must agree



Explicit continuity

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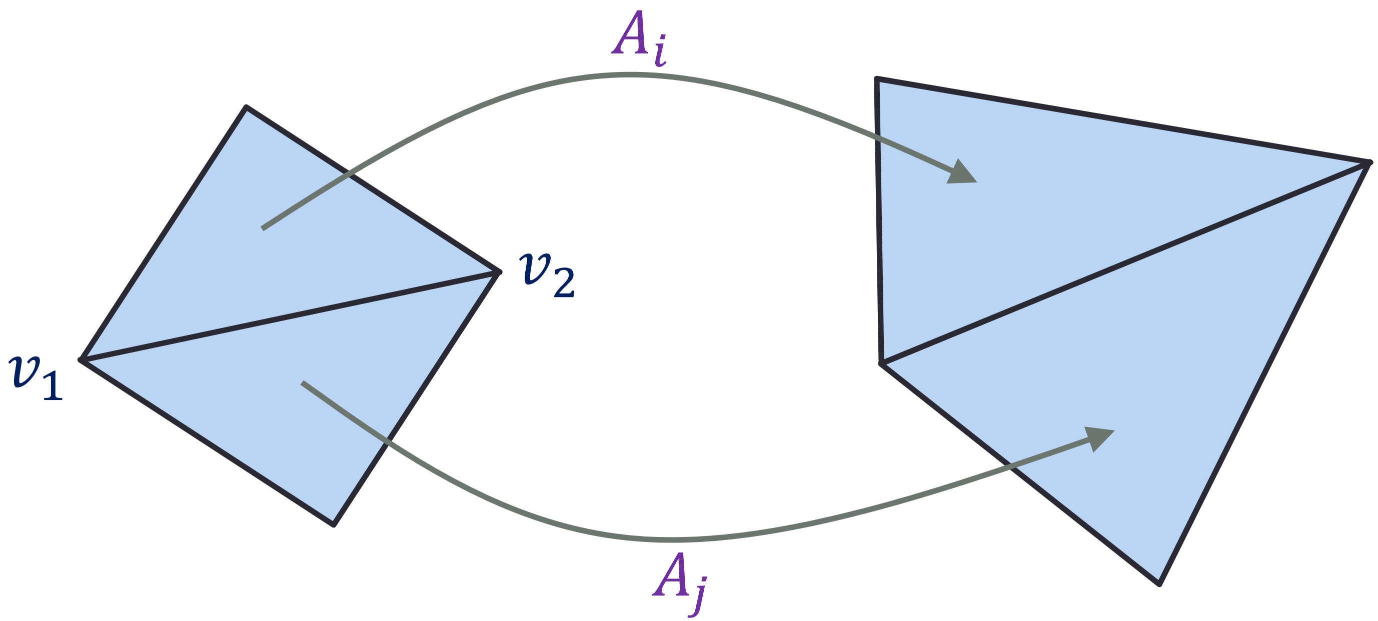
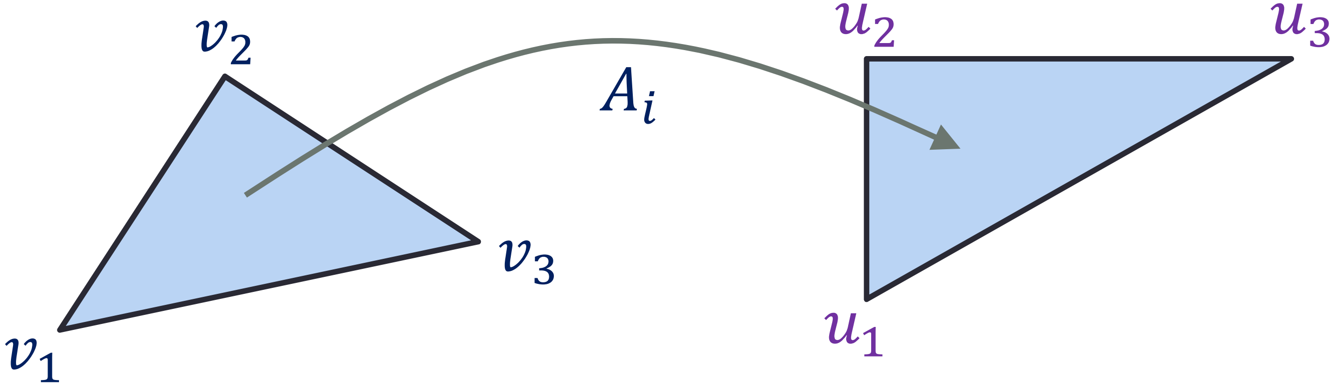


$$A_i v_1 = A_j v_1$$
$$A_i v_2 = A_j v_2$$

Implicit continuity



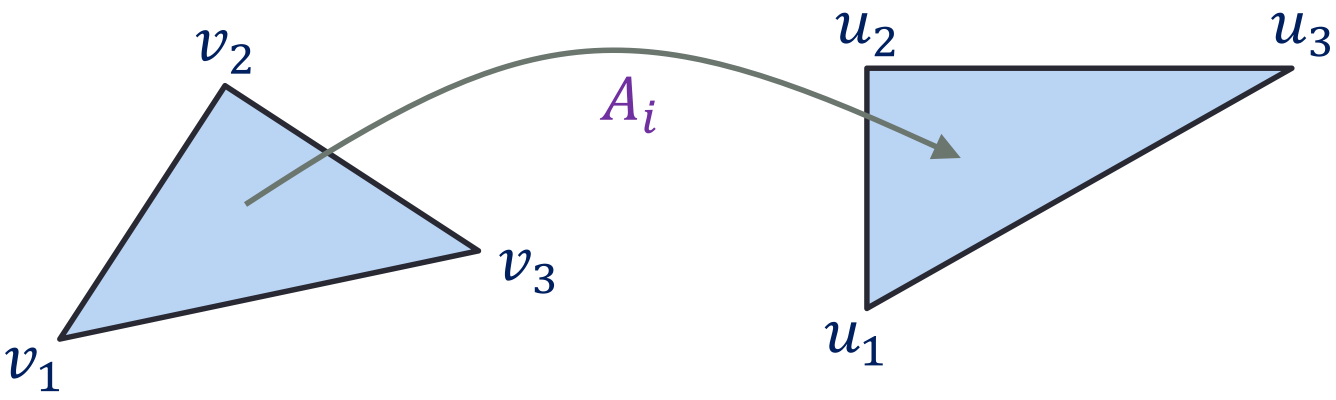
$$A_i \overline{[v_1\ v_2\ v_3]} = \overline{[u_1\ u_2\ u_3]}$$

$$A_i = \overline{[u_1\ u_2\ u_3]}\ \overline{[v_1\ v_2\ v_3]}^{\dagger}$$

$$A_i = A_i(U)$$

Linearly express A_i's in terms of U

Implicit continuity

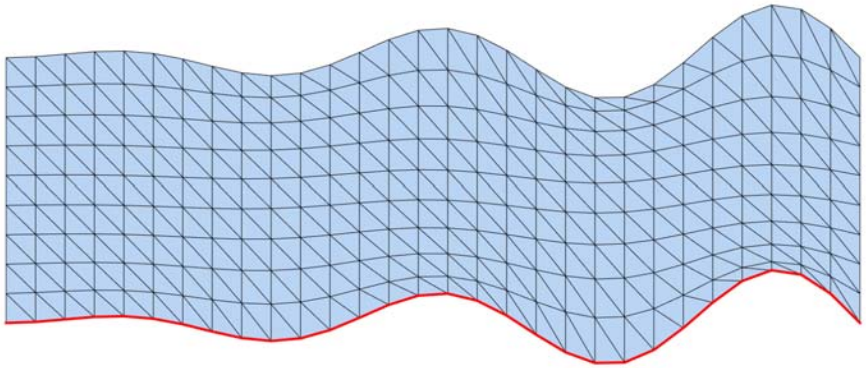


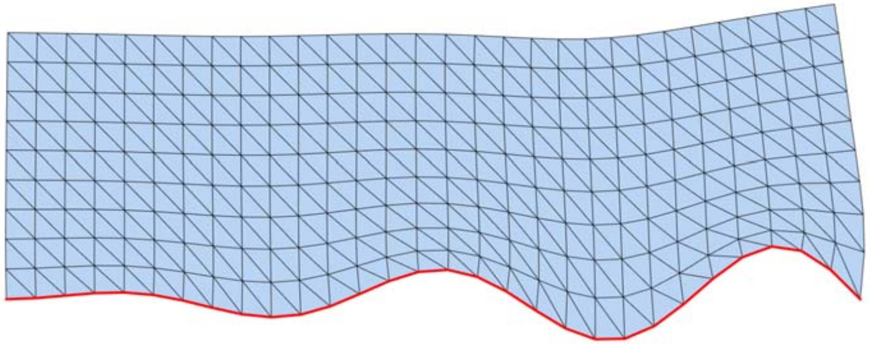
- Optimization variables: $u_1, u_2, \ldots, u_n \quad (U)$

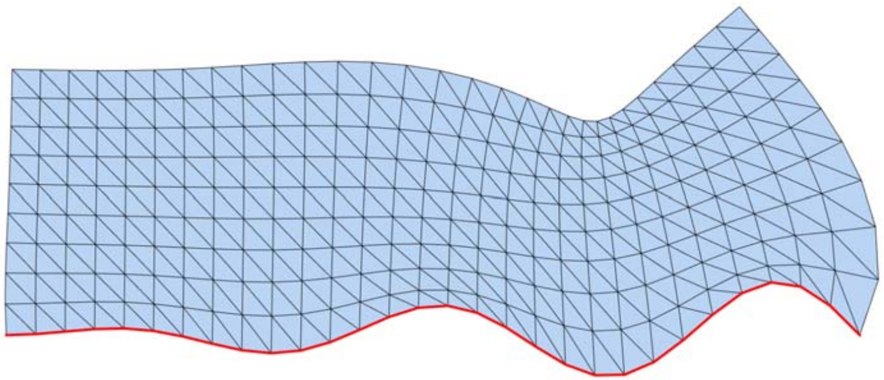
$$E(\Phi) = \sum_j f\left(A_j(U)\right)$$

Popular energies

$$\operatorname{argmin} \sum_j f(A_j)$$



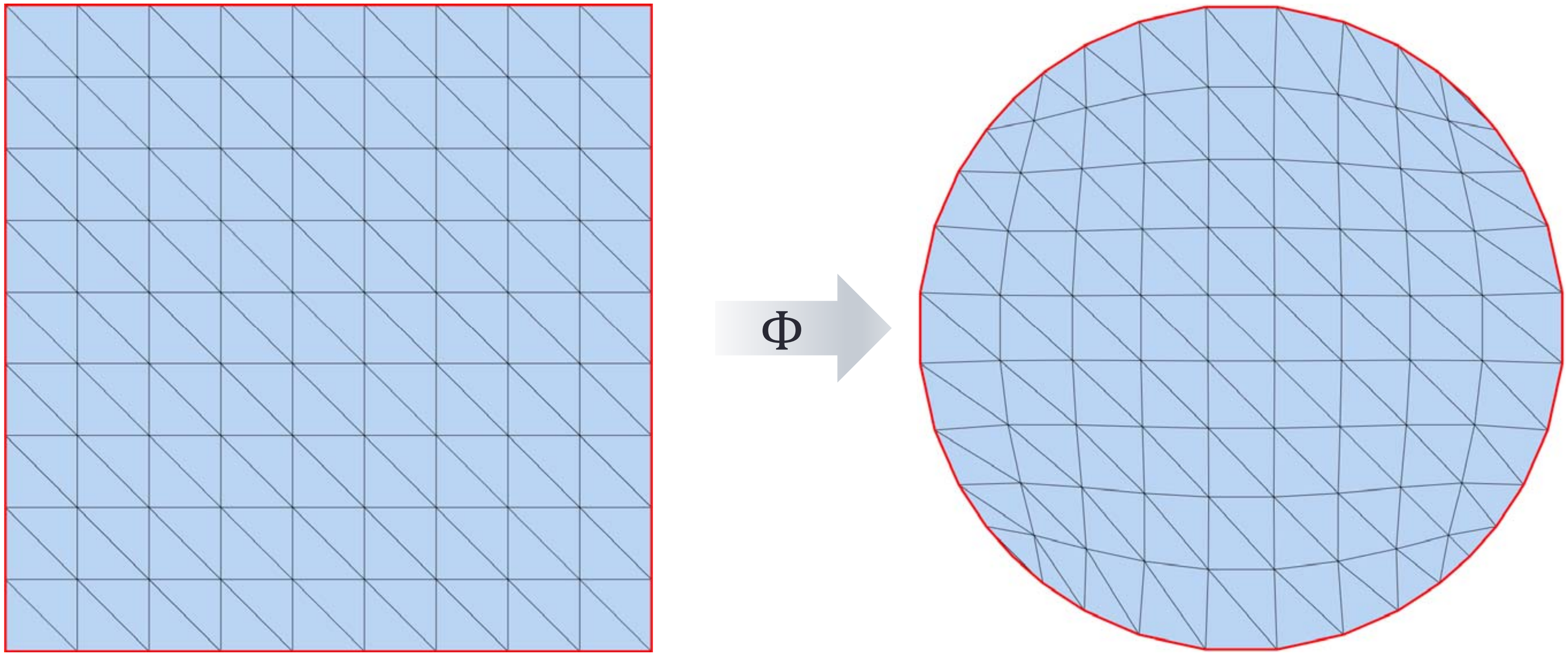




Dirichlet

area / volume

$$E_D = \sum_j w_j \|A_j\|_F^2$$



Dirichlet

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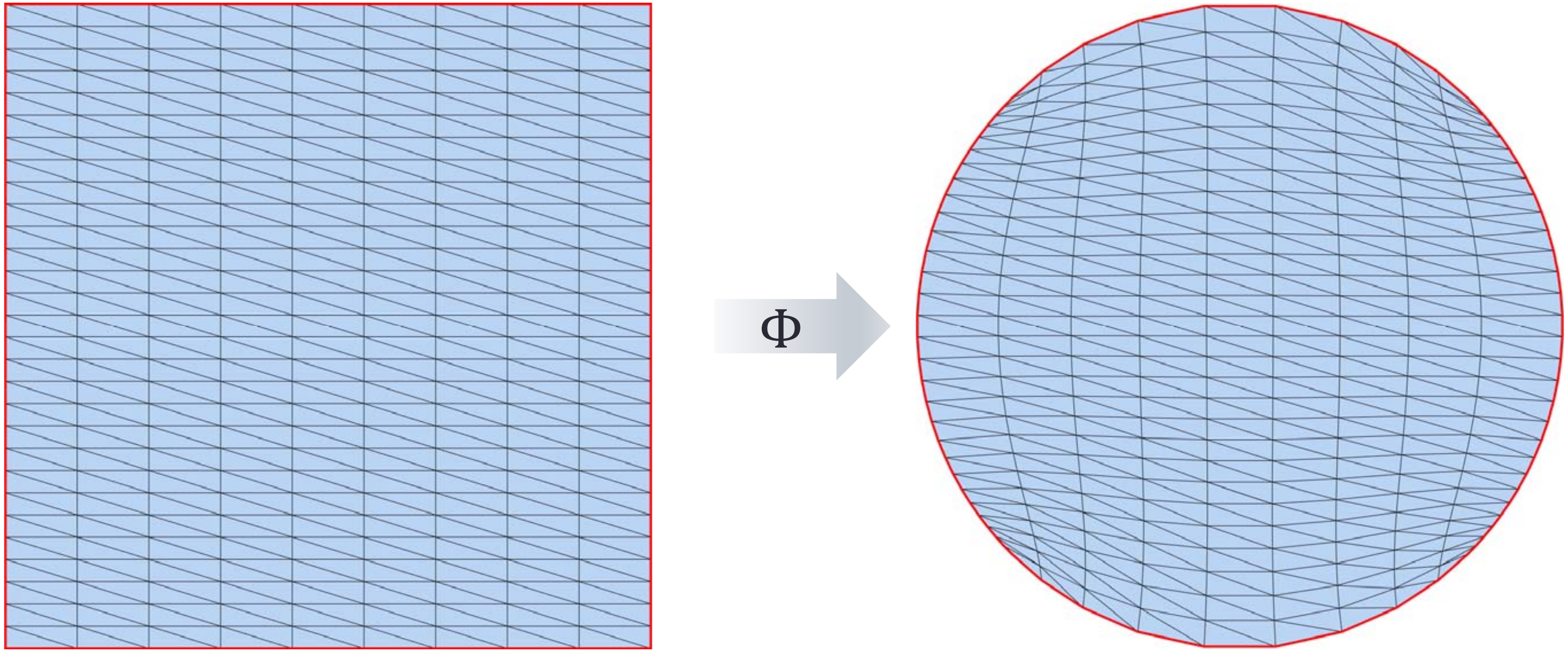

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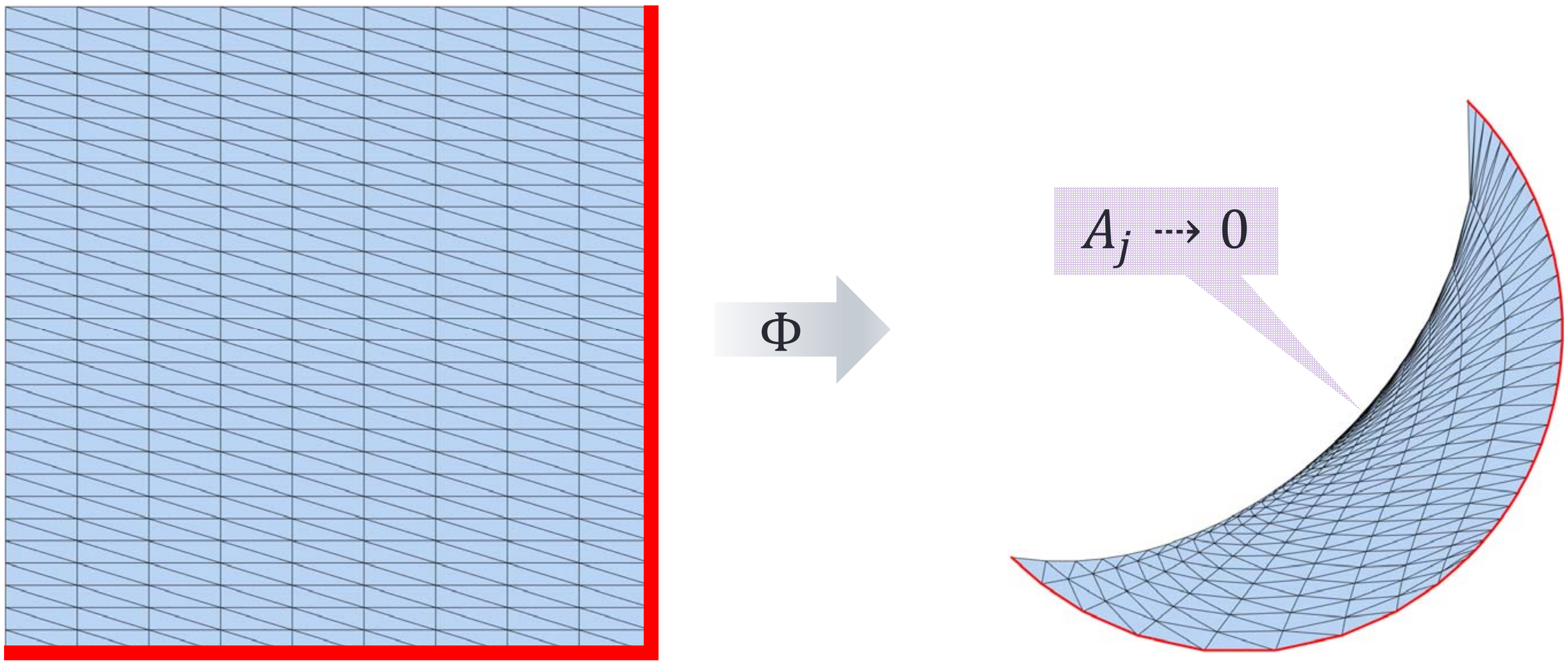
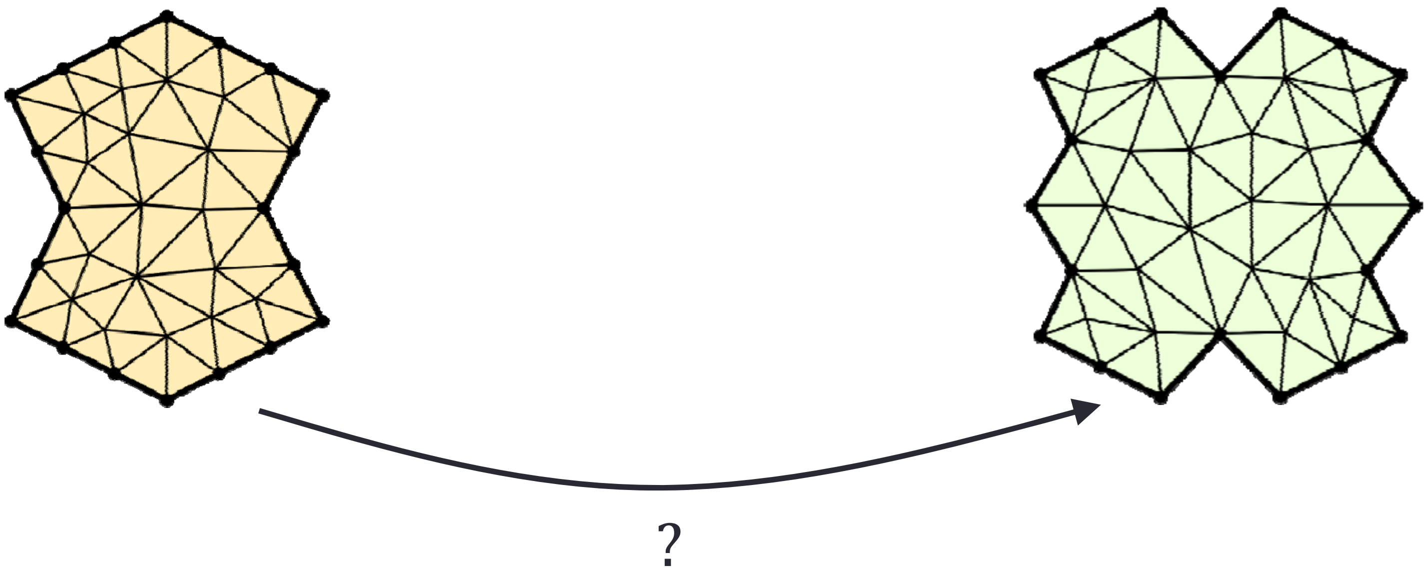
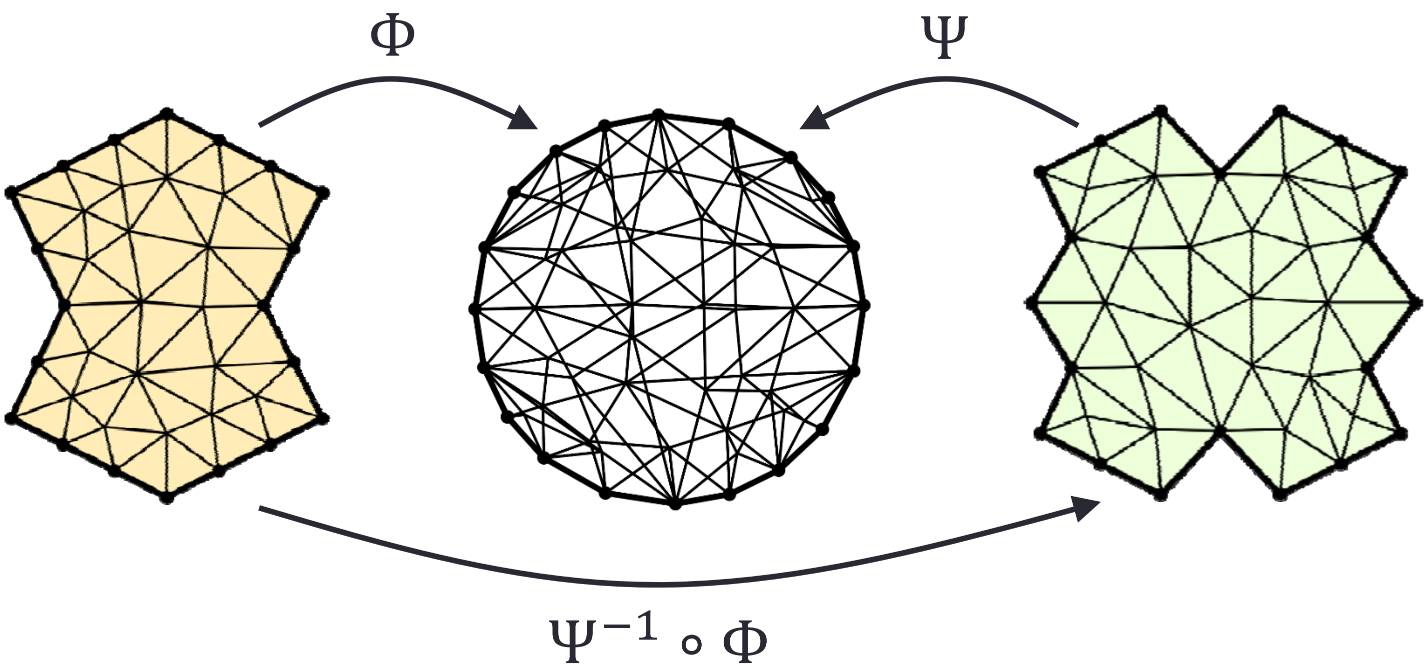


Dirichlet



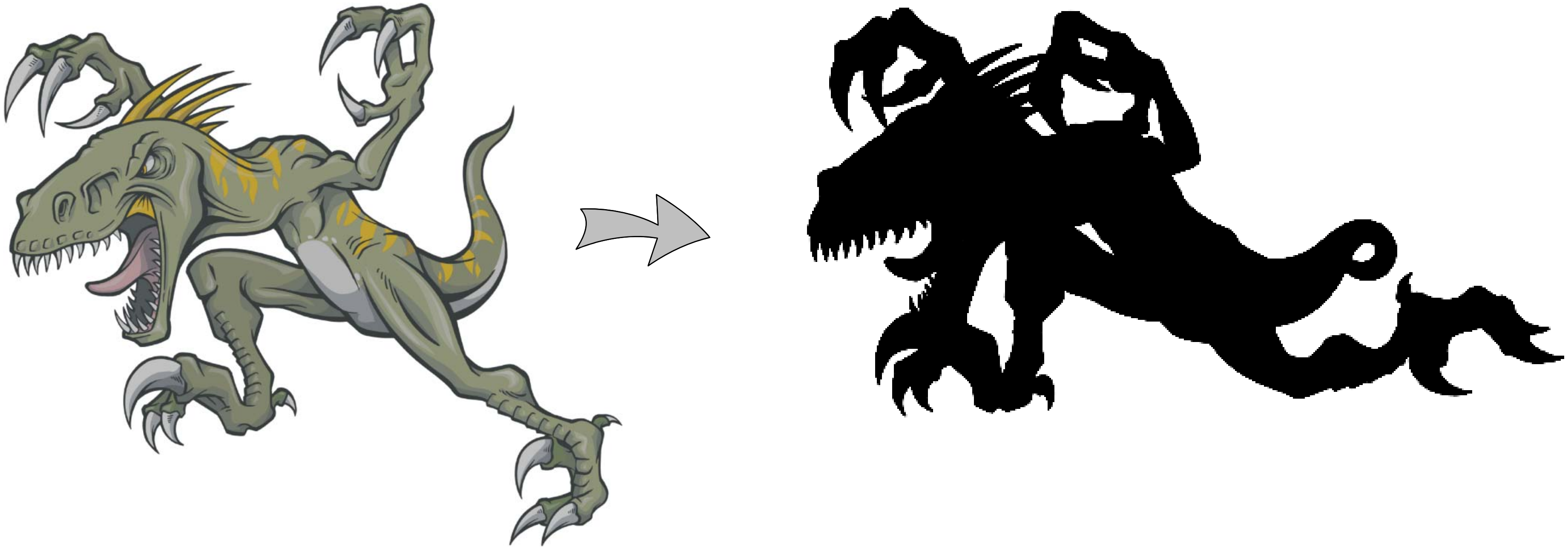
[Weber & Zorin 2014]

Dirichlet



[Weber & Zorin 2014]

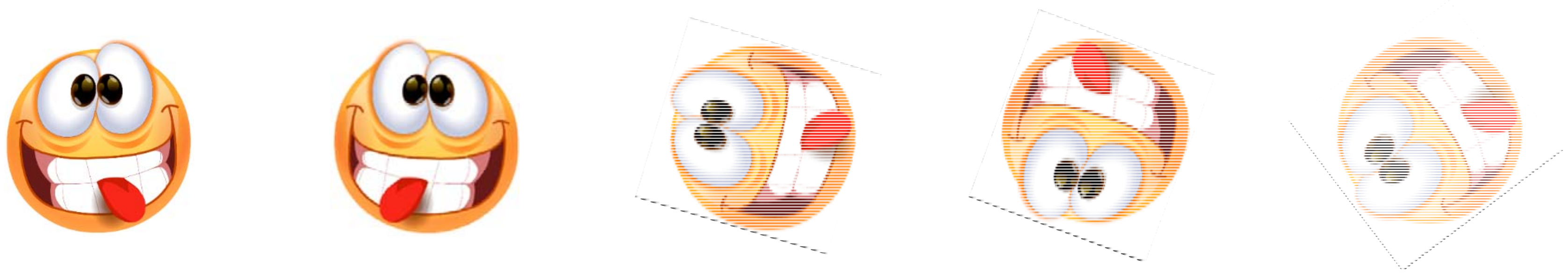
Dirichlet



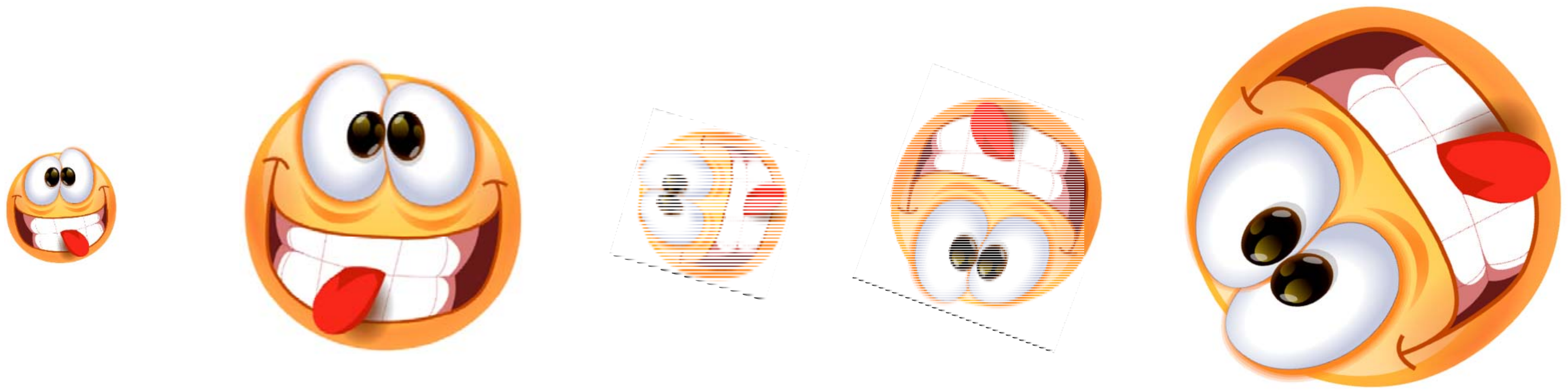
[Weber & Zorin 2014]

Orthogonal and Similarity

- R is <u>orthogonal</u> if $R^T = R^{-1}$
  (rotation if $\det R > 0$)



- S is a <u>similarity</u> if $S = \alpha R$



Closest R and S

- $\mathcal{R}(A)$ = closest orthogonal/rotation matrix to A
- $\mathcal{S}(A)$ = closest similarity matrix to A

- Computable using SVD/SSVD:

$$A = U\Sigma V^T; \qquad \Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$$

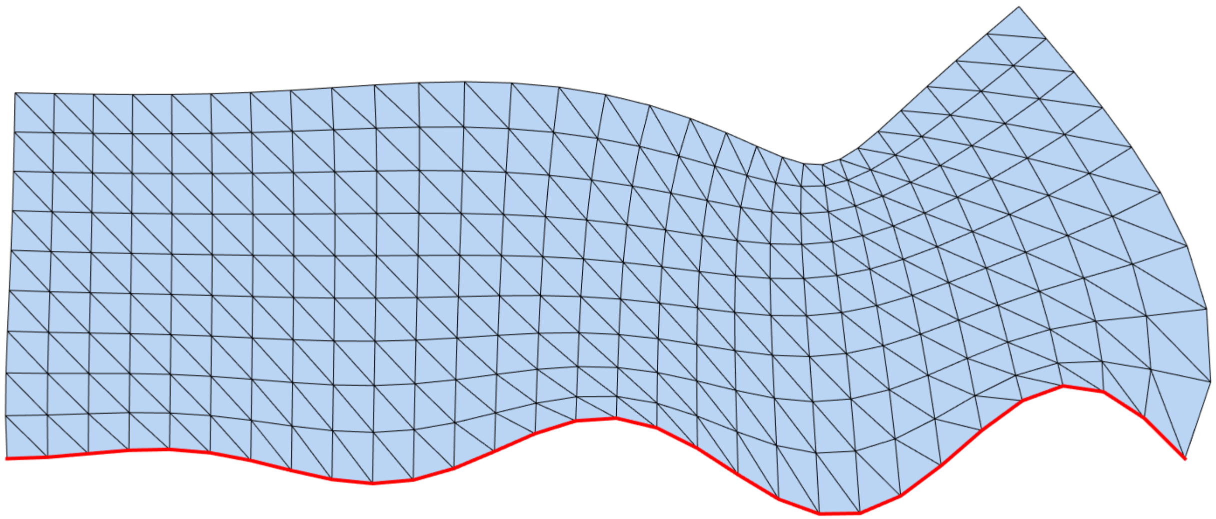
- $\mathcal{R}(A) = U\cancel{\Sigma}V^T = UV^T$
- $\mathcal{S}(A) = \bar{\sigma}UV^T$

mean of SVs

Least Squares Conformal Map (LSCM)

$$E_L = \sum_j w_j \left\| A_j - \mathcal{S}(A_j) \right\|_F^2$$

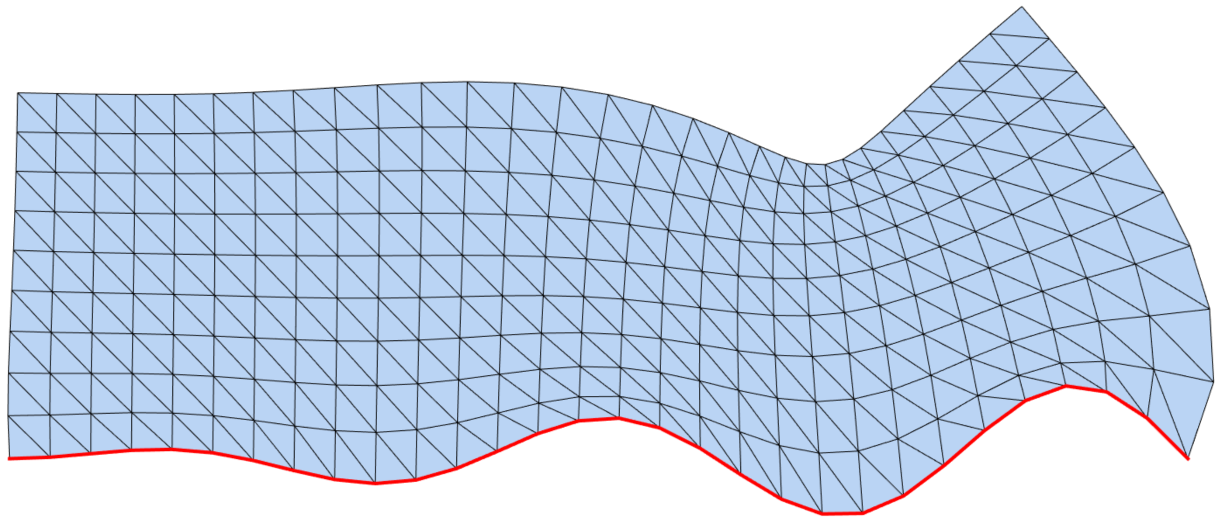
closest similarity



Least Squares Conformal Map (LSCM)

$$E_L = \sum_j w_j \left\| A_j - \mathcal{S}(A_j) \right\|_F^2$$

amount of non-similarity



Least Squares Conformal Map (LSCM)

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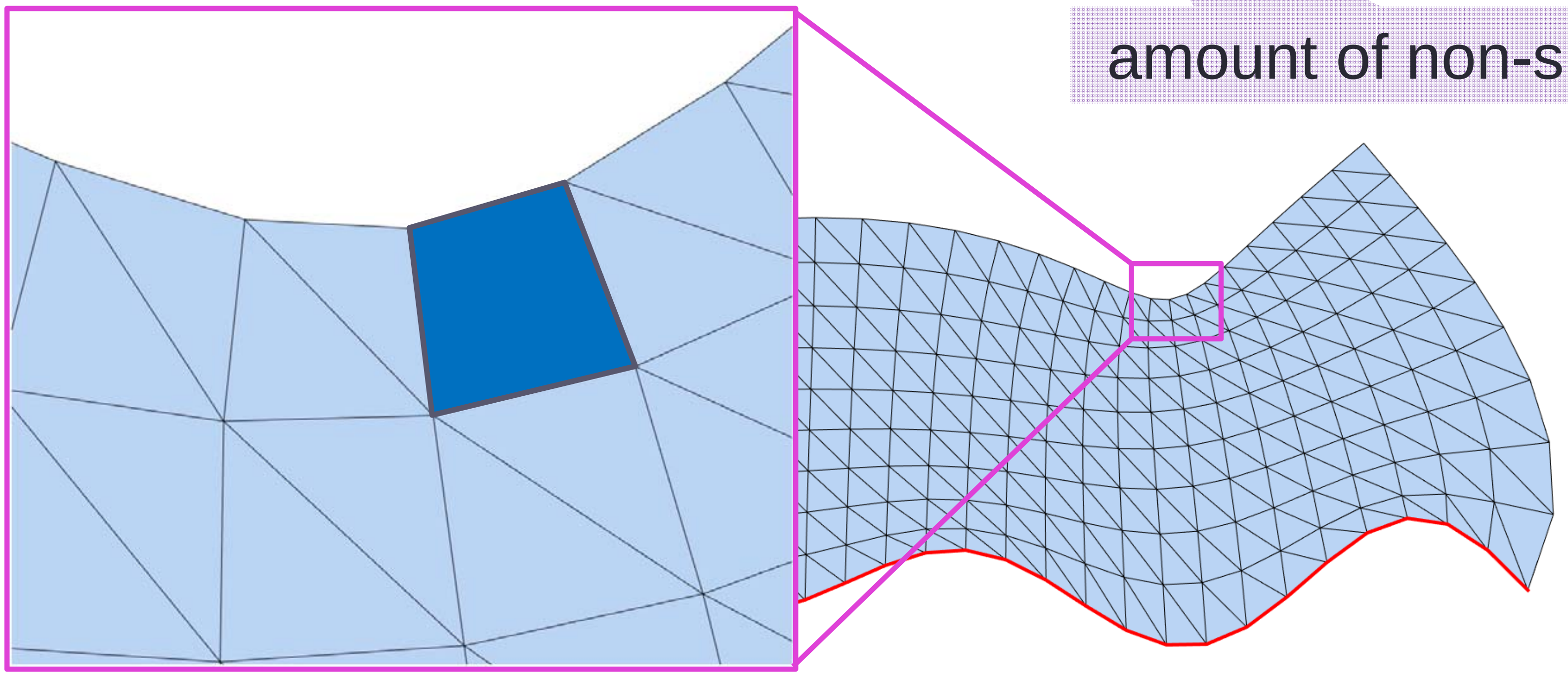
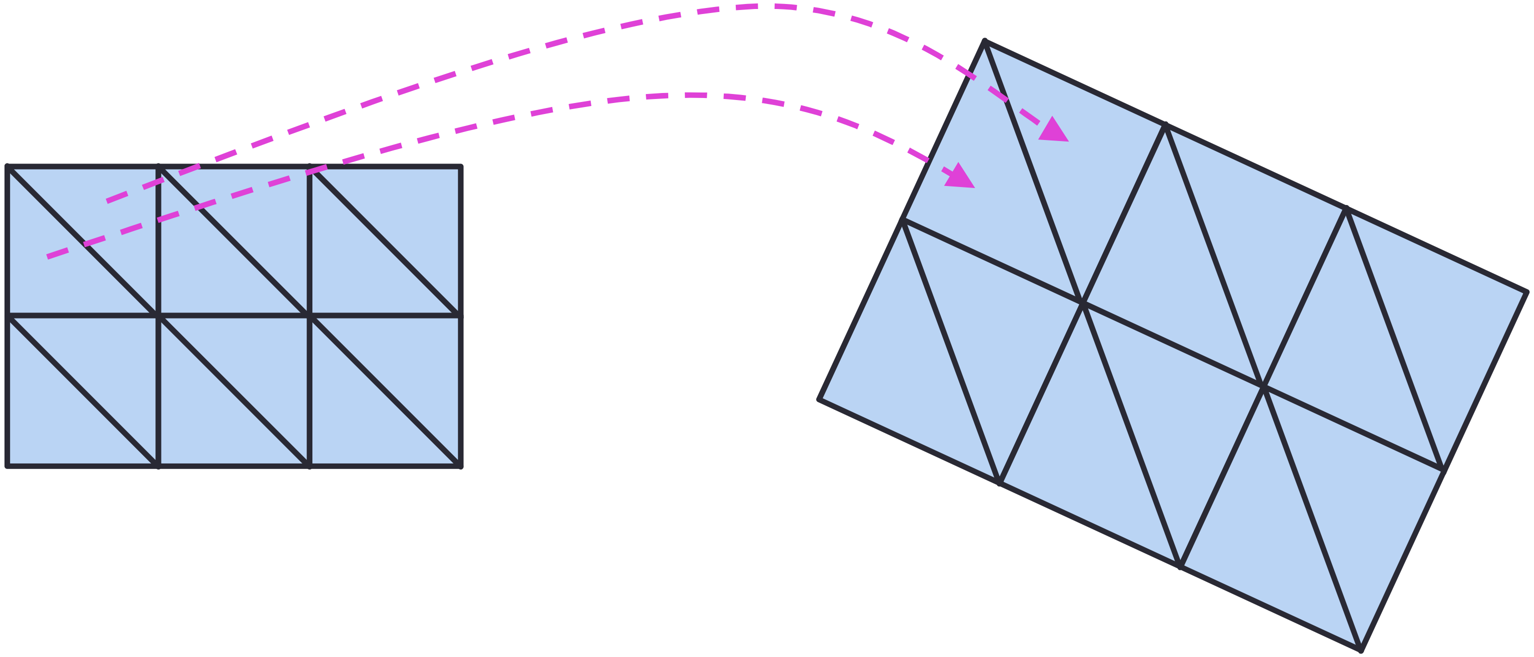


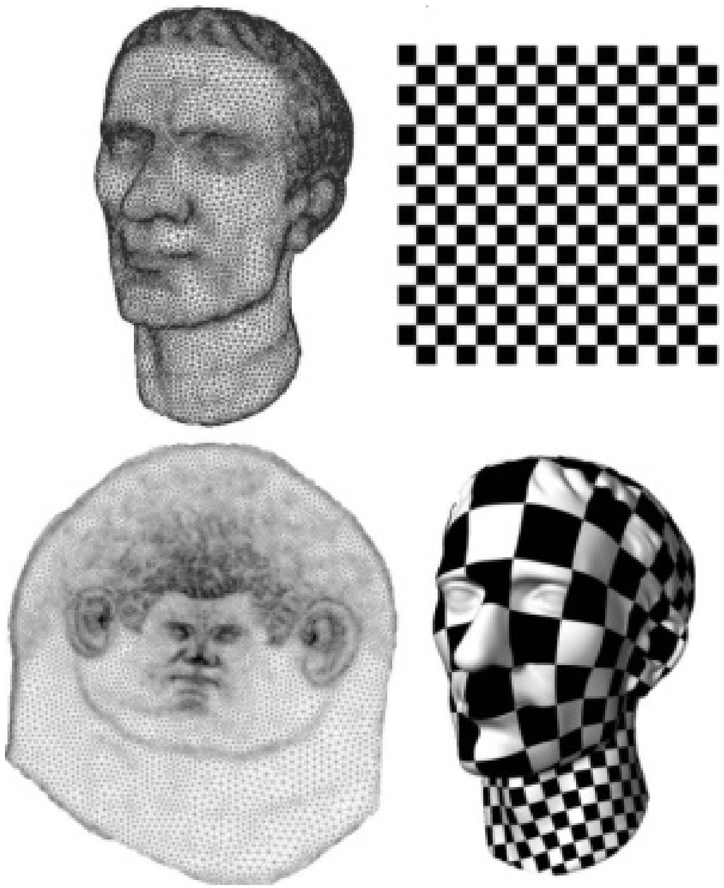
Least Squares Conformal Map (LSCM)

$$\sum_j w_j \left\| A_j - \mathcal{S}(A_j) \right\|_F^2 = 0$$



global similarity = discrete conformal maps

Least Squares Conformal Map (LSCM)

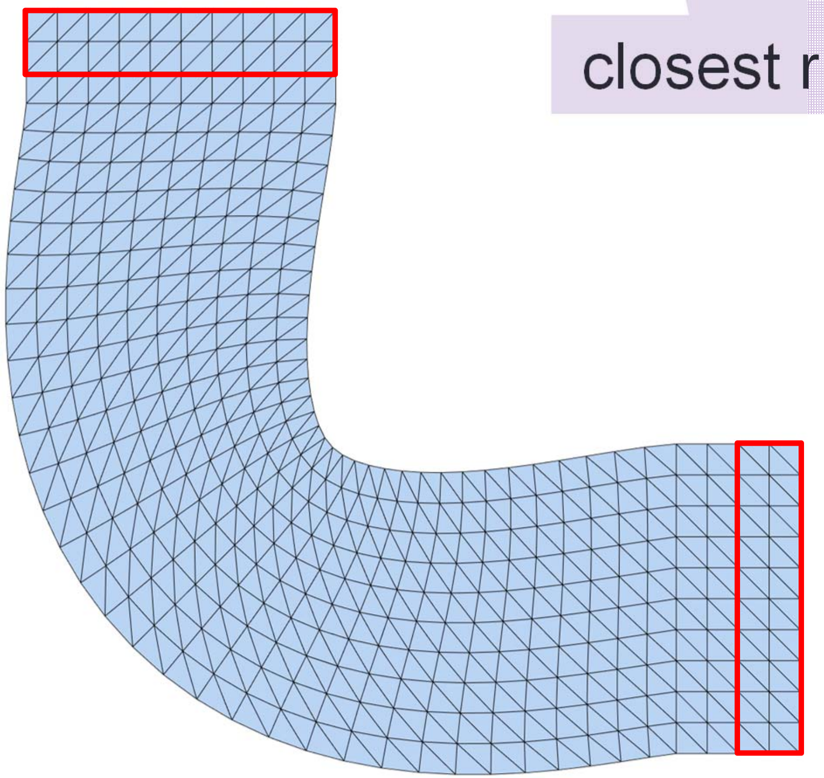


[Lévy et al. 2002]

As-Rigid-As-Possible (ARAP)

$$E_R = \sum_j w_j \|A_j - \mathcal{R}(A_j)\|_F^2$$

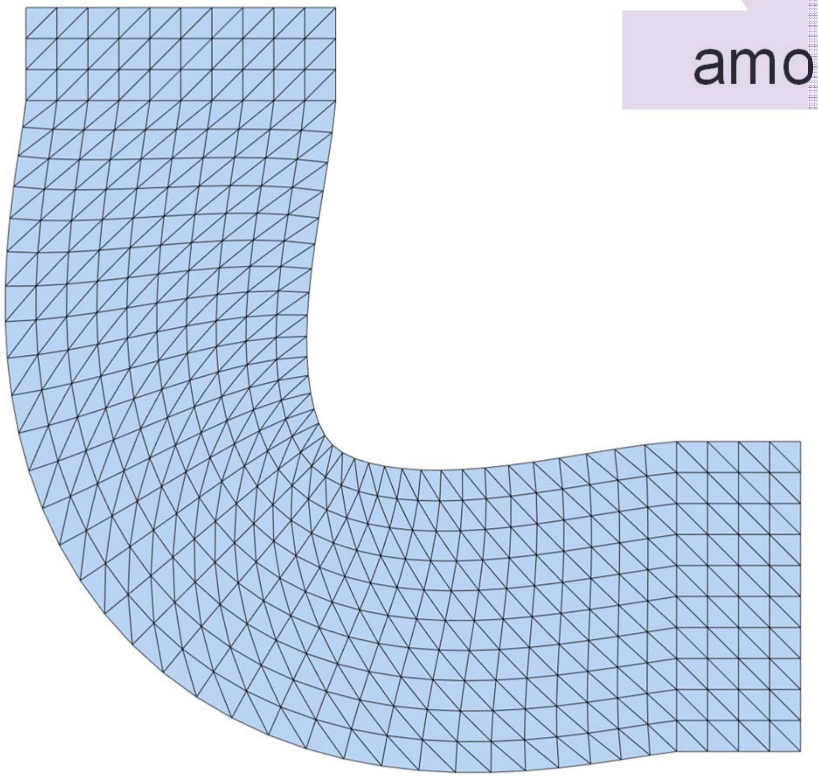
closest rigid transformation



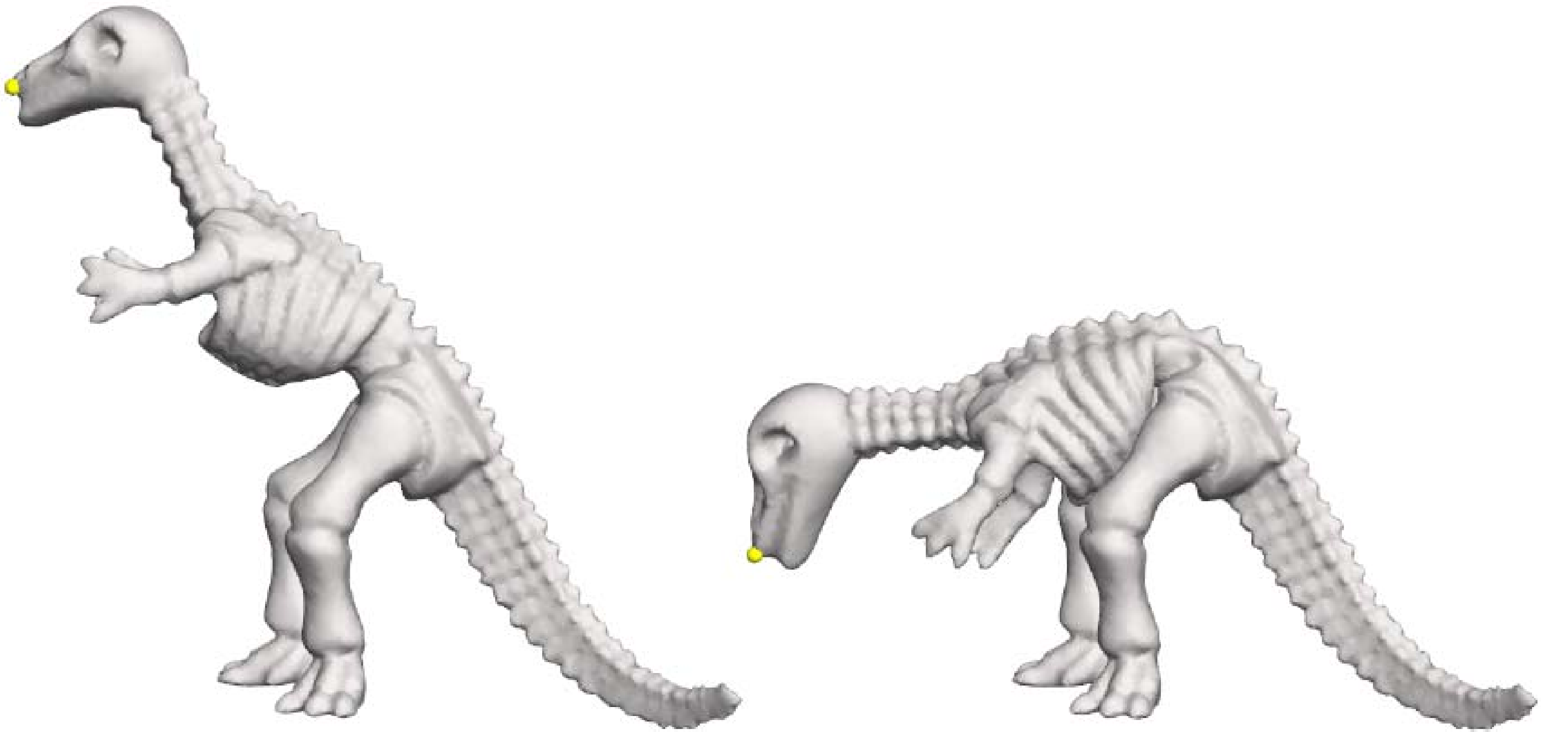
As-Rigid-As-Possible (ARAP)

$$E_R = \sum_j w_j \left\| A_j - \mathcal{R}(A_j) \right\|_F^2$$

amount of non-rigidity

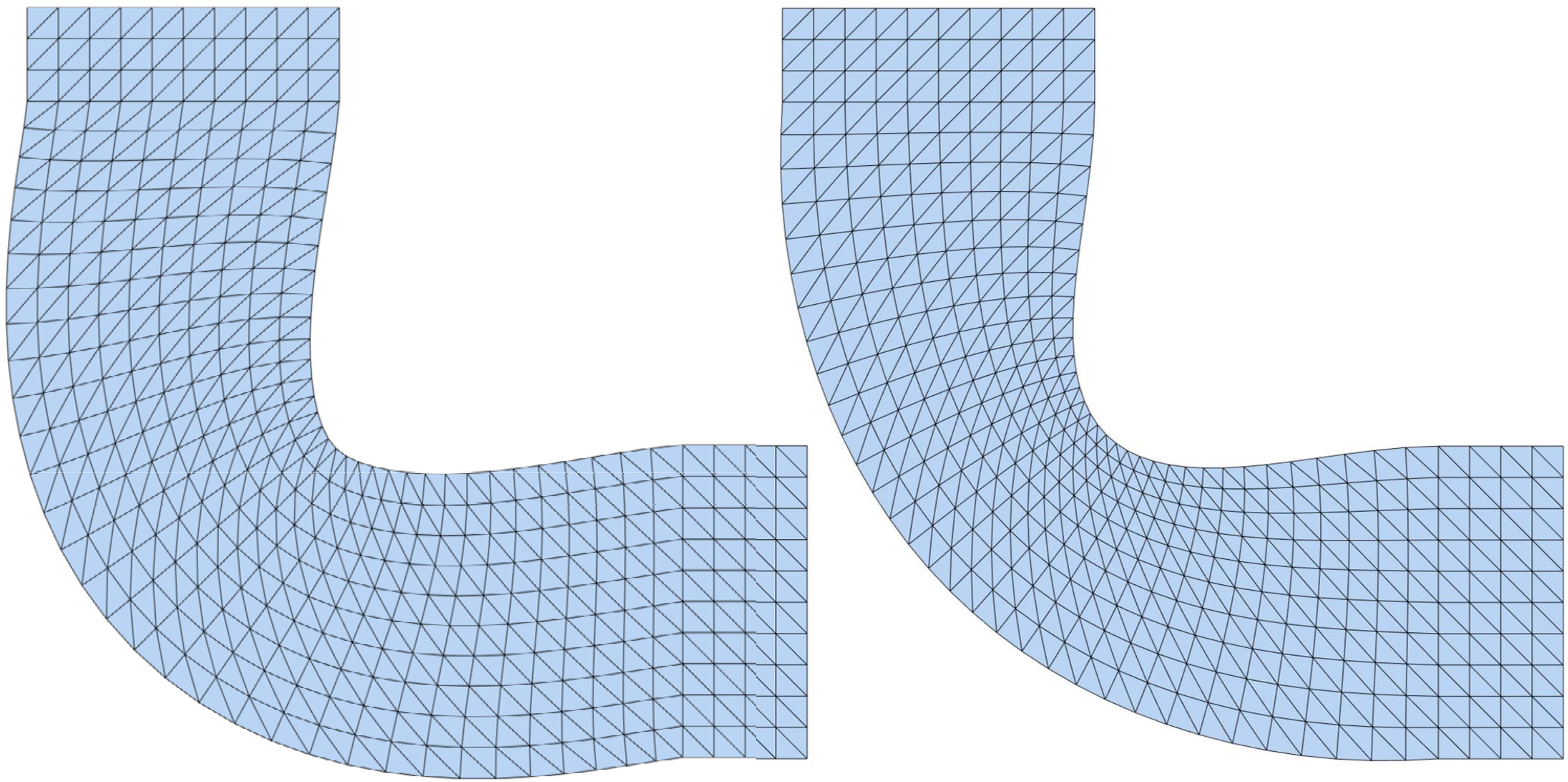


As-Rigid-As-Possible (ARAP)



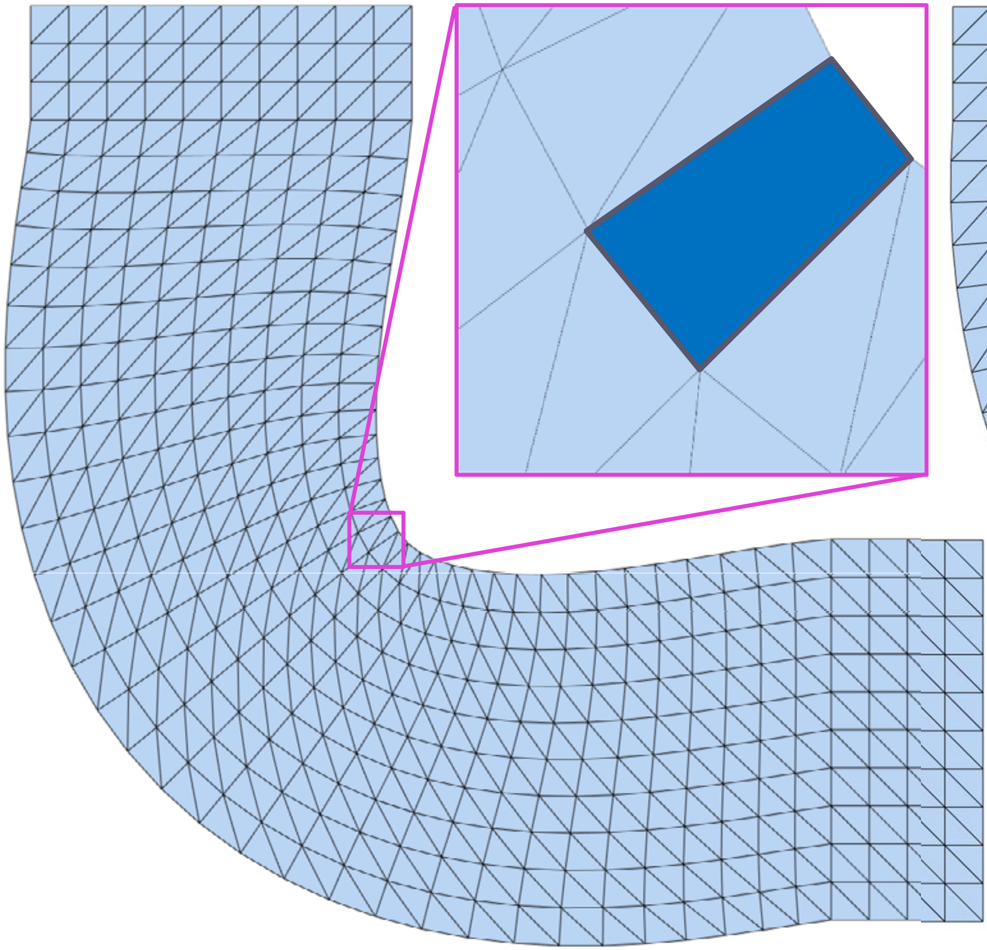
[Sorkine & Alexa 2007*; Chao et al. 2010]

ARAP vs. LSCM

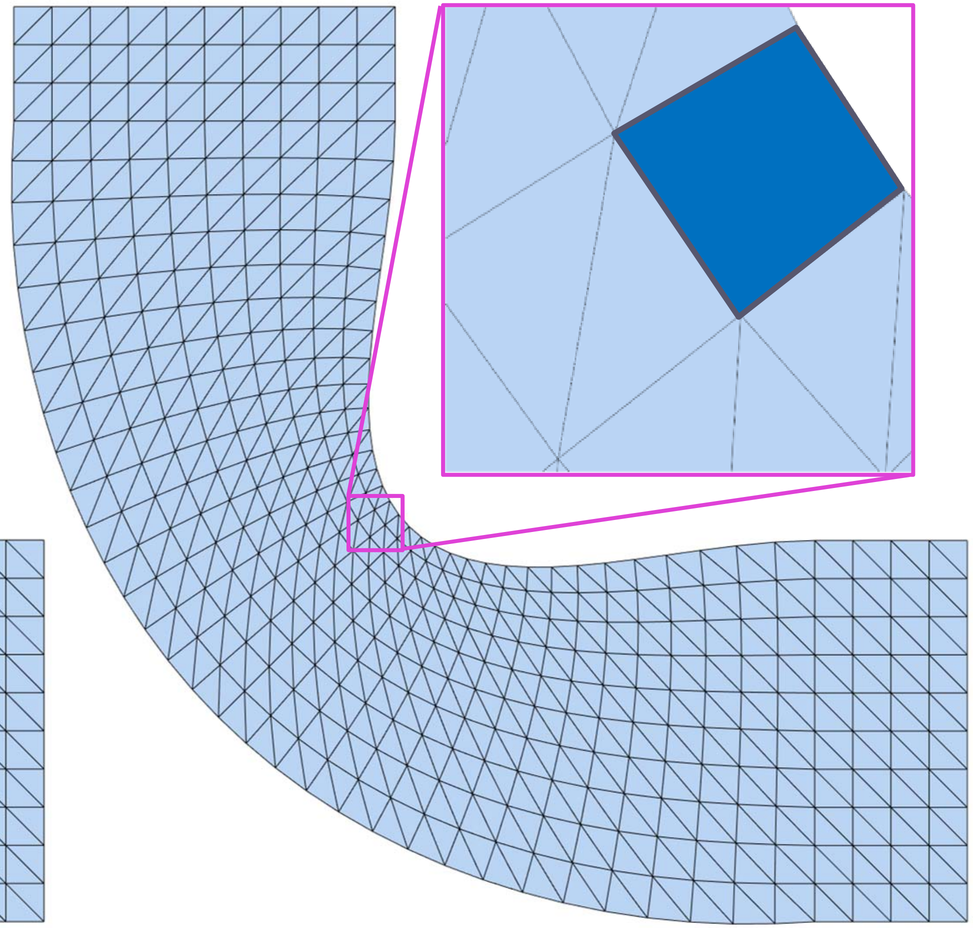


ARAP LSCM

ARAP vs. LSCM

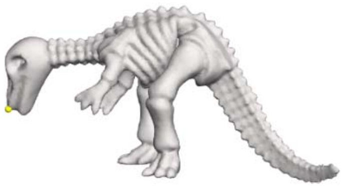


ARAP

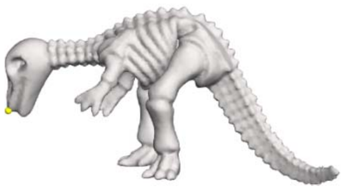


LSCM

Recap: Popular energies

Dirichlet		$\|A_j\|_F^2$
LSCM		$\|A_j - \mathcal{S}(A_j)\|_F^2$
ARAP		$\|A_j - \mathcal{R}(A_j)\|_F^2$

Recap: Popular energies

Dirichlet		$\left\|A_j\right\|_F^2$	Least squares
LSCM		$\left\|A_j - \mathcal{S}(A_j)\right\|_F^2$	
ARAP		$\left\|A_j - \mathcal{R}(A_j)\right\|_F^2$	

Closest S – 2d case

- $\mathcal{S}(A) = \overline{\sigma} U V^T$
- Takes a closed form:

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \frac{1}{2}\begin{bmatrix} a+d & c-b \\ b-c & a+d \end{bmatrix} + \frac{1}{2}\begin{bmatrix} a-d & c+b \\ b+c & -a+d \end{bmatrix}$$

similarity

anti-similarity

Recap: Popular energies

Dirichlet

$$\|A_j\|_F^2$$

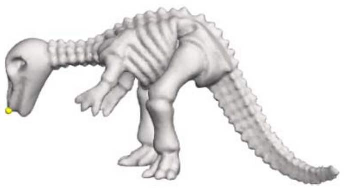
Least squares

LSCM

$$\|A_j - \mathcal{S}(A_j)\|_F^2$$

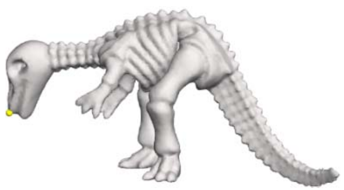
anti-similarity

ARAP

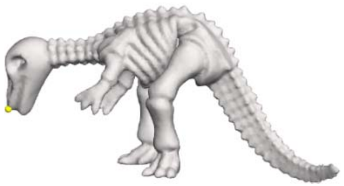


$$\|A_j - \mathcal{R}(A_j)\|_F^2$$

Recap: Popular energies

Dirichlet		$\|A_j\|_F^2$	Least squares
LSCM		$\|A_j - \mathcal{S}(A_j)\|_F^2$	**2d – Least squares**
ARAP		$\|A_j - \mathcal{R}(A_j)\|_F^2$	

Recap: Popular energies

Dirichlet		$\|A_j\|_F^2$	Least squares
LSCM		$\|A_j - \mathcal{S}(A_j)\|_F^2$	2d - least squares iterative approximation
ARAP		$\|A_j - \mathcal{R}(A_j)\|_F^2$	iterative approximation

Where's the difficulty?

$$E_R = \sum_j w_j \|A_j - \mathcal{R}(A_j)\|_F^2$$

- Not very friendly for direct minimization:

$$A - \mathcal{R}(A) = A - UV^T$$

via SVD of A

But $\mathcal{R}(A_j)$ is easy to compute…

Iterative approximation

$$E_R = \sum_j w_j \left\| A_j - \mathcal{R}(A_j) \right\|_F^2$$

- Iteratively:
  - Compute $R_j = \mathcal{R}(A_j)$
  - Global optimization (w.r.t. R_j)

Alternating optimization

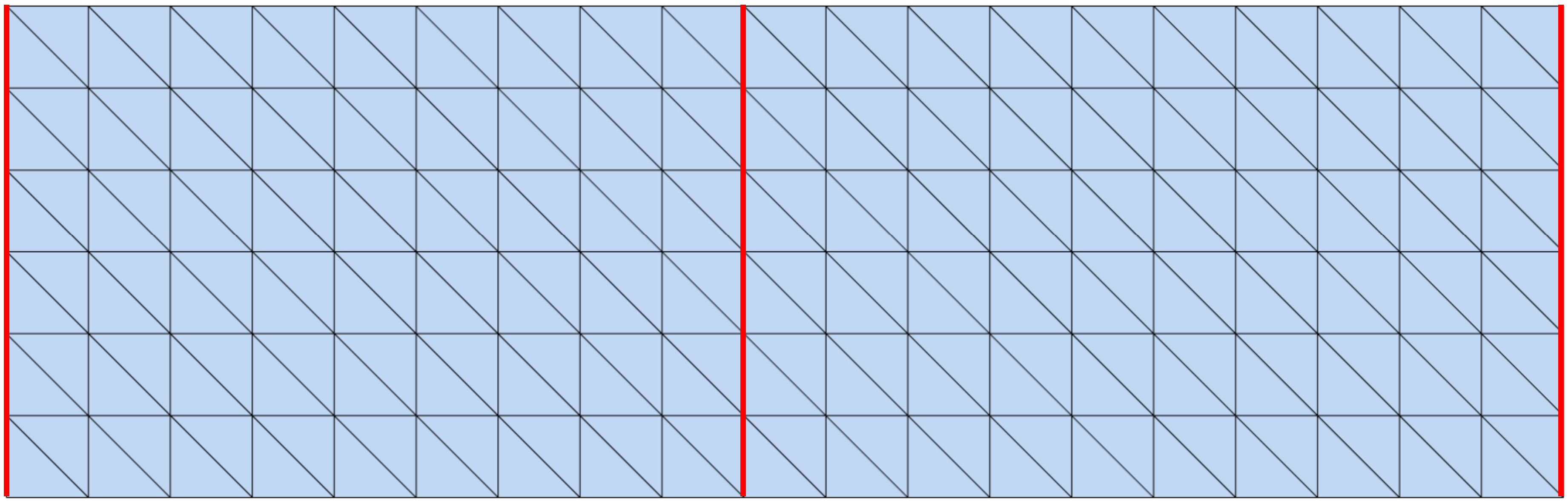
$$E_R = \sum_j w_j \left\| A_j - \mathcal{R}(A_j) \right\|_F^2$$

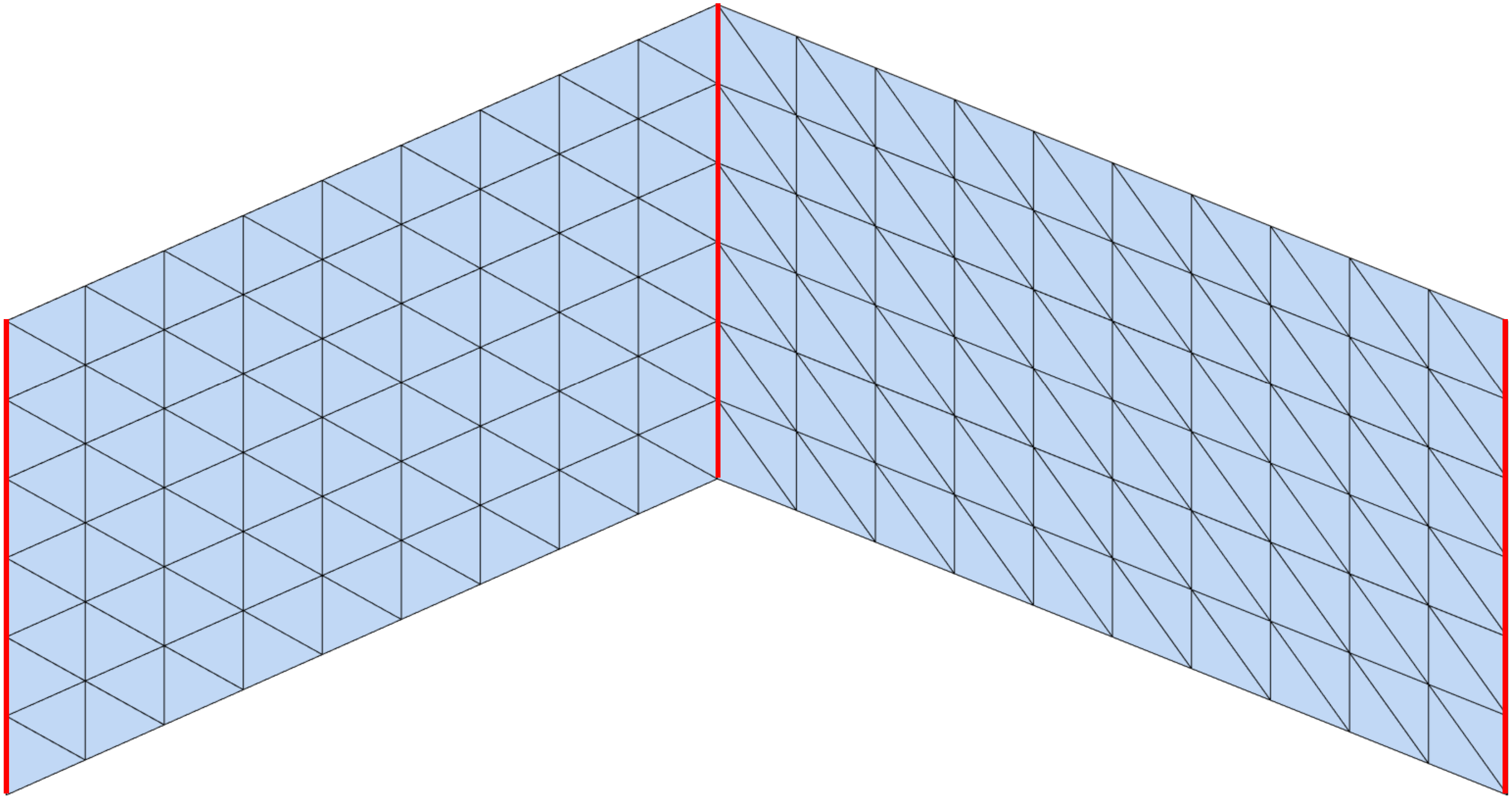
- Iteratively:
  - Compute and fix $R_j = \mathcal{R}(A_j)$ **Local** step
  - Minimize

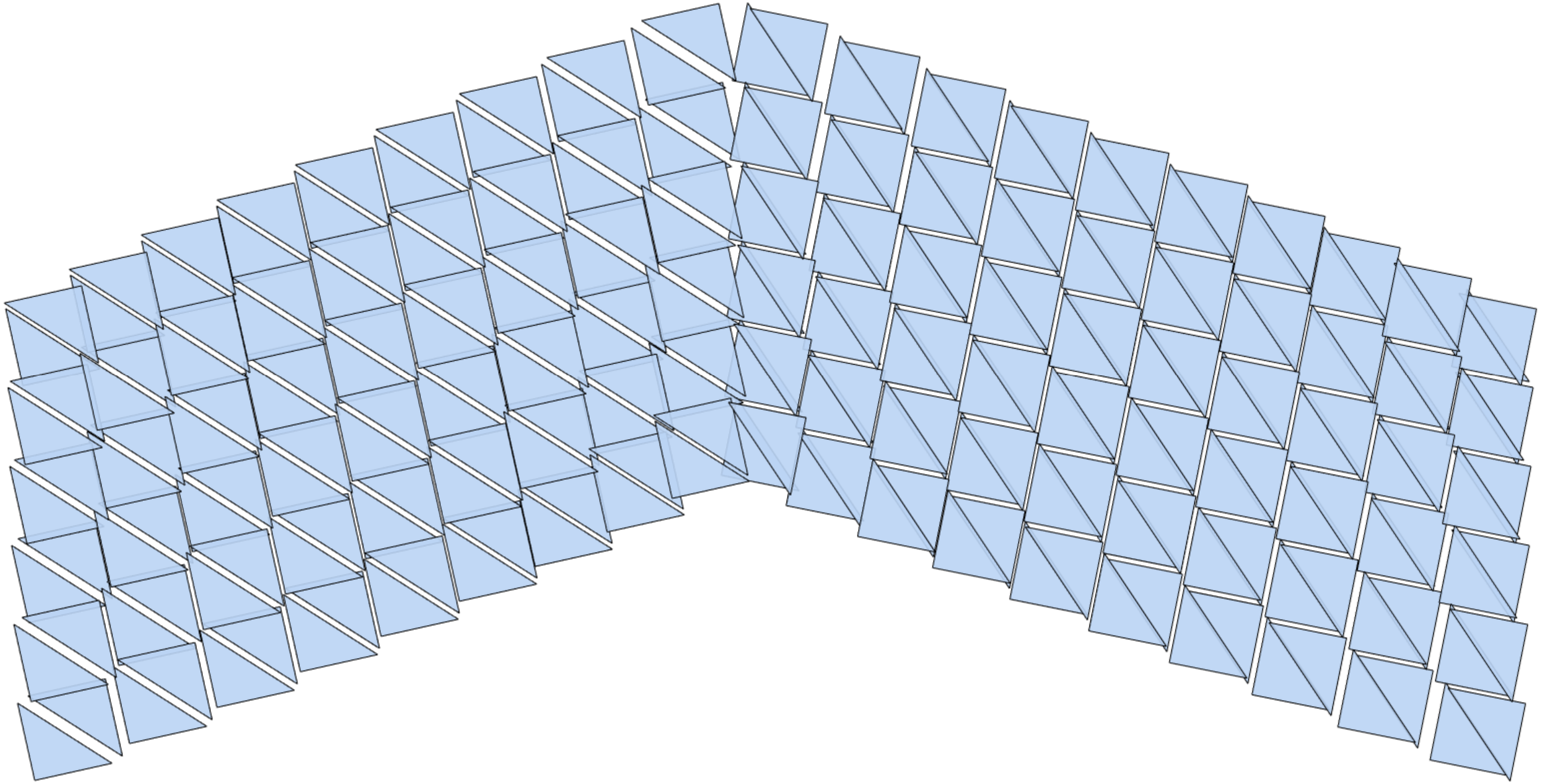
$$\sum_j w_j \left\| A_j - R_j \right\|_F^2$$

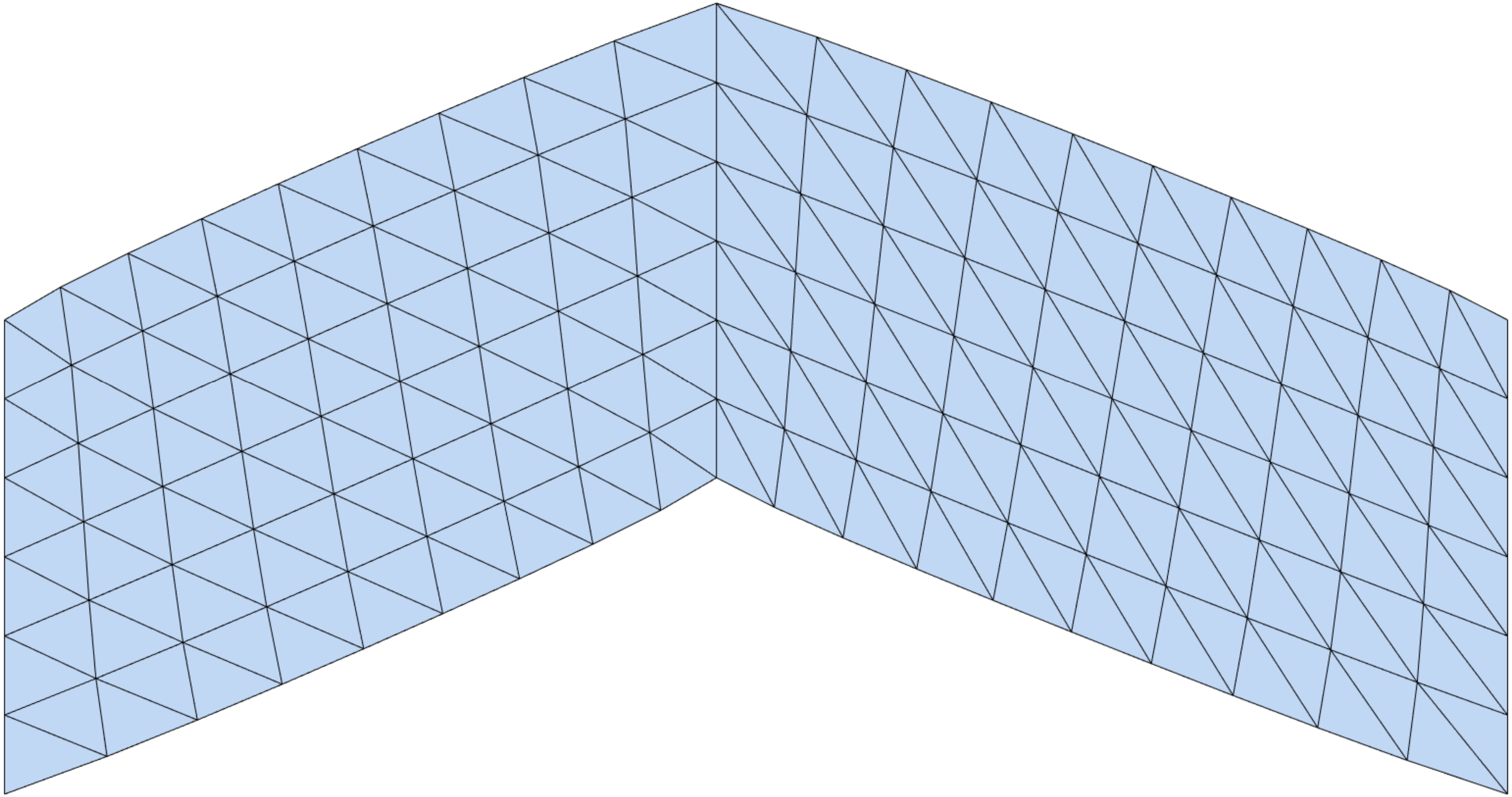
Global step

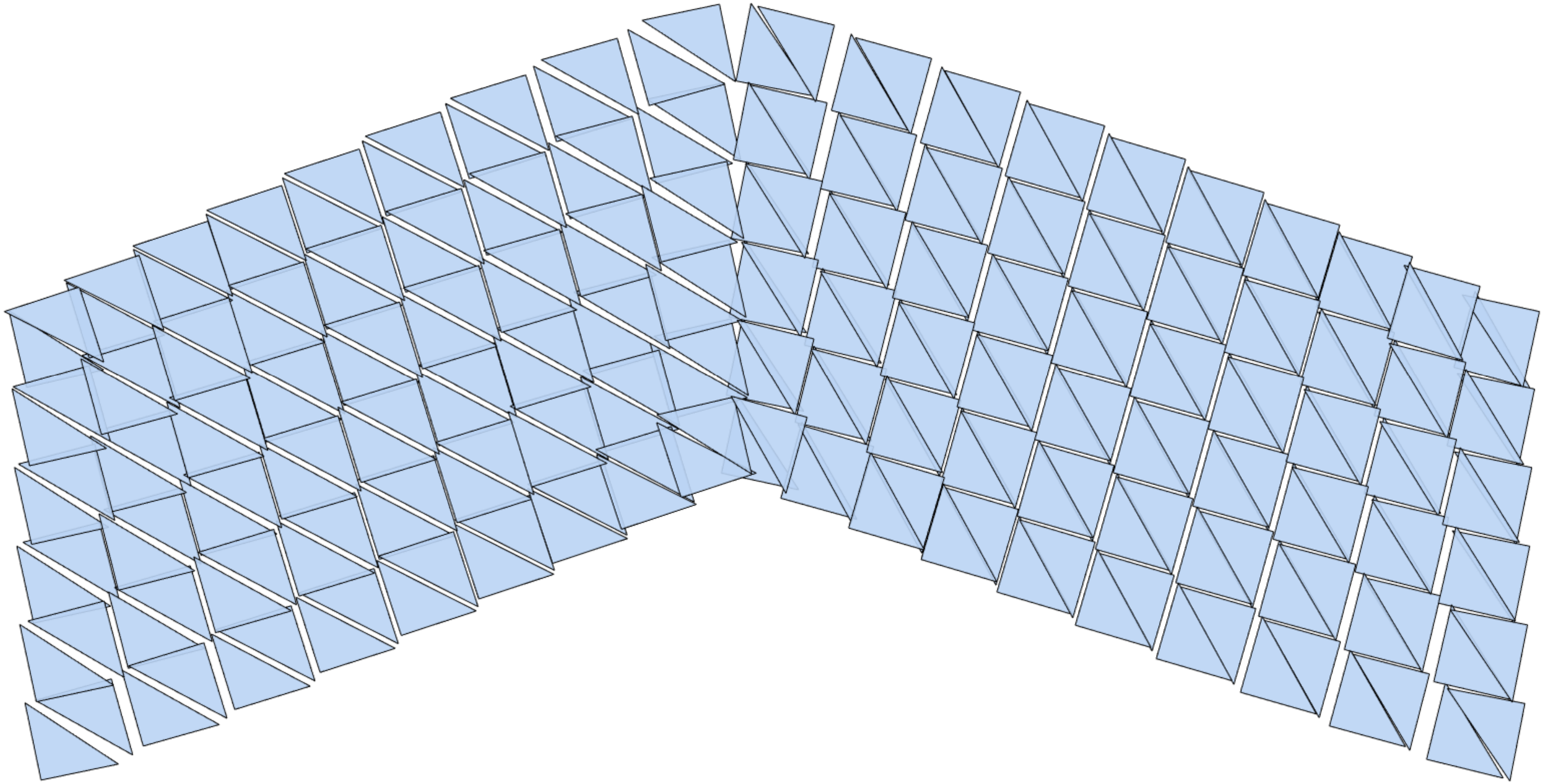
[Liu et al. 2008]

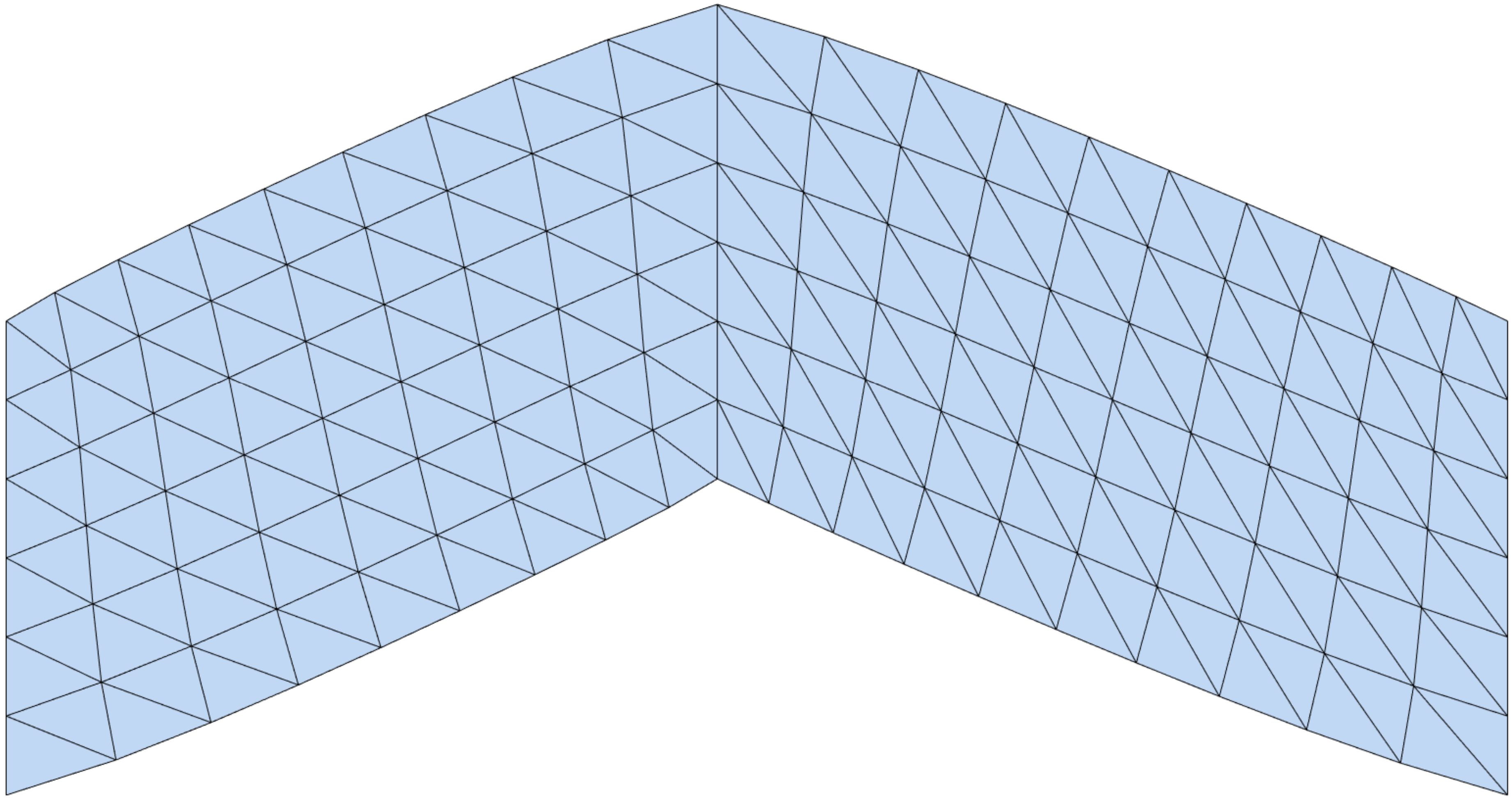


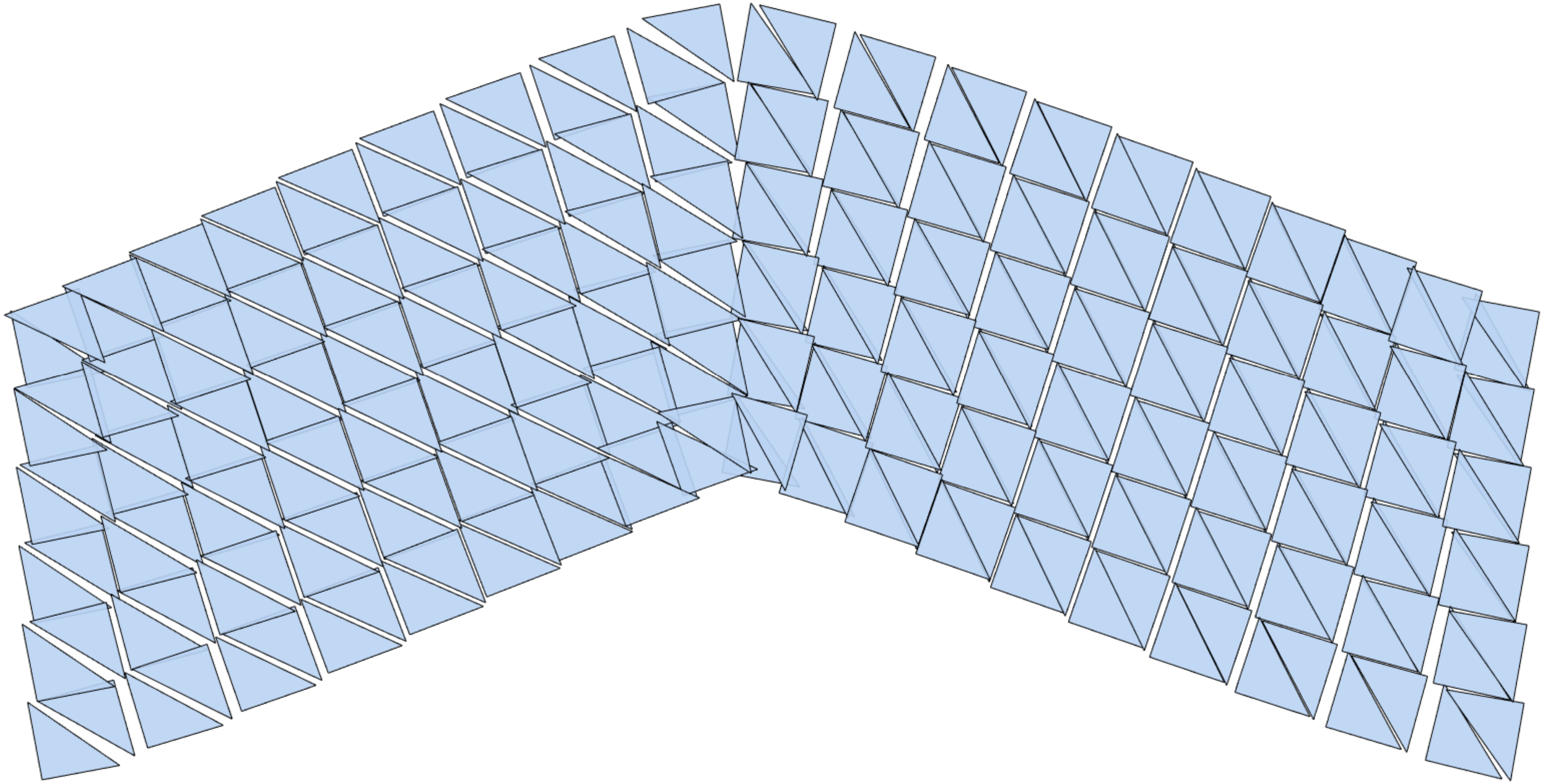


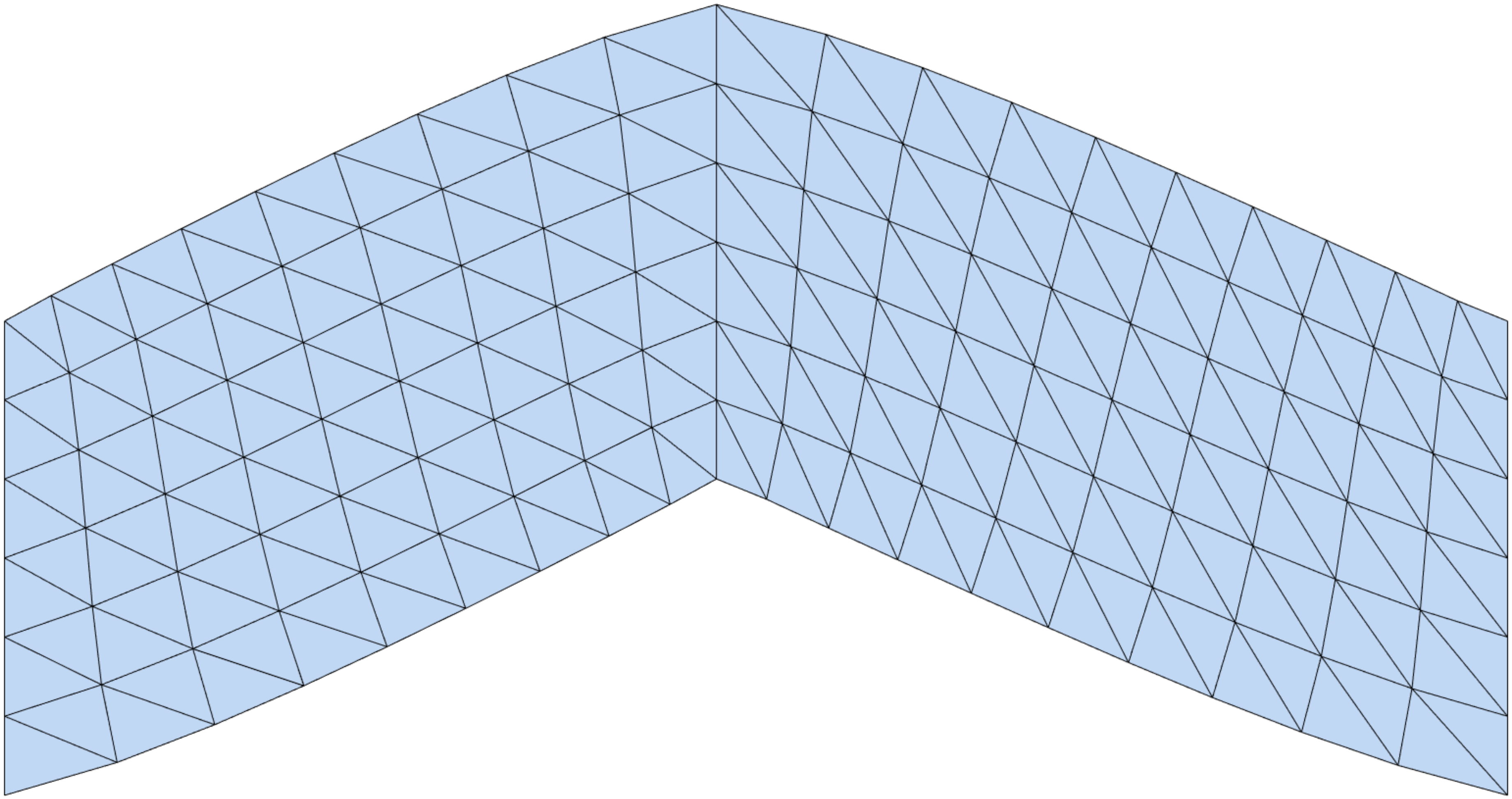


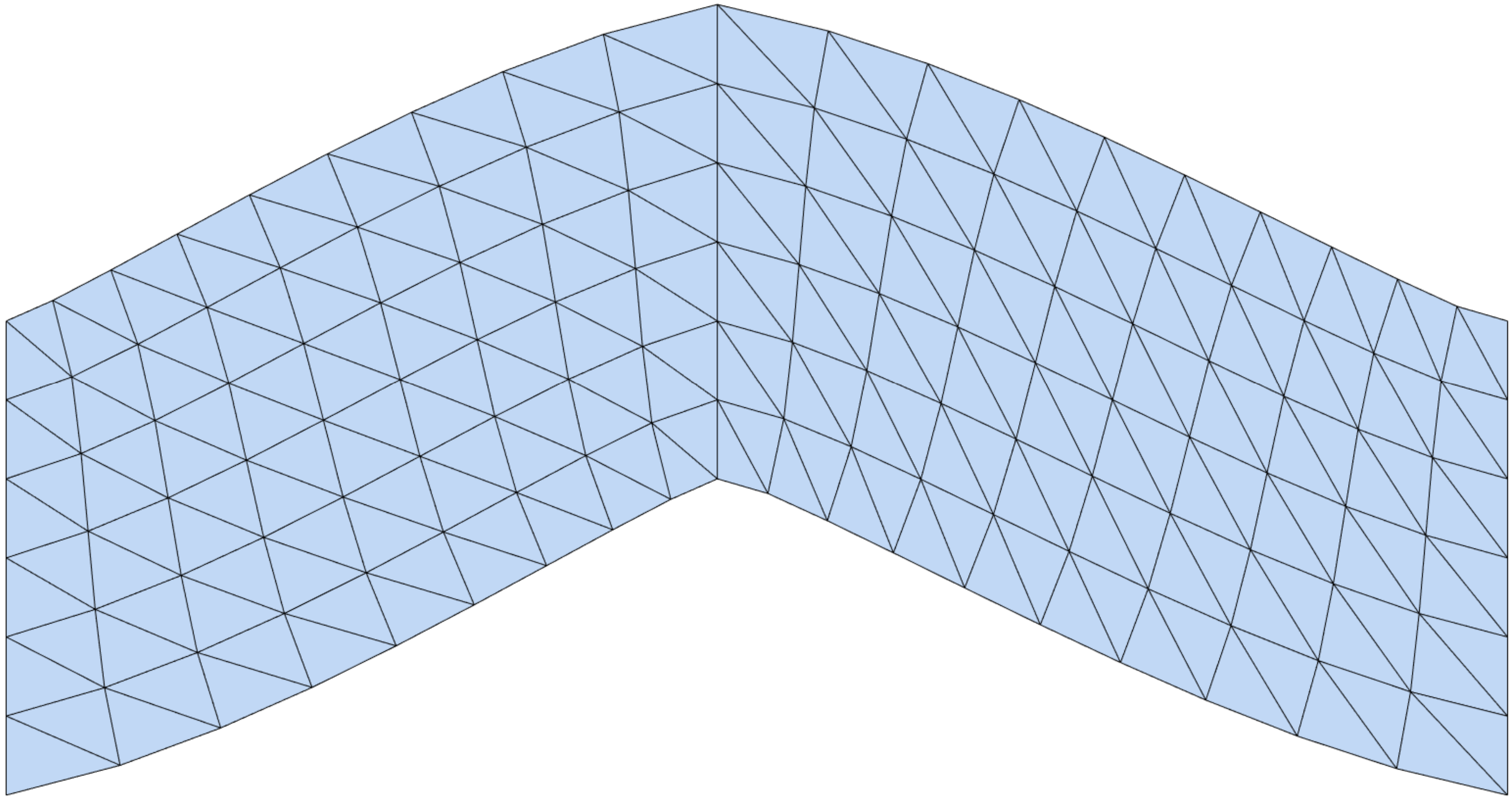


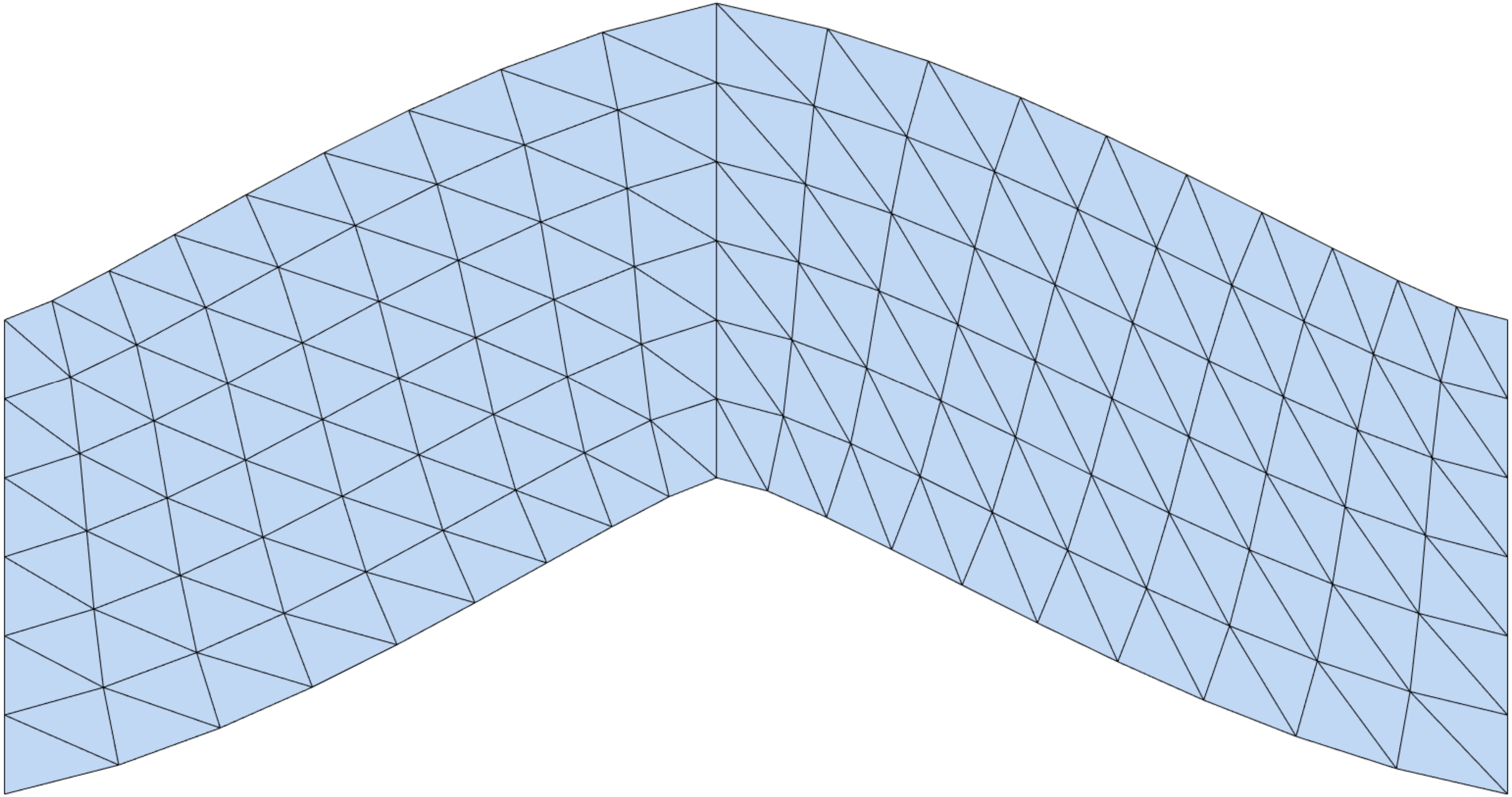


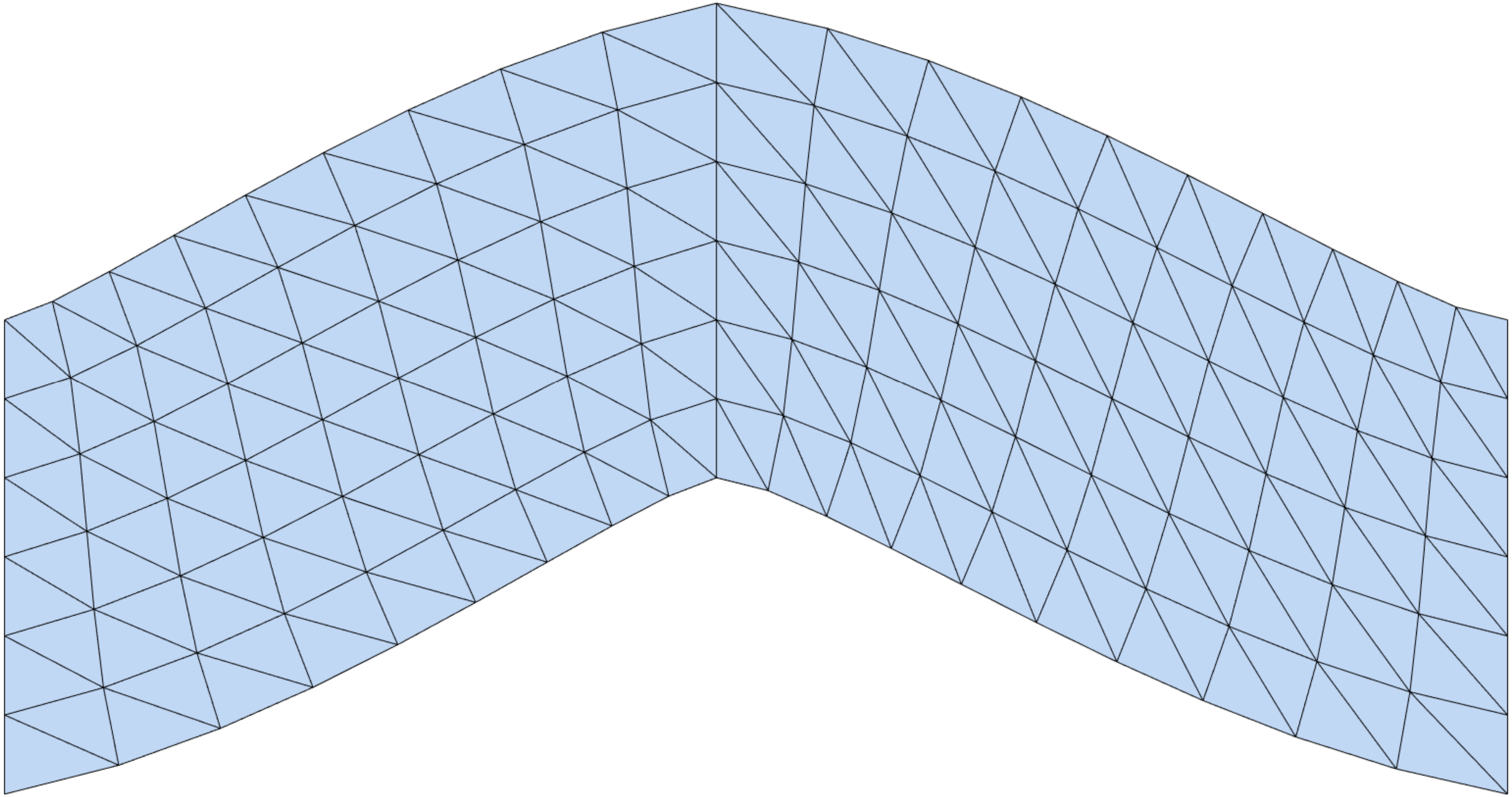


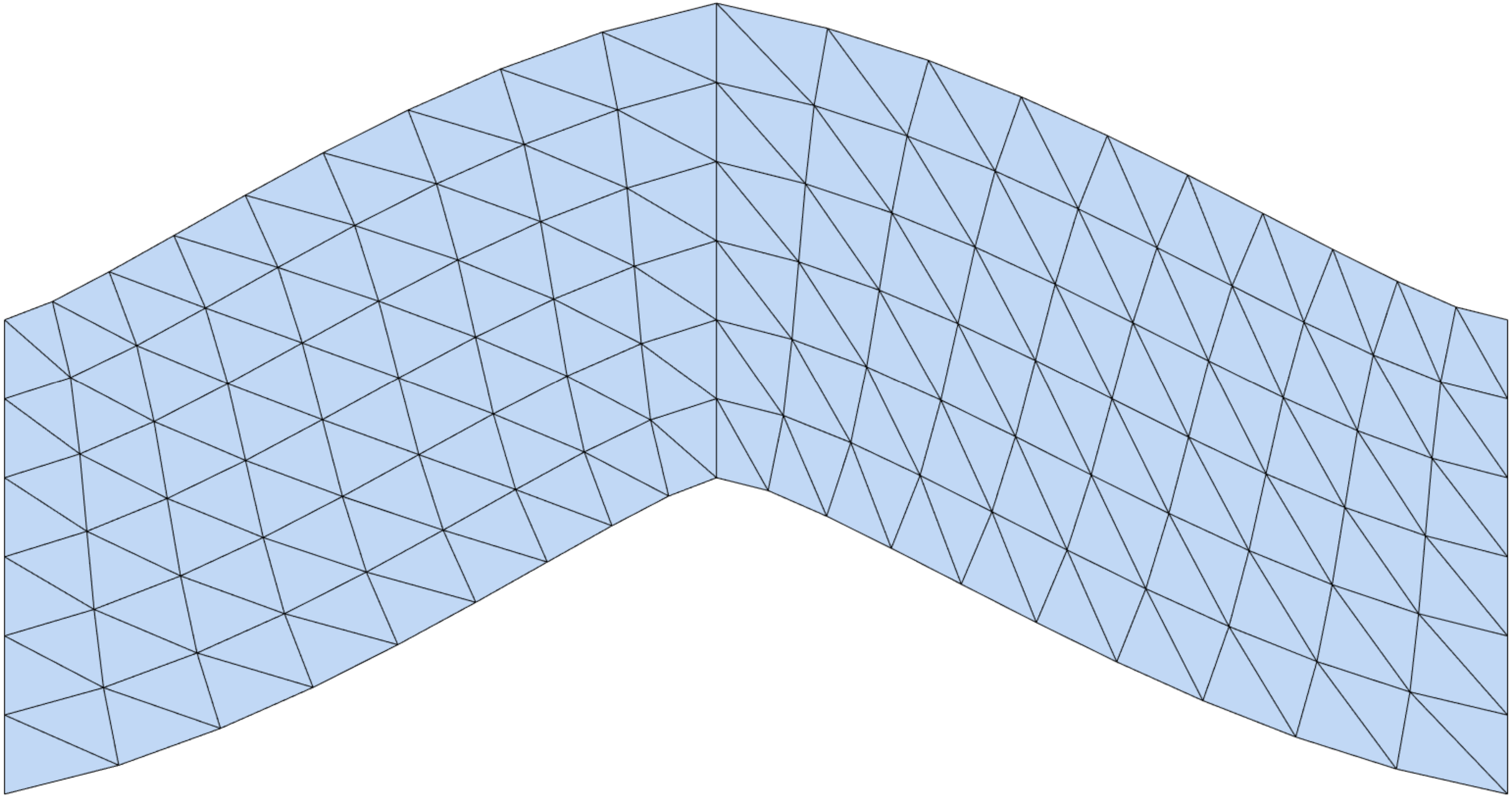






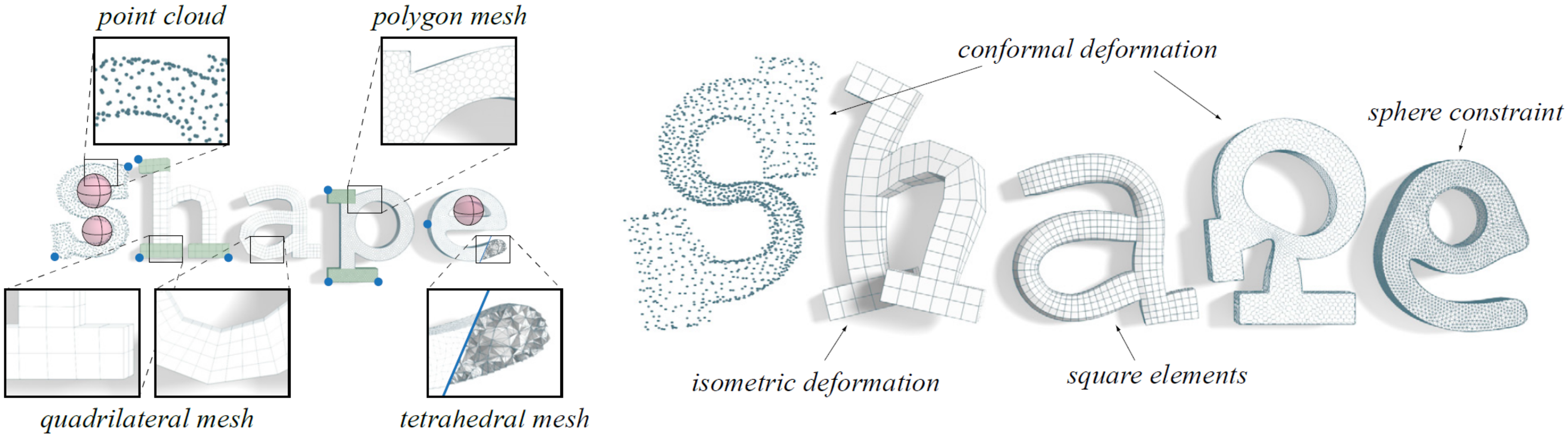






Alternating optimization

- Very general



[Bouaziz et al. 2012]

- Related jargon:
  gradient descent, global-local, alternating projections

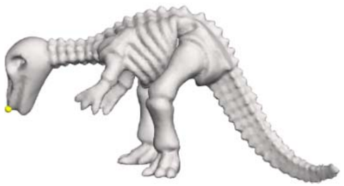
Higher order approximation

- Newton approach (2^{nd} order)

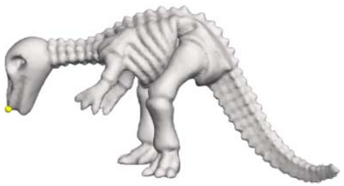
[Chao et al. 2010, Alec's Web Log]

http://www.alecjacobson.com/weblog/?p=4185

Singular values perspective

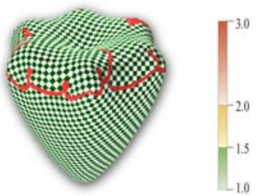
Dirichlet		$\|\|A\|\|_F^2$
LSCM		$\|\|A - \mathcal{S}(A)\|\|_F^2$
ARAP		$\|\|A - \mathcal{R}(A)\|\|_F^2$

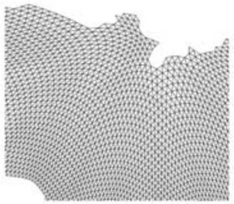
Singular values perspective

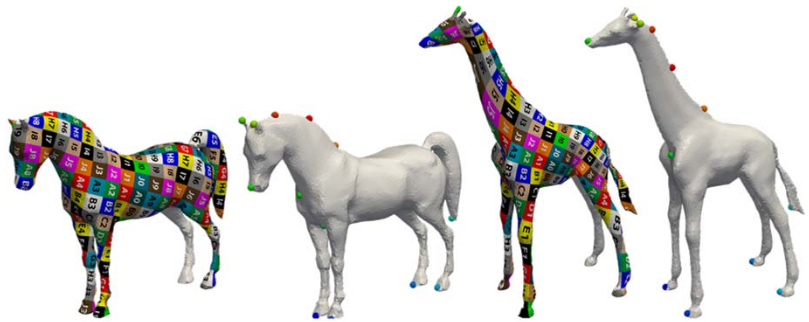
Dirichlet		$\|A\|_F^2$	$\sum_k \sigma_k^2$
LSCM		$\|A - \mathcal{S}(A)\|_F^2$	$\sum_k (\sigma_k - \overline{\sigma})^2$ (mean of SVs)
ARAP		$\|A - \mathcal{R}(A)\|_F^2$	$\sum_k (\sigma_k - 1)^2$

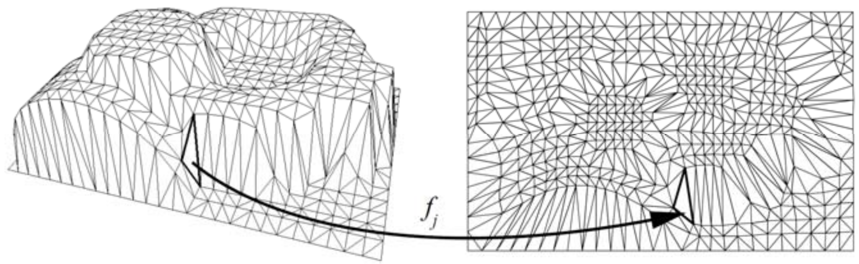
Problems in terms of SV

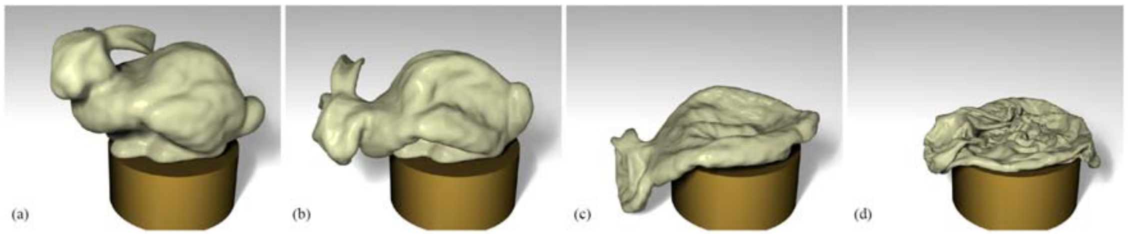
- Want to solve other "SV problems"

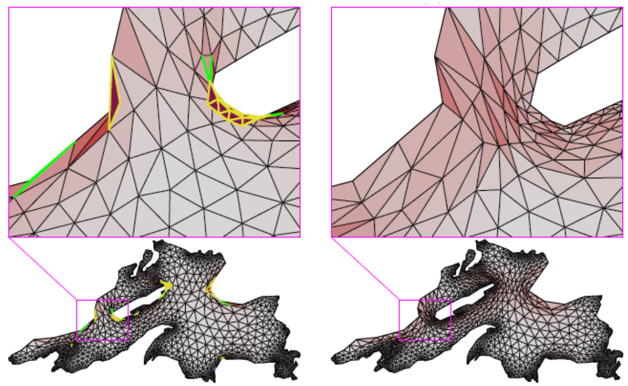












Where's the challenge?

Analytically:

- Singular values of A

$\Updownarrow$

- Eigenvalues of $A^T A$

$\Updownarrow$

- Roots of $p(\lambda) = \det(A^T A - \lambda I)$

$\Updownarrow$

- Roots of high order polynomial

Where’s the challenge?

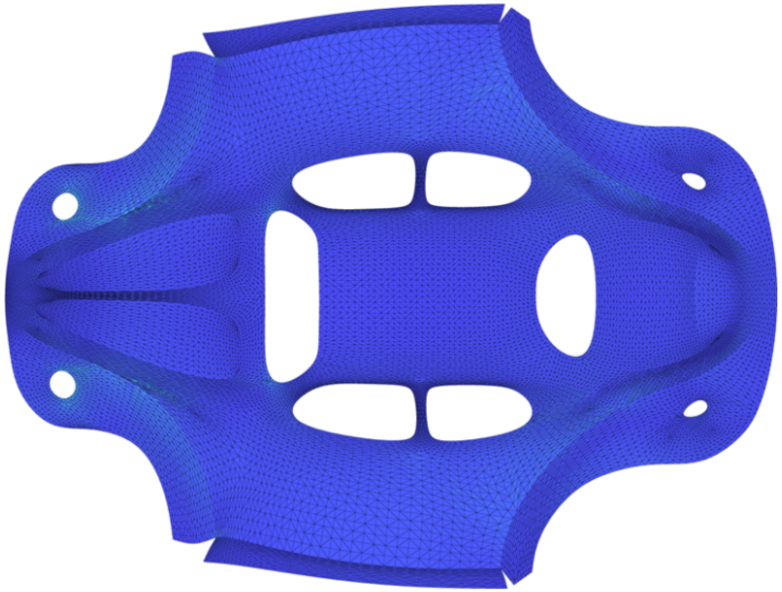
- Singular values are:
  - Non-linear
  - Non-convex
  - No closed form

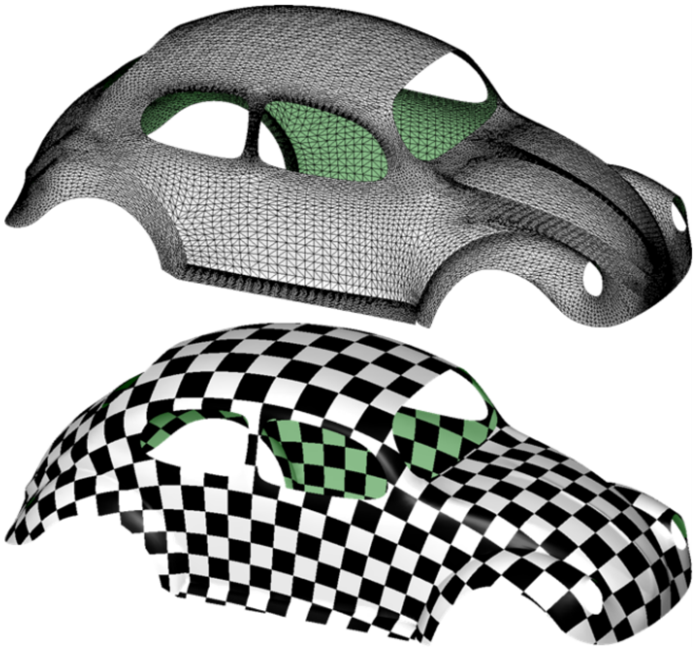
- Yet they’re awesome, so…

Miracles happen

- For example:
  - Dirichlet – $\sum_k \sigma_k^2$
  - LSCM – $\sum_k (\sigma_k - \overline{\sigma})^2$

→ Linear system of equations



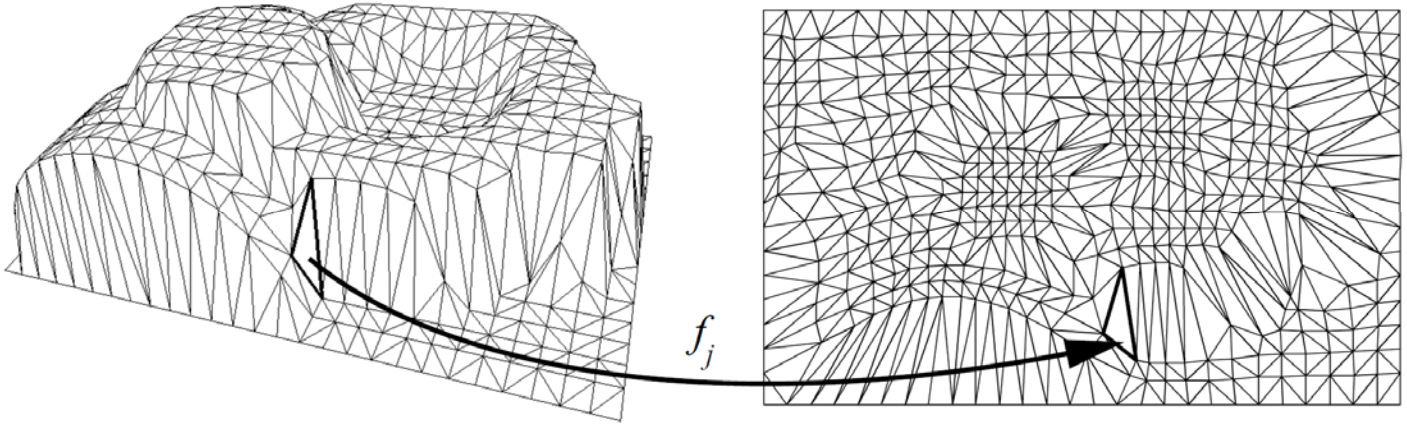


[Mullen et al. 2008]

Other SV energies

- Parameterization

$$\frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}$$



MIPS [Hormann & Greiner 2000]

Most Isometric Parameterization

Other SV energies

- Parameterization

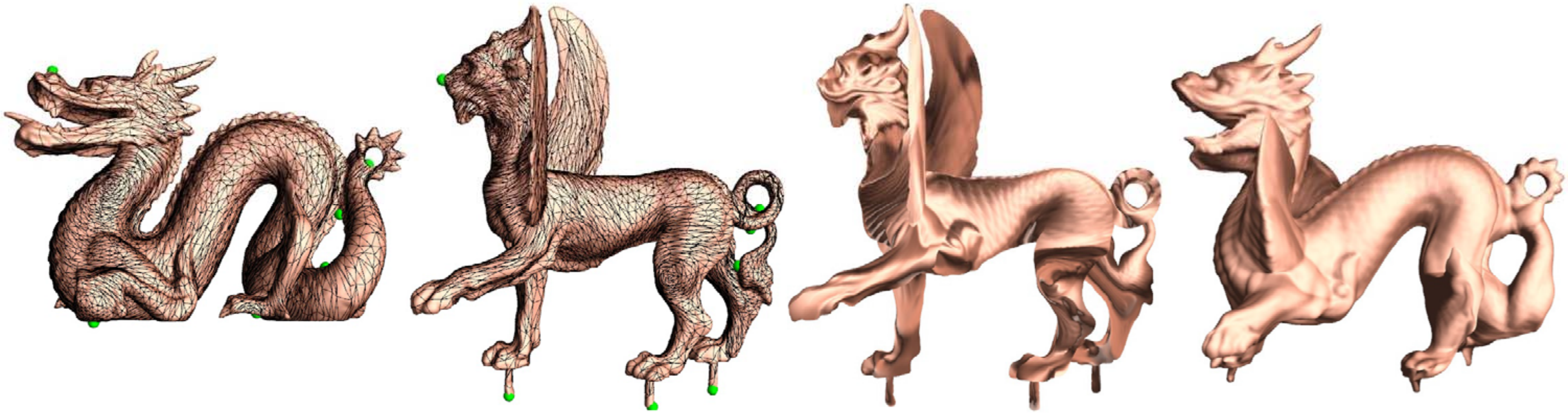
$$\max\left\{\sigma_1, \frac{1}{\sigma_2}\right\}$$

[Sorkine et al. 2000]

Other SV energies

- Surface mapping

$$\sigma_1^2 + \sigma_2^2 + \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$$

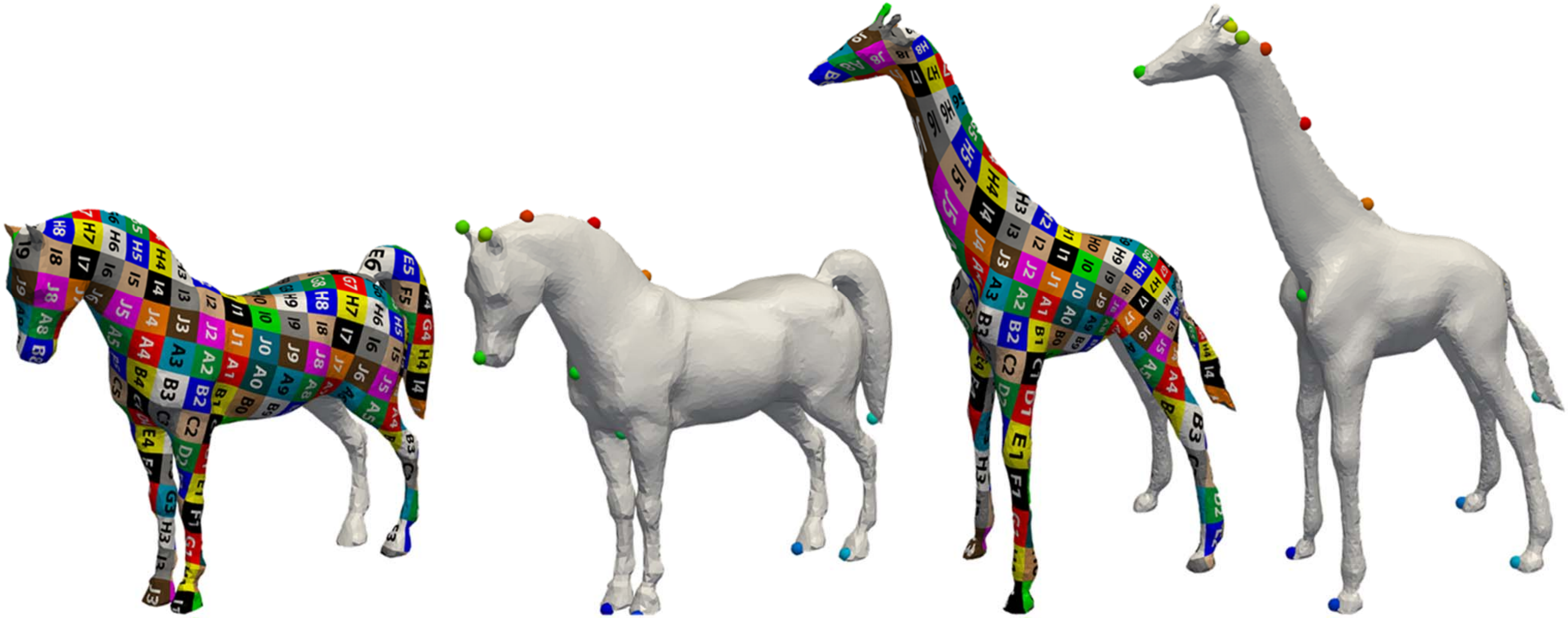


[Schreiner et al. 2014]

Other SV energies

- Surface mapping

$$\sqrt{\sigma_1^2 + \frac{1}{\sigma_2^2}}$$

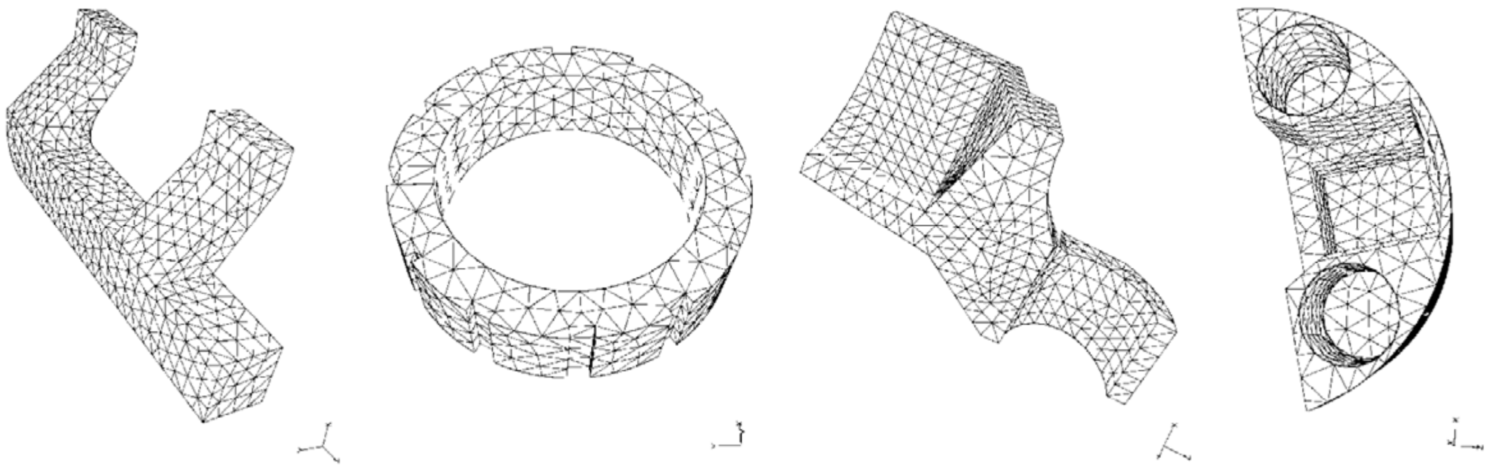


[Aigerman et al. 2014]

Other SV energies

- Volumetric mesh improvement

$$\frac{\sigma_1}{\sigma_3}$$

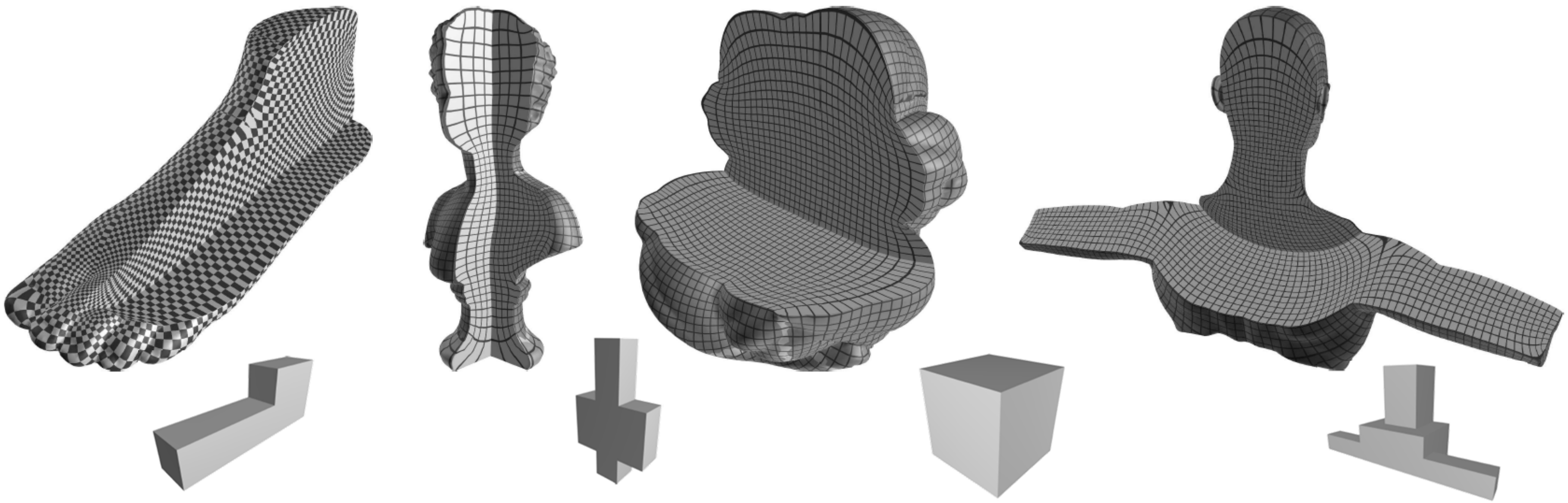


[Freitag & Knupp 2002]

Other SV energies

- Volume mapping

$$(\sigma_1 - \overline{\sigma})^2 + (\sigma_2 - \overline{\sigma})^2 + (\sigma_3 - \overline{\sigma})^2$$

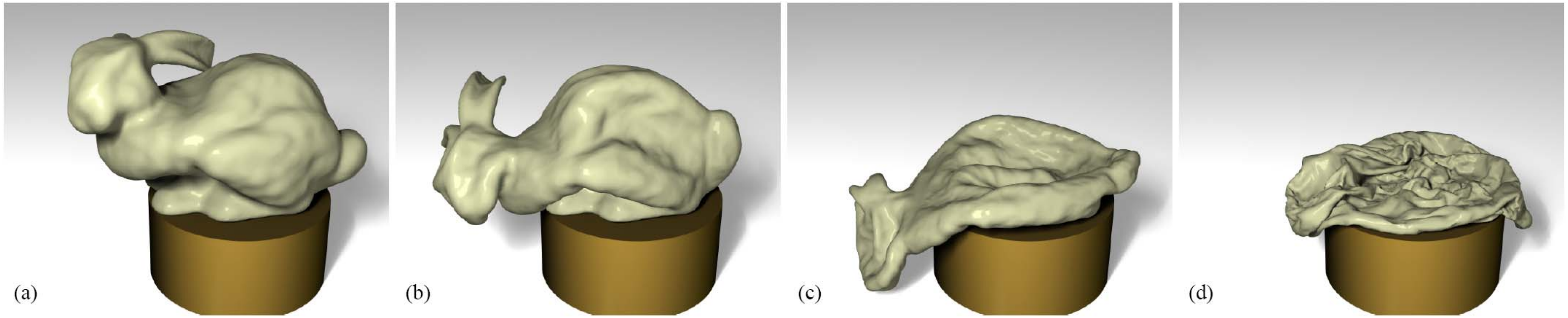


[Paillé & Poulin 2012]

SV constraints

- Strain limiting

$$\gamma \leq \sigma_1, \sigma_2, \sigma_3 \leq \Gamma$$



[Wang et al. 2010]

2-dimensional SVs

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

- Closed form:

$$\sigma_1 = \sigma_{max} = \frac{1}{2}\left(\sqrt{(b+c)^2 + (a-d)^2} + \sqrt{(b-c)^2 + (a+d)^2}\right)$$

$$\sigma_2 = \sigma_{min} = \frac{1}{2}\left(\sqrt{(b+c)^2 + (a-d)^2} - \sqrt{(b-c)^2 + (a+d)^2}\right)$$

[⋯; Lipman 2012; Smith & Schaefer 2015]

2-dimensional SVs

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \mathcal{S}(A) + \mathcal{S}^{\perp}(A)$$

$$\frac{1}{2}\begin{bmatrix} a+d & c-b \\ b-c & a+d \end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix} a-d & c+b \\ b+c & -a+d \end{bmatrix}$$

2-dimensional SVs

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \mathcal{S}(A) + \mathcal{S}^{\perp}(A)$$

- Closed form:

$$\sigma_1 = \sigma_{max} = \frac{1}{\sqrt{2}}\left(\|\mathcal{S}^{\perp}(A)\|_F + \|\mathcal{S}(A)\|_F\right)$$

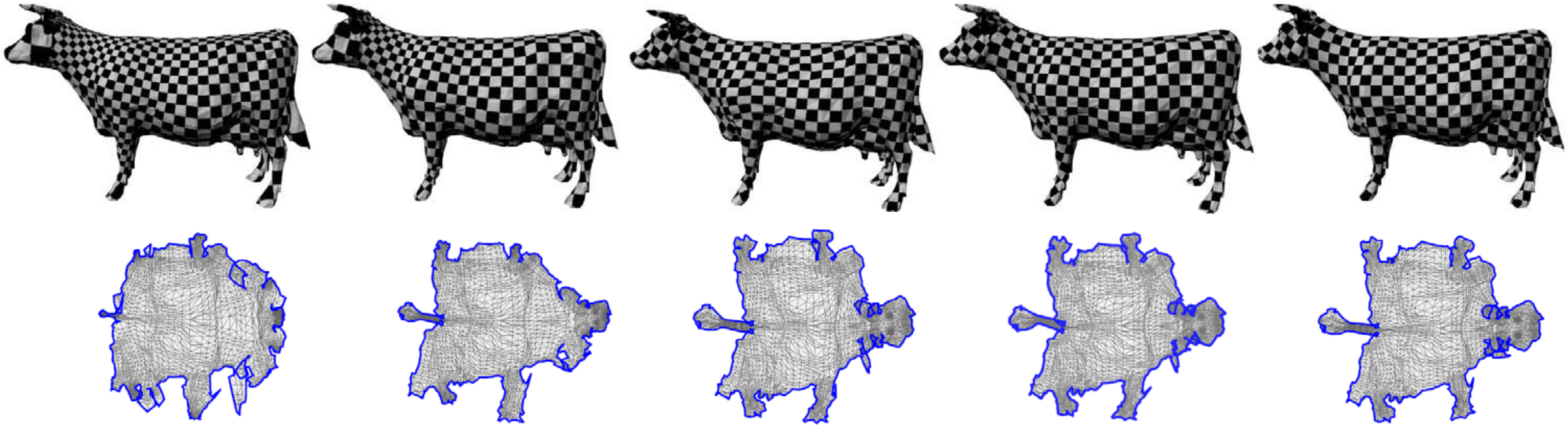
$$\sigma_2 = \sigma_{min} = \frac{1}{\sqrt{2}}\left(\|\mathcal{S}^{\perp}(A)\|_F - \|\mathcal{S}(A)\|_F\right)$$

[Lipman 2012; Smith & Schaefer 2015]

2-dimensional SVs

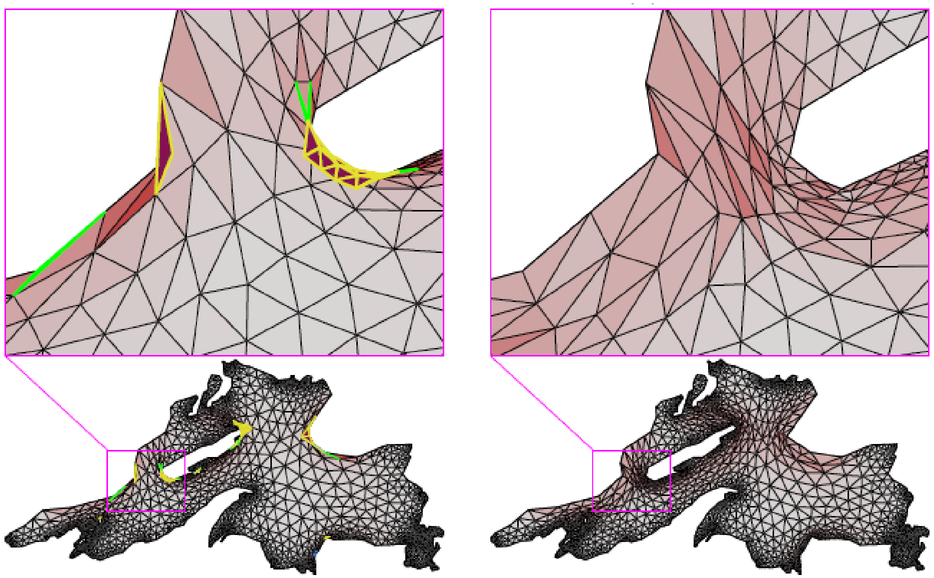
- Parameterization

$$\sigma_1^2 + \sigma_2^2 + \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$$



[Smith & Schaefer 2015]

2d SV constraints (+energy)



[Lipman 2012]

Approximate via a sequence of convex programs

Detour - convex programming

$$\underset{x}{\operatorname{argmin}} \quad u^T x$$

$$\text{s.t.} \quad Ax = b$$

$$x \in K$$

Convex

- Convex is "good":
  - Global minimum
  - Guarantees
  - Efficient algorithms (usually)

Convex = Simple?

- Cone of positive semidefinite matrices

$$\{A : x^T Ax \geq 0 \;\; for\ all\ x\}$$

"easy"

Convex = Simple?

- Cone of copositive matrices

$$\{A : x^T A x \geq 0 \;\; for\ all\ x \geq 0\}$$

“very difficult”

Detour - convex programming

$$\underset{x}{\operatorname{argmin}} \quad u^T x$$

$$\text{s.t.} \quad Ax = b$$

$$x \in K$$

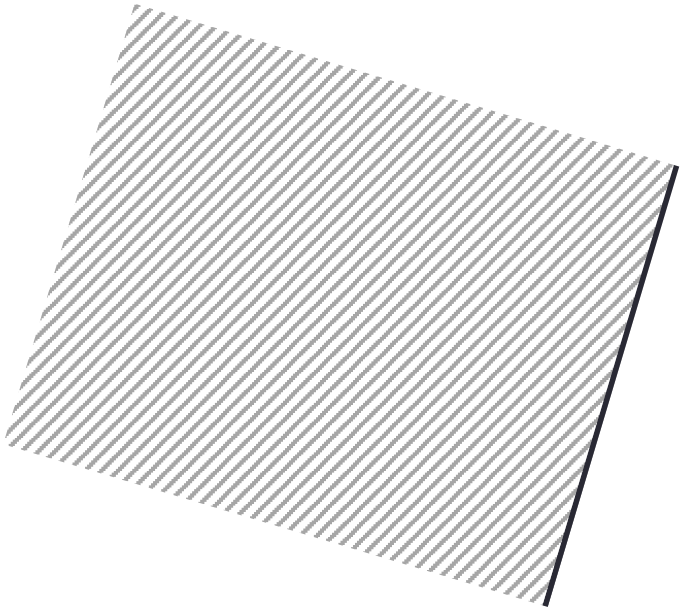
Convex

- Convex is "good":
  - Global minimum
  - Guarantees
  - Efficient algorithms (usually)

Standard convex conic programs

- Linear inequalities

$$c^T \mathrm{x} \leq d$$

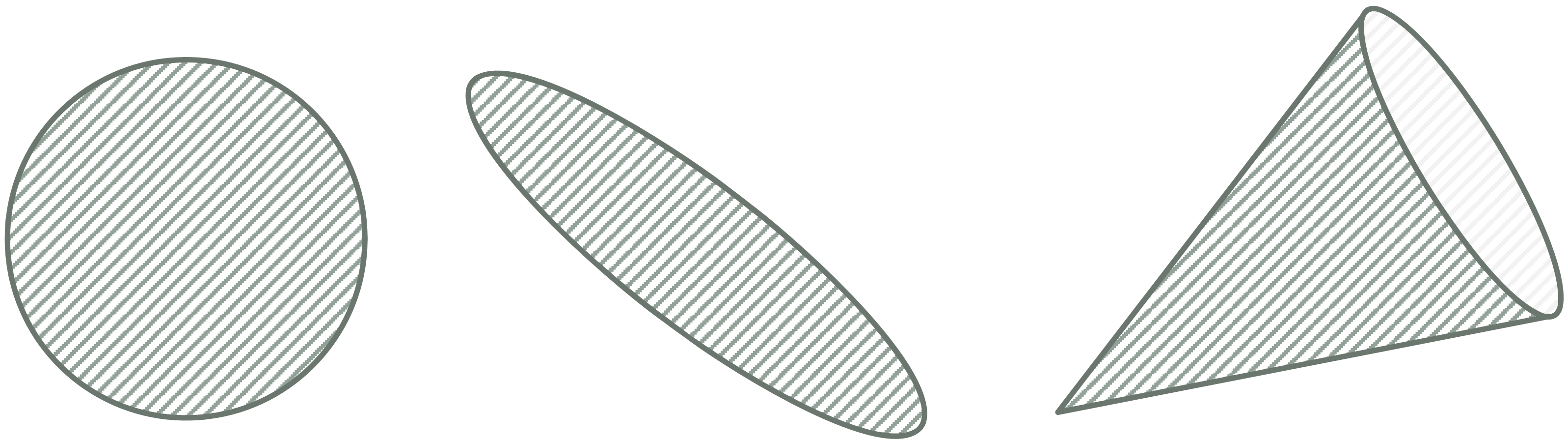


⇒ linear programming (LP)

Standard convex conic programs

- Second order (ice cream) cones

$$\|c^T x - d\|_2 \leq e^T x - f$$



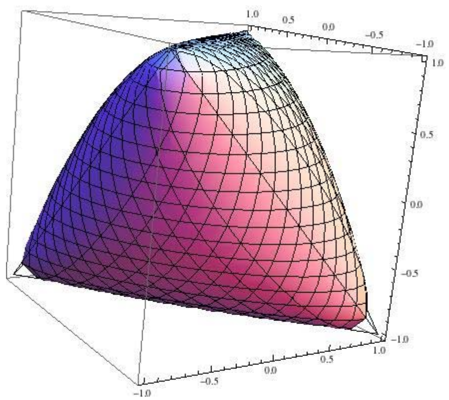
⇒ second order cone programming (SOCP)

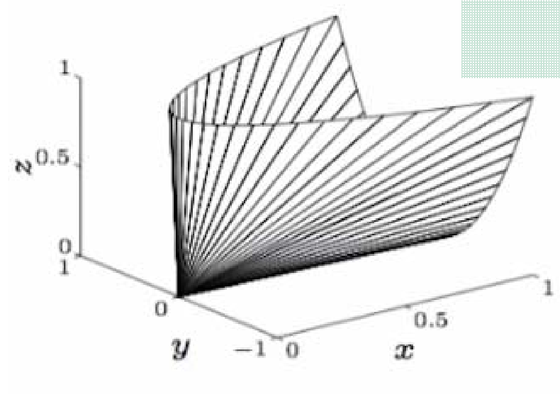
Standard convex conic programs

- Linear matrix inequalities (LMIs)

$$\operatorname{mat}(c^T \mathrm{x} - d) \succcurlyeq 0$$

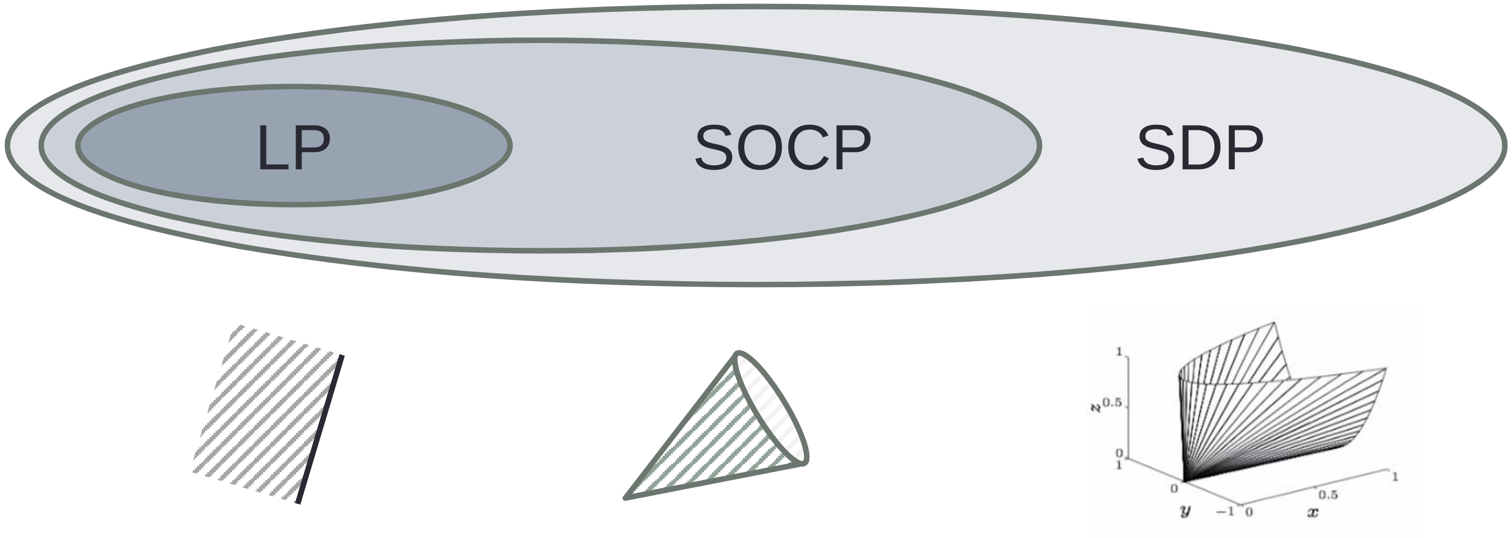
symmetric and
$\lambda_i \geq 0$





⇒ semidefinite programming (SDP)

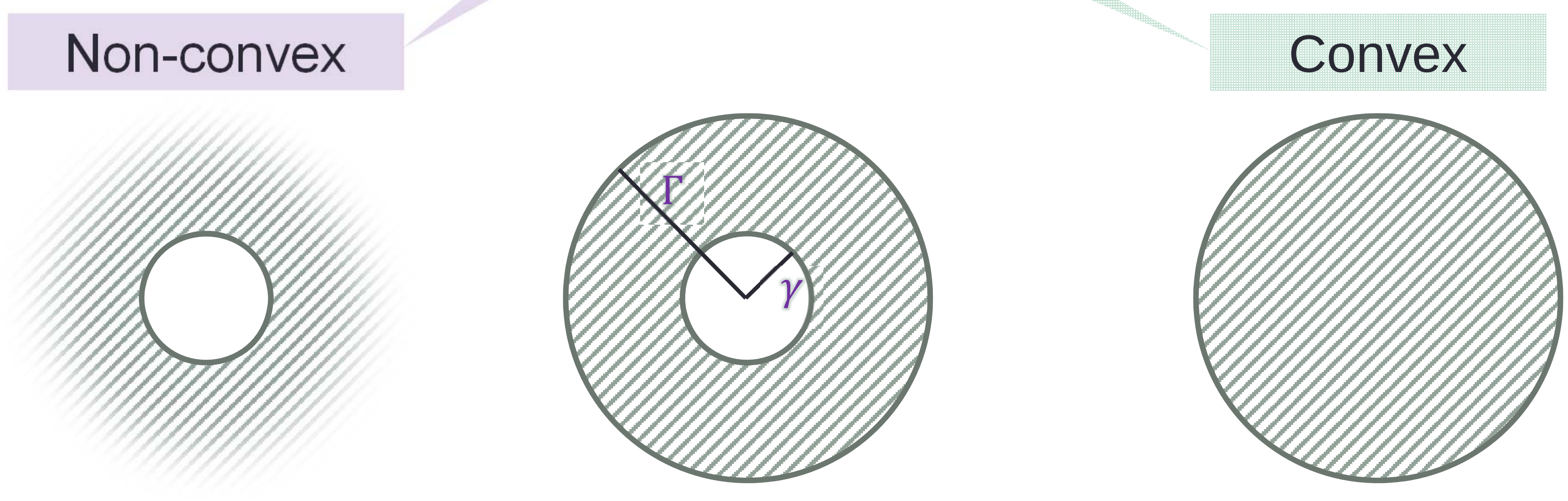
Hierarchy



Bounding SVs

- Simplest constraint

$$\gamma \leq \sigma_2 \leq \sigma_1 \leq \Gamma$$



[Lipman 2012]

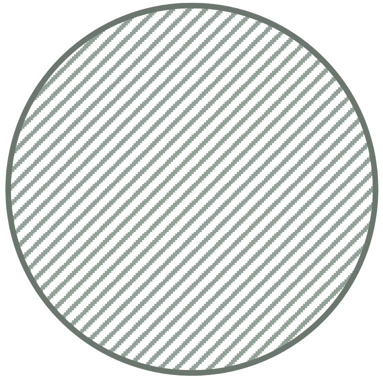
Upper bound

$$\sigma_{max} \leq \Gamma$$

Upper bound

$$\frac{1}{\sqrt{2}}\left(\|\mathcal{S}^{\perp}(A)\|_F + \|\mathcal{S}(A)\|_F\right) \leq \Gamma$$

- SOC <u>convex</u> constraint



Lower bound

$$\gamma \leq \sigma_{min}$$

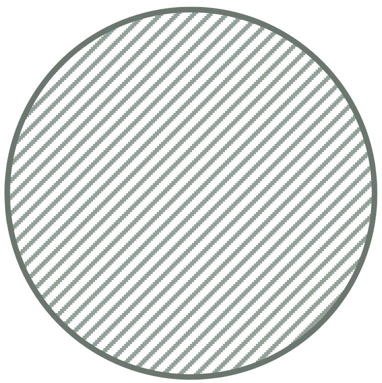
Lower bound

$$\gamma \leq \frac{1}{\sqrt{2}}\left(\|\mathcal{S}^{\perp}(A)\|_F - \|\mathcal{S}(A)\|_F\right)$$

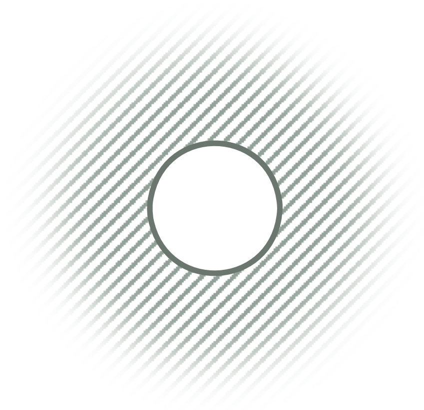
$$\|\mathcal{S}(A)\|_F + \sqrt{2}\gamma \leq \|\mathcal{S}^{\perp}(A)\|_F$$

$$\|\mathcal{S}(A)\|_F + \sqrt{2}\gamma \leq s \leq \|\mathcal{S}^{\perp}(A)\|_F$$

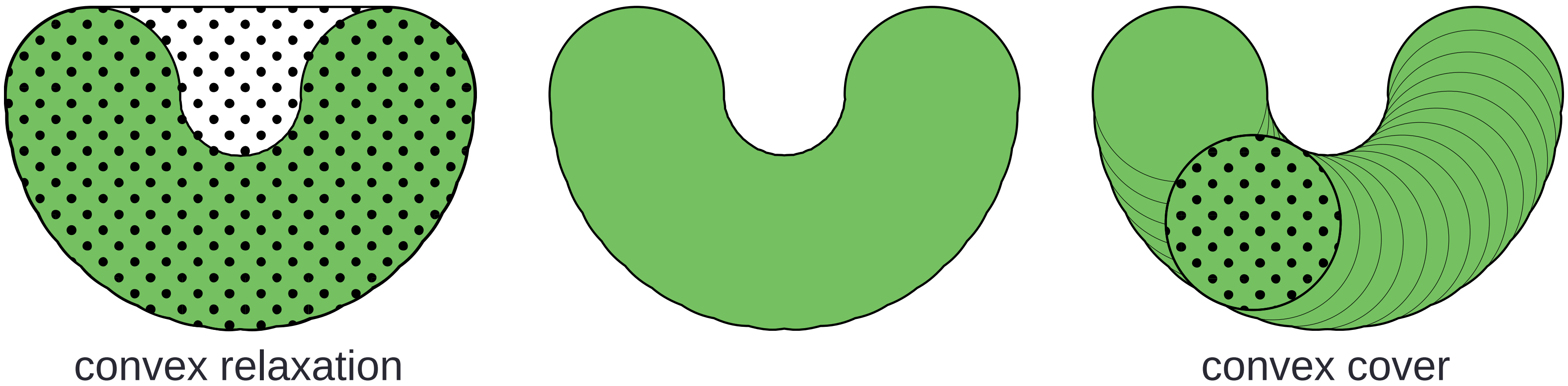
Convex
(norm ball)



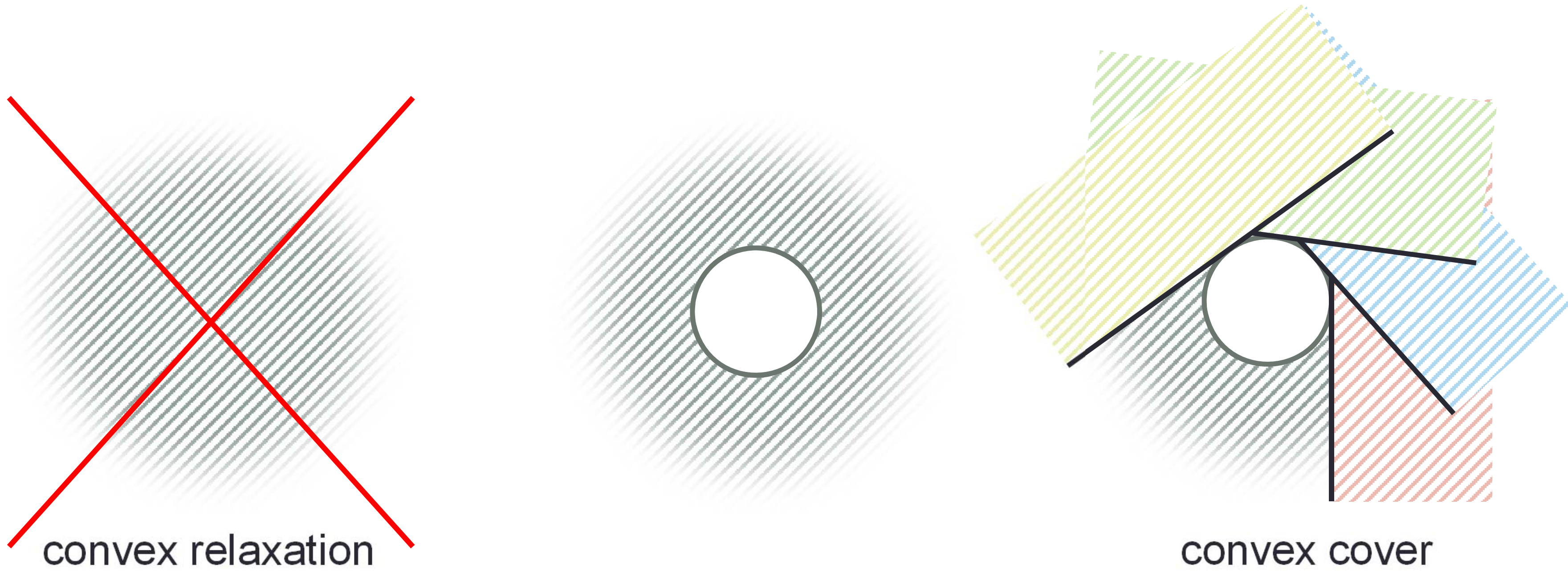
non-convex
(comp. of norm ball)



Convex relaxation vs. restriction



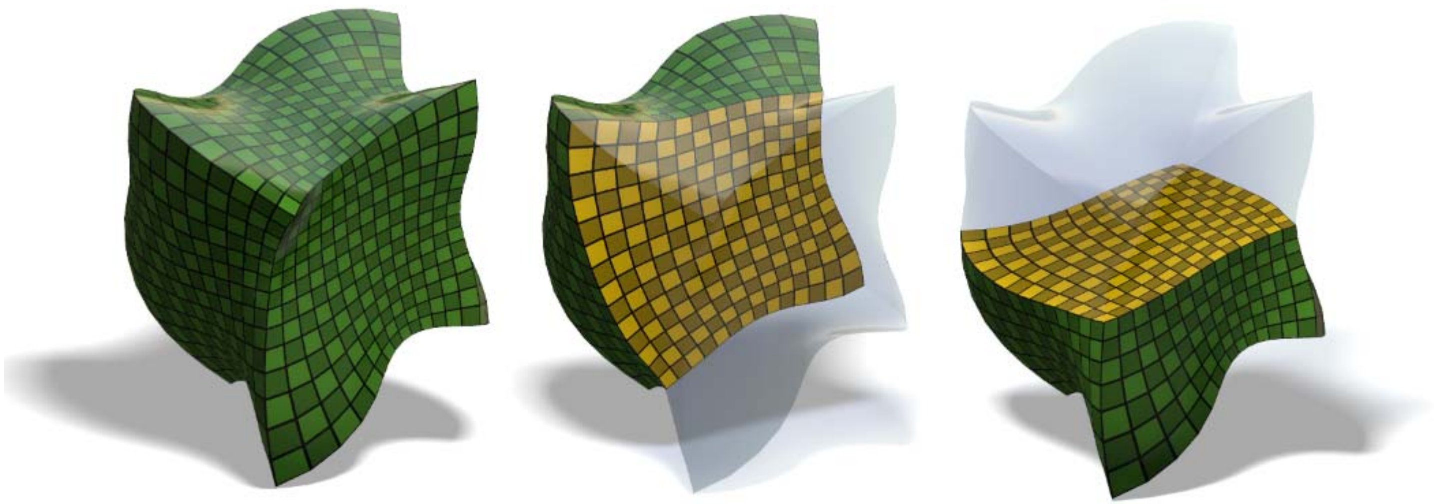
In our case



- Replace $s \leq \|\mathcal{S}^{\perp}(A)\|_F$ with a linear inequality

$$s \leq \langle \mathcal{S}^{\perp}(A), \boldsymbol{U} \rangle$$

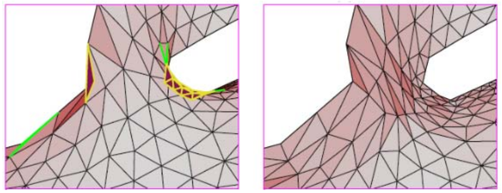
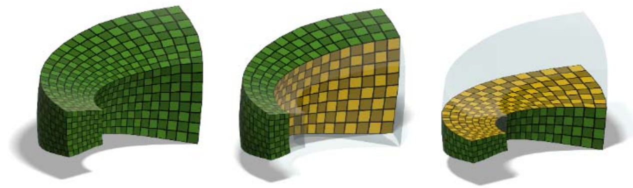
SV constraints (+energy)



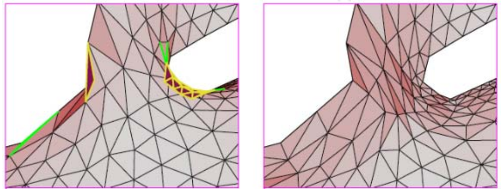
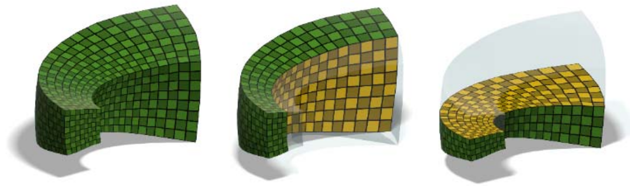
[Kovalsky et al. 2014]

Approximate via a sequence of convex programs

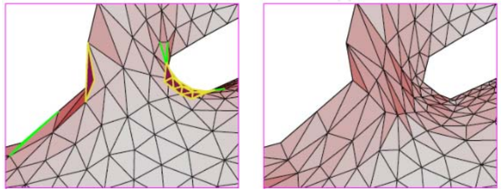
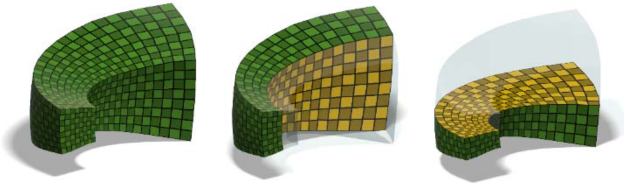
2d vs. 3d

2-d	3-d (and higher)
	

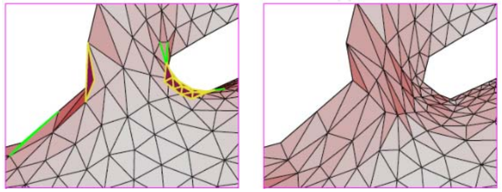
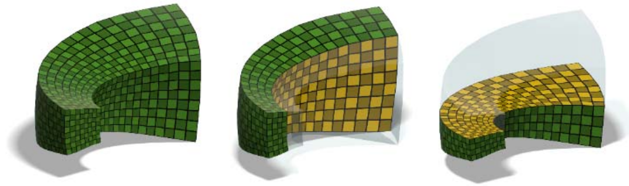
2d vs. 3d

2-d	3-d (and higher)
	
$\mathcal{S}(A)$ has a closed **linear** form	$\mathcal{S}(A)$ is non-linear

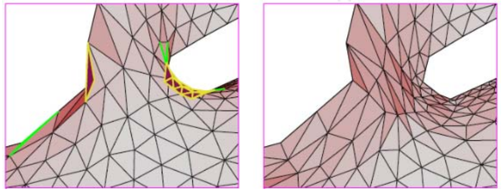
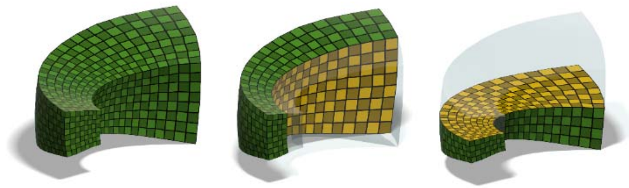
2d vs. 3d

2-d	3-d (and higher)
	
$\mathcal{S}(A)$ has a closed **linear** form	$\mathcal{S}(A)$ is non-linear
$R_1 R_2 = R_2 R_1$	$R_1 R_2 \neq R_2 R_1$

2d vs. 3d

2-d	3-d (and higher)
	
$\mathcal{S}(A)$ has a closed **linear** form	$\mathcal{S}(A)$ is non-linear
$R_1R_2 = R_2R_1$	$R_1R_2 \neq R_2R_1$
σ_1, σ_2 have a closed form	σ_i's = roots of $\geq 6^{th}$ degree polynomial

2d vs. 3d

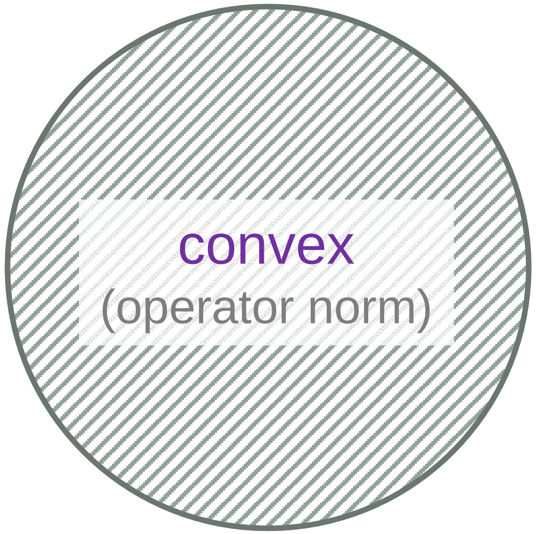
2-d	3-d (and higher)
	
$\mathcal{S}(A)$ has a closed **linear** form	$\mathcal{S}(A)$ is non-linear
$R_1 R_2 = R_2 R_1$	$R_1 R_2 \neq R_2 R_1$
σ_1, σ_2 have a closed form	σ_i's = roots of $\geq 6^{th}$ degree polynomial
SOCP	**SDP**

Bounding SVs

$$\boldsymbol{\gamma} \leq \sigma_3 \leq \sigma_2 \leq \sigma_1 \leq \boldsymbol{\Gamma}$$

Upper bound

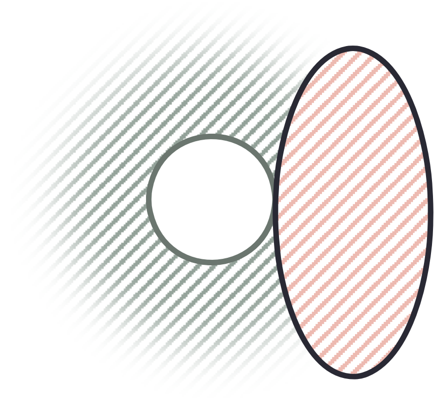
$\sigma_{\max}(A) \leq \Gamma$ $\Longleftrightarrow$ $\begin{pmatrix} \Gamma I & A \\ A^T & \Gamma I \end{pmatrix} \succcurlyeq 0$



Upper bound

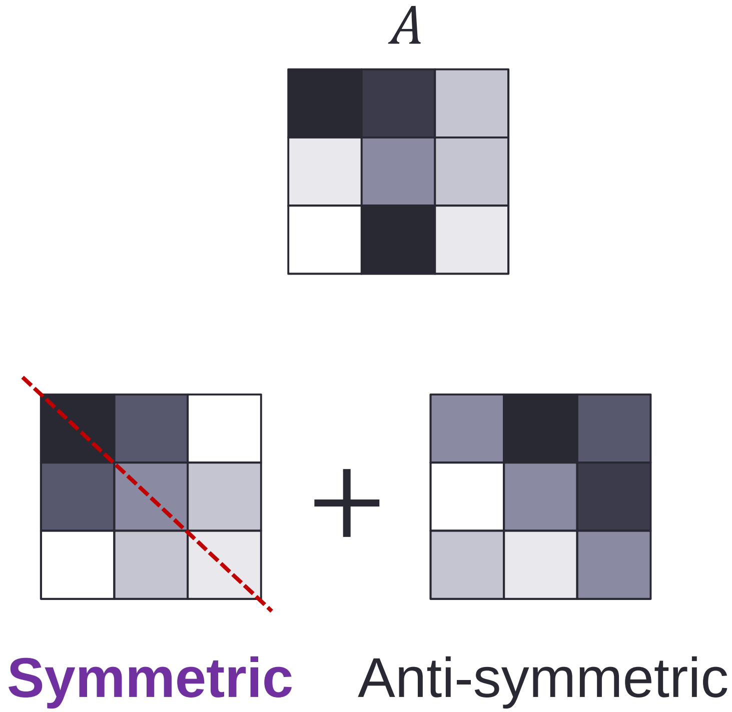
$$\gamma \leq \sigma_{\min}(A)$$

non-convex



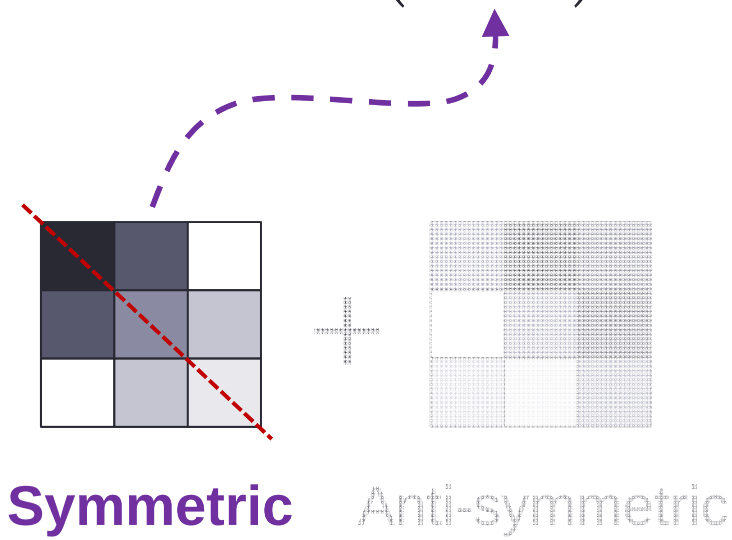
cut a convex subset?

Upper bound



Upper bound

$$\gamma \le \sigma_{\min}(A) \quad \Leftarrow \quad \gamma \le \sigma_{\min}\left(\frac{A + A^T}{2}\right)$$



Lower bound

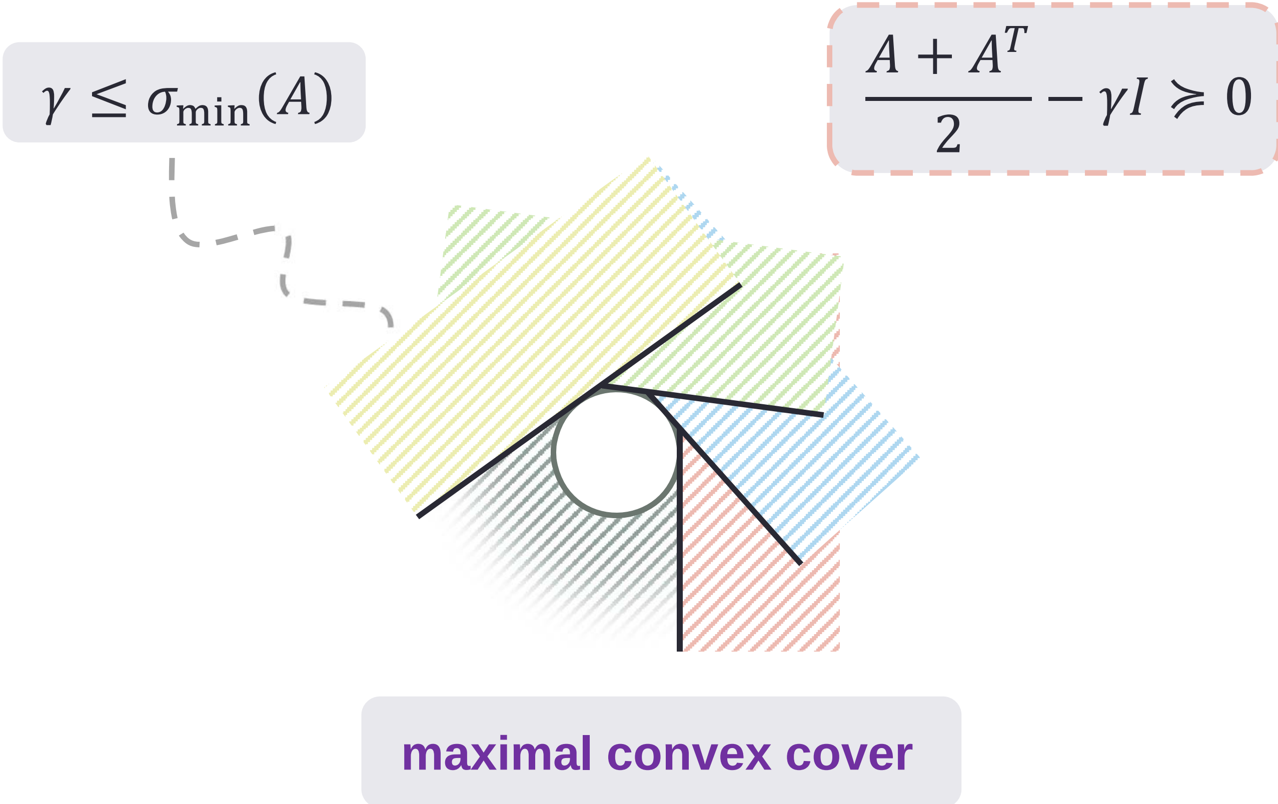
$$\gamma \leq \sigma_{\min}(A) \quad \Leftarrow \quad \gamma \leq \sigma_{\min}\left(\frac{A + A^T}{2}\right)$$

Lower bound

$\gamma \leq \sigma_{\min}(A)$ $\Longleftarrow$ $\dfrac{A + A^T}{2} - \gamma I \succcurlyeq 0$

maximal convex subset

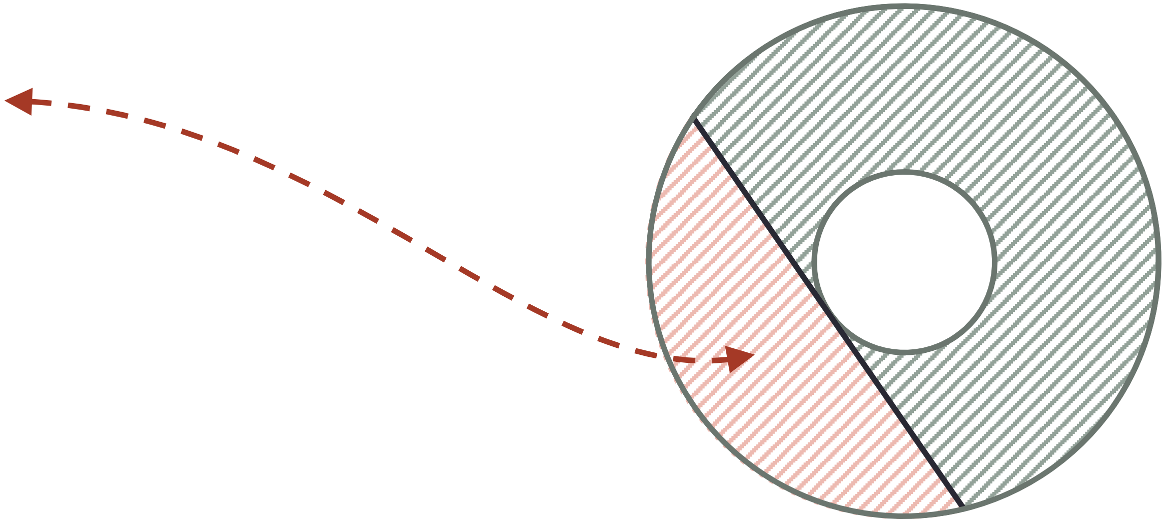
Maximal Convex Cover



Recap

2d – SOCP
n-d – SDP

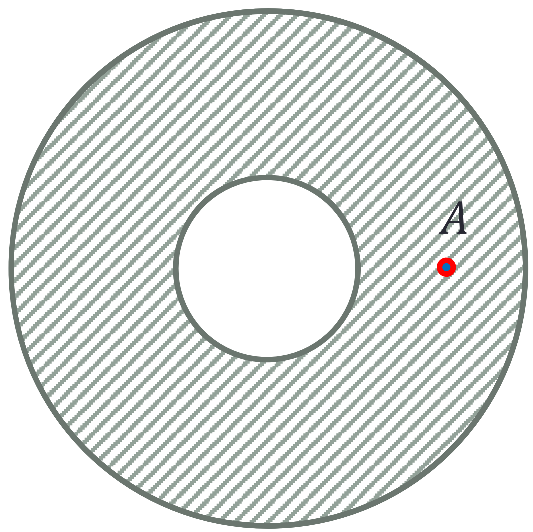
$\gamma \leq \sigma(A) \leq \Gamma$



Sufficient for more elaborate convex SV constraints + energy…

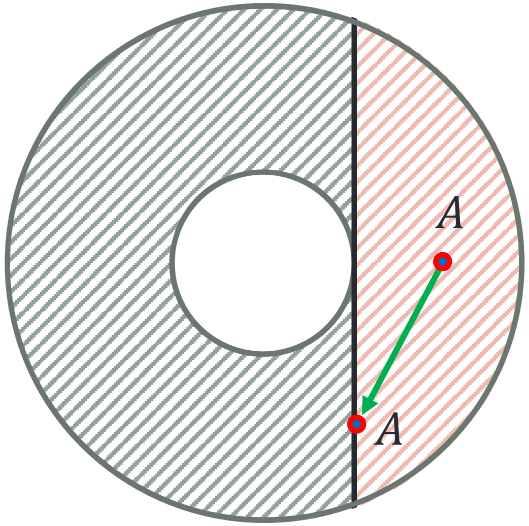
Optimization with SV constraints

- Start with feasible A



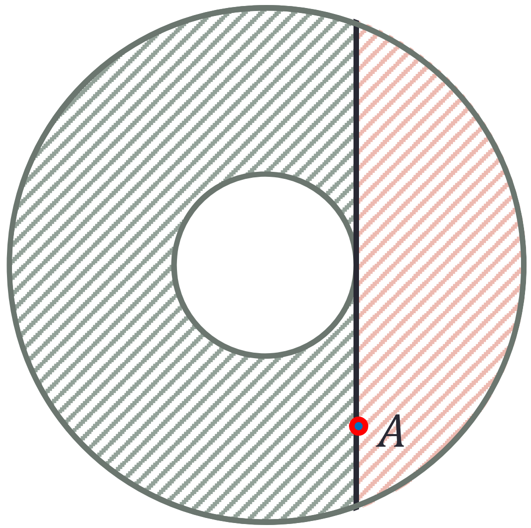
Optimization with SV constraints

- Start with feasible A
- Repeat:
  - Optimize over a maximal convex restriction (SDP)



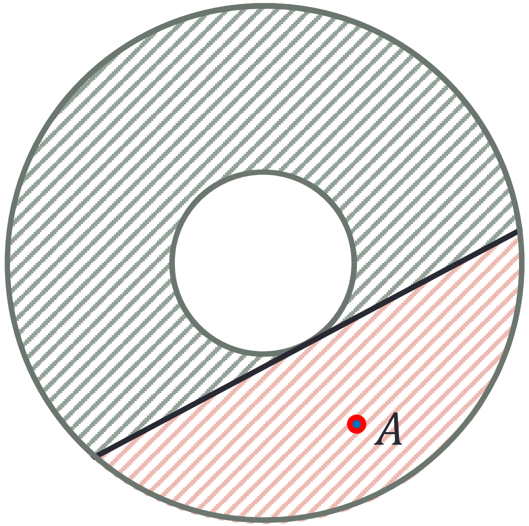
Optimization with SV constraints

- Start with feasible A
- Repeat:
  - Optimize over a maximal convex restriction (SDP)
  - Update rotation (centering)



Optimization with SV constraints

- Start with feasible A
- Repeat:
  - Optimize over a maximal convex restriction (SDP)
  - Update rotation (centering)



Optimization with SV constraints

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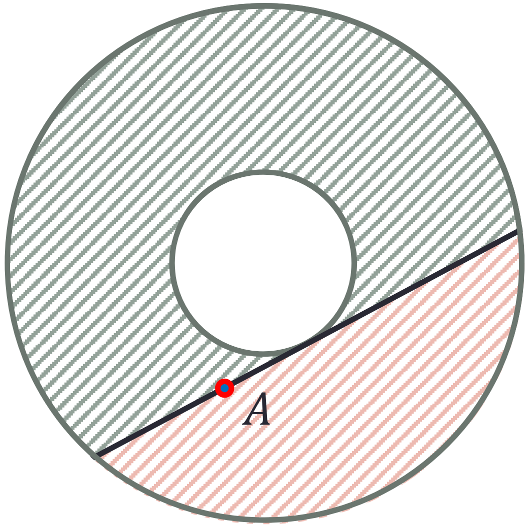
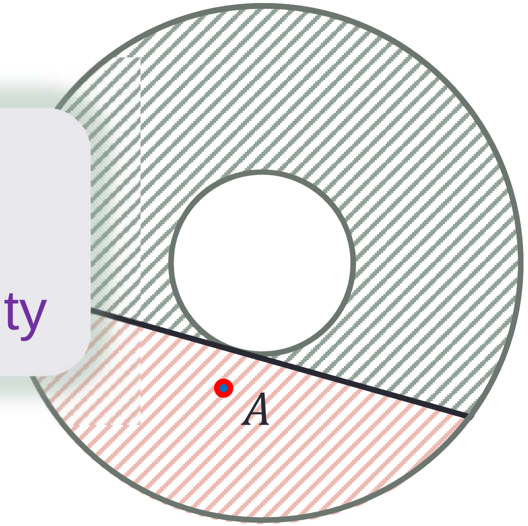


Optimization with SV constraints

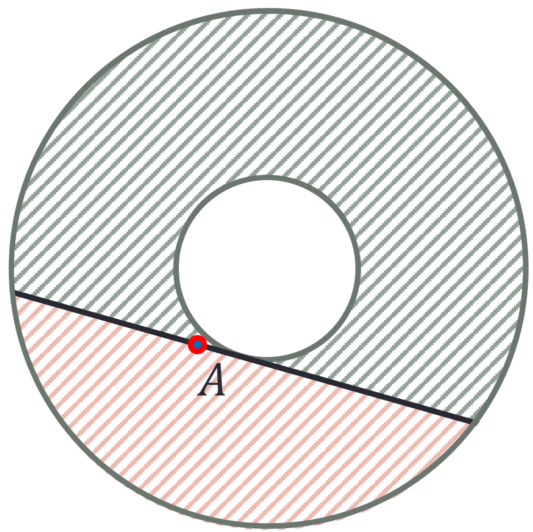
- Start with feasible A
- Repeat:
  - Optimize over a maximal convex restriction (SDP)
  - Update rotation (centering)



Every iteration
energy decreases
guaranteed feasibility

Optimization with SV constraints

- Start with feasible A
- Repeat:
  - Optimize over a maximal convex restriction (SDP)
  - Update rotation (centering)



Optimization with SV constraints

- Start with feasible A
- Repeat:
  - Optimize over a maximal convex restriction (SDP)
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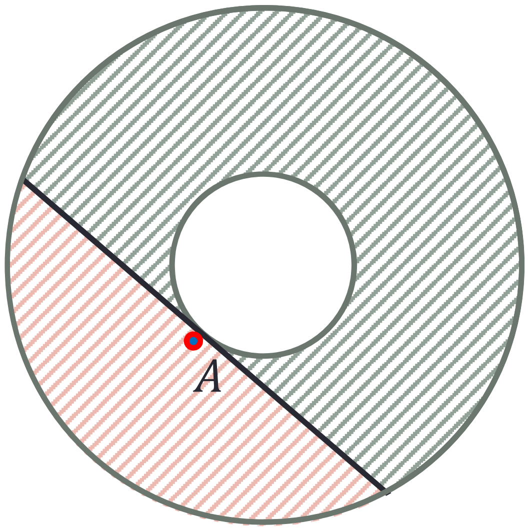

Optimization with SV constraints

- Start with feasible A
- Repeat:
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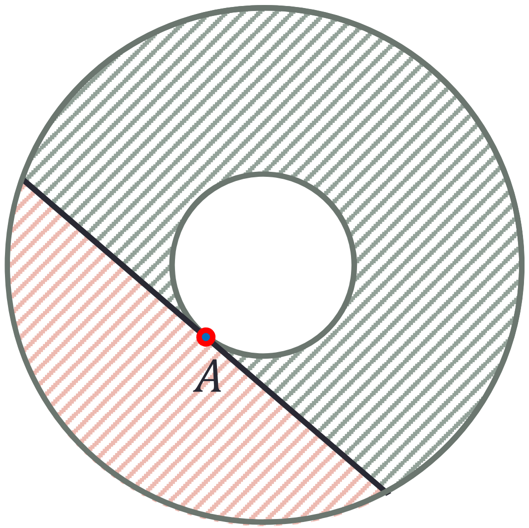
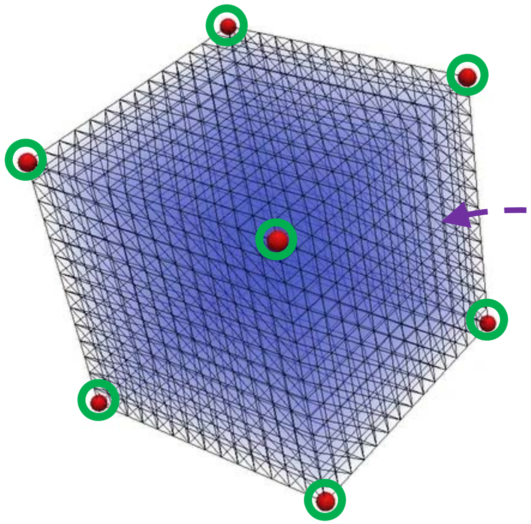
  - Update rotation (centering)

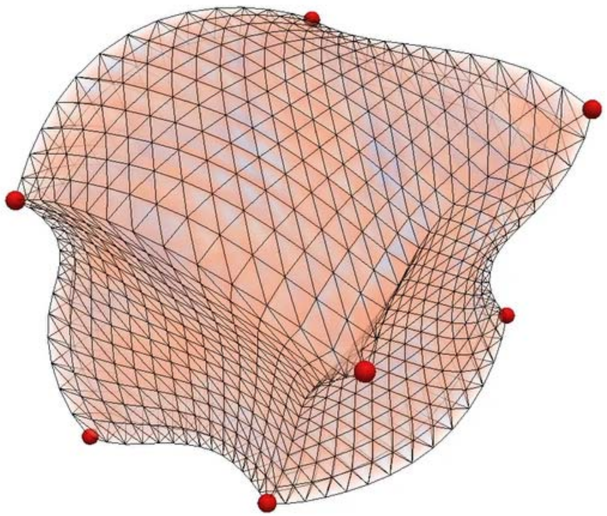


Extremal Quasiconformal Mappings



$$\textbf{minimize}\left(\max_i \frac{\sigma_{\max}(A_i)}{\sigma_{\min}(A_i)}\right)$$

Extremal Quasiconformal Mappings



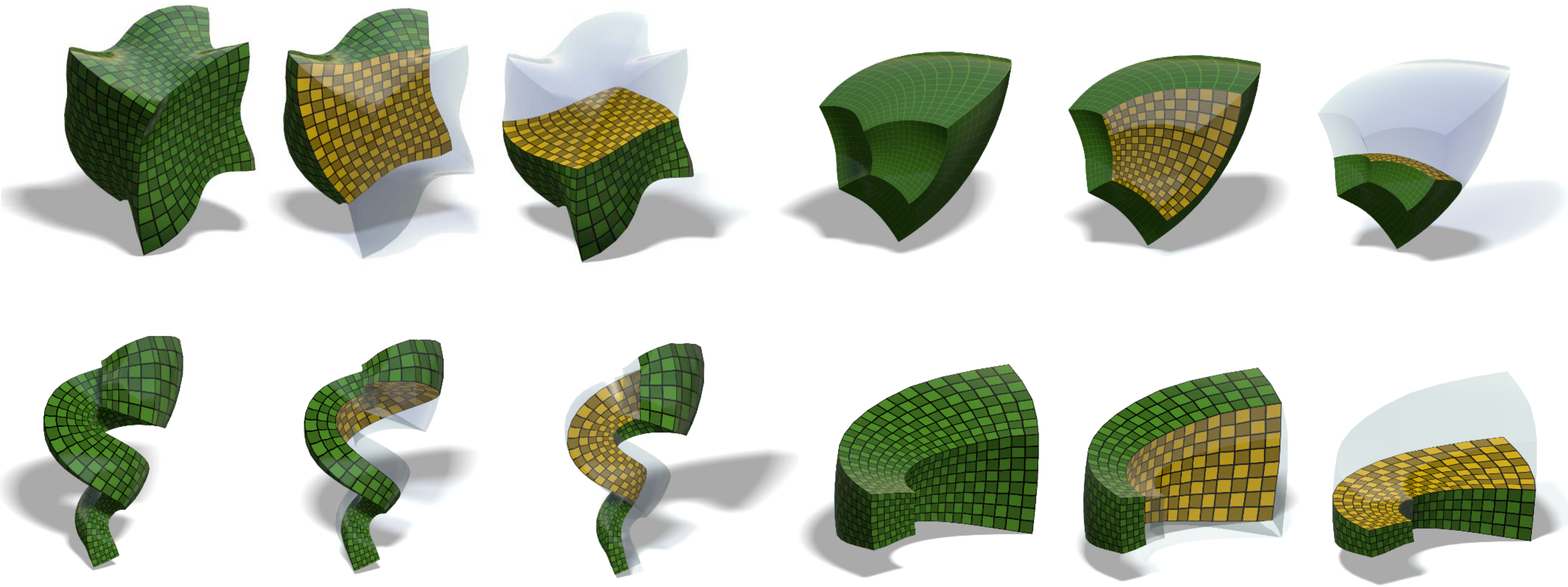
$$\textbf{minimize}\left(\max_i \frac{\sigma_{\max}(A_i)}{\sigma_{\min}(A_i)}\right)$$

"Most Conformal Mapping"

- Well studied in 2D [Weber et al. 2012]
- Little known in 3D…

histogram of distortions

Extremal Quasiconformal Mappings



Extremal Quasiconformal Mappings