SEMI-DISCRETE ISOTHERMIC SURFACES

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ABSTRACT. We study mappings of the form $x : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^3$ which can be seen as a limit case of purely discrete surfaces, or as a semi-discretization of smooth surfaces. In particular we discuss circular surfaces, isothermic surfaces, conformal mappings, and dualizability in the sense of Christoffel. We arrive at a semidiscrete version of Koenigs nets and show that in the setting of circular surfaces, isothermicity is the same as dualizability. We show that minimal surfaces constructed as a dual of a sphere have vanishing mean curvature in a certain well-defined sense, and we also give an incidence-geometric characterization of isothermic surfaces.

Contents

1. Introduction

An important topic in discrete differential geometry is the study of smooth surface parametrizations $g : \mathbb{R}^2 \to \mathbb{R}^3$ which arise as limits of discrete mappings (*nets*) of the form $x : \epsilon\mathbb{Z}^2 \to \mathbb{R}^3$, as $\epsilon \to 0$. This approach was initiated by R. Sauer, whose work is summarized in his textbook [Sau70]. For instance, the *conjugate* parametrizations g characterized by the condition

$$\{\partial_1 g, \partial_2 g, \partial_{12} g\} \text{ linearly dependent}$$

arise as limits of nets with planar faces (Q-nets). This planarity is equivalently expressed in terms of forward differences as

$$\{\Delta_1 x, \Delta_2 x, \Delta_{12} x\} \text{ linearly dependent.}$$

A systematic treatment of the d-dimensional case, and especially of the important concepts of consistency and integrability, is contained in the recent monograph

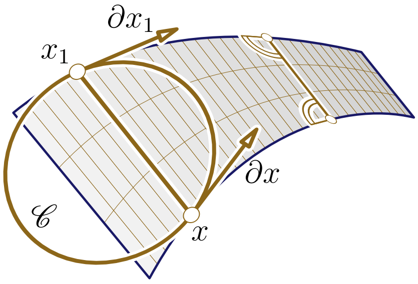

FIGURE 1: A circular semidiscrete surface $x : \{0,1\} \times \mathbb{R} \to \mathbb{R}^3$ with corresponding pairs of points and a circle $\mathscr{C}$. Assuming developability, equal angles characterize the circular property.

[BS09a]. One important point is that sequences of smooth surfaces (transformations of surfaces) can be seen as the limit of higher-dimensional nets where only some discrete parameters converge to continuous ones, while others remain discrete. The low-dimensional case of mappings $x : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^3$ has not received attention from the viewpoint of surface transformations, but has been employed in geometry processing [PSB$^+$08]: Such a semidiscrete surface is *conjugate*, if

$$\{\partial x, \Delta x, \partial \Delta x\} \text{ linearly dependent},$$

where ∂x is the derivative w.r.t. the continuous parameter, and

$$\Delta x = x_1 - x$$

is the forward difference w.r.t. the discrete parameter. The notation x_1 indicates an index shift: $x_1(k,t) = x(k+1,t)$.

Applications of such surfaces come from the following fact: The conjugacy condition immediately implies that the surface consisting of the line segments $x(k,t)x(k+1,t)$ is *developable*. A semidiscrete conjugate surface therefore can be seen as a union of developable strips.

Another interesting class of surfaces which occurs in all three categories is the *circular* ones: The smooth curvature line parametrizations correspond to the circular nets which are characterized by each face having a circumcircle. The semidiscrete version is defined as follows:

DEFINITION 1. *A semidiscrete surface x is circular, if for each corresponding pair of points x, x_1 there is a circle $\mathscr{C}$ passing through these points and being tangent to ∂x, ∂x_1 there (see Figure 1). We consider these circles as a mapping*

$$\mathscr{C} : \mathbb{Z} \times \mathbb{R} \to \{\textit{circles}\}.$$

As regards mappings between surfaces, their rather rigid combinatorics prevents semidiscrete surfaces from enjoying the rich variety of such mappings which their smooth counterparts have. However, they are not as restricted in this matter as their fully discrete colleagues. *Parameter transforms*, i.e., invertible self-mappings, are of the kind $x(k,t) \mapsto x(k_0 \pm k, \psi(t))$, for some local diffeomorphism ψ on the real line. Another kind of mapping which plays an important role in both the smooth and discrete categories is the topic of the next section.

Semidiscrete mappings already occur in the transformation theory of surfaces which was built mainly in the 19th century. In fact one could treat our semidiscrete surfaces as transformations of curves. Assuming this viewpoint, circular semidiscrete surfaces correspond to Ribaucour transformations.

2. Conformal mapping of semidiscrete surfaces

A mapping of smooth surfaces is conformal if tangent vectors retain their angles, and their lengths are multiplied by a factor which depends on the point only. In the semidiscrete case we first consider *circular* surfaces only, where we require only a suitable transformation rule for lengths of derivatives, disregarding the angle completely.

Definition 2. *The mapping $x(k,t) \mapsto x'(k,t)$ of circular semidiscrete surfaces is* conformal, *if there is a function $\nu : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}^+$, $\nu = e^u$, such that*

$$\begin{aligned} \|\partial x\| &= \nu \, \|\partial x'\| = e^u \, \|\partial x'\| \,, \\ \|\Delta x\| &= \sqrt{\nu\nu_1}\, \|\Delta x'\| = e^{\frac{u+u_1}{2}} \, \|\Delta x'\| \,. \end{aligned}$$

Example 1. It turns out that any Möbius transform μ induces a conformal mapping between surfaces: First observe that μ maps circles to circles, so for circular x, also $x' = \mu \circ x$ is circular. Further we have the property that there exists a positive function ρ such that $\|\mu(a) - \mu(b)\|^2 = \rho(a)\rho(b)\|a-b\|^2$, for all $a, b \in \mathbb{R}^3$ (see for instance [SSP08]). By differentiation we get $\|d_a\mu(v)\|^2 = \rho(a)^2\|v\|^2$, where v is a tangent vector attached to a. It follows that Def. 2 is fulfilled with $\nu = \rho\circ x$:

$$\|\Delta x'\|^2 = (\rho\circ x_1)(\rho\circ x)\|\Delta x\|^2, \quad \|\partial x'\|^2 = (\rho\circ x)^2\|\partial x\|^2.$$

It turns out that infinitesimally, Möbius transforms and conformal mappings are the same:

Lemma 3. *The mapping $x(k,t) \mapsto x'(k,t)$ of circular semidiscrete surfaces is conformal, if and only if each pair of corresponding circles $\mathscr{C}(k,t)$, $\mathscr{C}'(k,t)$ is mapped via a Möbius transformation. I.e., there is $\mu : \mathbb{Z}\times\mathbb{R} \to$ Möb such that for each (k,t),*

$$(\mathscr{C}, x, x_1, \partial x, \partial x_1) \overset{\mu, d\mu}{\longmapsto} (\mathscr{C}', x', x'_1, \partial x', \partial x'_1). \qquad (1)$$

Proof. There is a Möbius transform $\mu : \mathscr{C} \to \mathscr{C}'$ such that $(x, x_1, \partial x) \mapsto (x', x'_1, \partial x')$. This is an infinitesimal or limit version of the well known property that a Möbius transformation on a circle is uniquely determined by the image of three points. Using the terminology of Def. 2, factors $\nu = \frac{\|\partial x'\|}{\|\partial x\|}$ and $\sqrt{\nu\nu_1} = \frac{\|\Delta x'\|}{\|\Delta x\|}$ determine the factor $\nu_1 = \frac{\|\partial x'_1\|}{\|\partial x_1\|}$, so the length of $\partial x'_1$ is already determined by the mapping of $x, x_1, \partial x$. By Example 1, the same applies to the Möbius transform μ: We have $\nu_1 = \frac{\|d\mu(\partial x_1)\|}{\|\partial x_1\|}$, which implies $\partial x'_1 = \pm d\mu(\partial x_1)$. Since the two tangent vectors $\partial x, \partial x_1$ have opposite orientation on the circle $\mathscr{C}$, and the same holds true for $\mathscr{C}'$, we have finally shown that $\partial x_1' = d\mu(\partial x_1)$. The converse (Möbius $\Longrightarrow$ conformal) is obvious and follows directly from Example 1. ☐

3. Duality

Both the smooth and discrete categories exhibit the feature of Christoffel duality, which relates spheres with minimal surfaces. In the discrete category, the analytic property of possessing such a dual is equivalent to the incidence-geometric property of being a Koenigs net [BS09b]. We wish to find out how this theorem manifests itself in the semidiscrete category.

In the class of circular surfaces, the following equivalences emerge: The natural definition of Christoffel dual leads us to a class of *dualizable* surfaces. They coincide with those conformal to trivial surfaces (the *isothermic* surfaces), and they further turn out to be characterized by a semidiscrete version of an incidence-geometric condition which in the discrete case characterizes Koenigs (dualizable) nets. We start with the definition of duality:

Definition 4. *Conjugate semidiscrete surfaces x, x^* are dual to each other, if there is a function $\nu : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^+$ such that*

$$\partial x^* = -\frac{1}{\nu^2}\partial x, \quad \Delta x^* = \frac{1}{\nu\nu_1}\Delta x. \tag{2}$$

Duality obviously is an equivalence relation. The following is a semidiscrete analogue of the original defining equation of Koenigs nets:

Lemma 5. *A conjugate semidiscrete surface x possesses a dual if and only if there is $\nu : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^+$ such that*

$$\Delta\partial x = \frac{\nu_1}{\nu}\,\partial x - \frac{\nu}{\nu_1}\,\partial x_1 + \partial \log(\nu\nu_1)\Delta x. \tag{3}$$

Proof. The proof consists of expanding the 'Schwarz' compatibility condition $\Delta(\partial x^*) = \partial(\Delta x^*)$ referring to the expressions for Δx^*, ∂x^* given by (2):

$$\Delta\partial x^* = \frac{1}{\nu^2}\partial x - \frac{1}{\nu_1^2}\partial x_1, \quad \partial\Delta x^* = \frac{1}{\nu\nu_1}\partial\Delta x - \frac{\nu_1\partial\nu + \nu\partial\nu_1}{(\nu\nu_1)^2}\Delta x.$$

☐

Lemma 6. *Consider the plane spanned by vectors $\partial x, \partial x_1, \Delta x$. If x is dualizable, then the vectors $\partial x, \partial x_1$ lie to the same side of the straight line spanned by Δx.*

Proof. Vector product of (3) with Δx yields $(1+\frac{\nu}{\nu_1})\,\partial x_1 \times \Delta x = (1+\frac{\nu_1}{\nu})\,\partial x \times \Delta x$. This implies the following quotient of parallel vectors which we need later:

$$(\partial x_1 \times \Delta x) : (\partial x \times \Delta x) = \nu_1 : \nu. \tag{4}$$

Since $\nu, \nu_1 > 0$, ∂x and ∂x_1 lie to one side of Δx. ☐

Corollary 7. *For dualizable surfaces, all developable ruled surface strips bounded by curves x, x_1 are free from singularities.*

Proof. It is well known [PW01] that the singularities of the strip occur in the point r given by

$$r = x + \alpha \Delta x, \quad \text{with} \quad \alpha = -\frac{\partial x \times \Delta x}{\Delta \partial x \times \Delta x} = \Big(1 - \frac{\partial x_1 \times \Delta x}{\partial x \times \Delta x}\Big)^{-1}. \tag{5}$$

By Lemma 6, $\alpha \notin (0,1)$ and r lies outside the segment xx_1. □

Proposition 8. *The dual of a conjugate surface is unique up to translation and scaling. In particular the function ν required by Definition 4 is unique up to multiplication with a constant, if it exists.*

Proof. Assume that both ν and $\widetilde{\nu}$ fulfill (3). Subtracting these two equations from each other yields

$$0 = (\nu_1\widetilde{\nu} - \widetilde{\nu}_1\nu)(\nu_1\widetilde{\nu}_1\partial x + \nu\widetilde{\nu}\partial x_1) + \nu\nu_1\widetilde{\nu}\widetilde{\nu}_1\Big(\frac{\partial\nu}{\nu} + \frac{\partial\nu_1}{\nu_1} - \frac{\partial\widetilde{\nu}}{\widetilde{\nu}} - \frac{\partial\widetilde{\nu}_1}{\widetilde{\nu}_1}\Big)\Delta x. \tag{6}$$

Lemma 6 implies linear independence of $\{\nu_1\widetilde{\nu}_1\partial x + \nu\widetilde{\nu}\partial x_1, \Delta x\}$, so

$$\frac{\nu_1}{\nu} - \frac{\widetilde{\nu}_1}{\widetilde{\nu}} = 0 \quad \text{and} \quad \partial \log \frac{\nu\nu_1}{\widetilde{\nu}\widetilde{\nu}_1} = 0.$$

The first equation means that $\Delta(\nu/\widetilde{\nu}) = 0$. We differentiate it and observe $\nu/\nu_1 = \widetilde{\nu}/\widetilde{\nu}_1$ again, which yields

$$0 = \frac{\partial\nu_1}{\nu} - \frac{\nu_1\partial\nu}{\nu^2} - \frac{\partial\widetilde{\nu}_1}{\widetilde{\nu}} + \frac{\widetilde{\nu}_1\partial\widetilde{\nu}}{\widetilde{\nu}^2} = \frac{\nu_1}{\nu}\partial \log \frac{\nu_1\widetilde{\nu}}{\nu\widetilde{\nu}_1}.$$

Comparison of the two vanishing $\partial \log$ terms shows that $\partial \log(\nu/\widetilde{\nu}) = 0$. In total we have now shown that the quotient $\nu/\widetilde{\nu}$ is constant. □

4. Isothermic surfaces

In the smooth category, the notion 'isothermic surface' refers to a principal curvature line parametrization which is conformal. Equivalently we may require that any curvature line parametrization can be made conformal by a parameter transform of the special kind $(u, v) \mapsto (\psi_1(u), \psi_2(v))$. Our definition of isothermic semidiscrete surfaces is guided by this property.

Definition 9. *A circular semidiscrete surface x is isothermic, if some mapping $x(k,t) \mapsto (\beta_1(k), \beta_2(t))$ is conformal.*

Corollary 10. *For circular semidiscrete surfaces isothermicity is equivalent with the existence of scalar functions ν, σ, τ with*

$$\|\Delta x\|^2 = \sigma\nu\nu_1, \quad \|\partial x\|^2 = \tau\nu^2 \quad \textit{where } \partial\sigma = 0,\ \Delta\tau = 0 \tag{7}$$

(i.e., σ, τ depend on the discrete and the continuous variable only).

Proof. By definition of conformality, there exists $\nu : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^+$ with $\|\partial x\| = \nu|\partial\beta_2|$ and $\|\Delta x\| = \sqrt{\nu\nu_1}|\Delta\beta_1|$. Let $\tau = (\partial\beta_2)^2$ and $\sigma = (\Delta\beta_1)^2$. The converse is analogous: we define $\beta_1 = \sum \sqrt{\sigma}$ and $\beta_2 = \int \sqrt{\tau}$. □

We proceed to show that among circular surfaces, isothermicity and dualizability are equivalent.

THEOREM 11. *For circular surfaces, dualizability and isothermicity are equivalent. The natural mapping of x to its dual x^* is a conformal mapping.*

PROOF. We start by assuming dualizability of x and the existence of a suitable function ν. It is obvious that the mapping $x(k,t) \mapsto x^*(k,t)$ fulfills the definition of conformal mapping. We let

$$\sigma = \frac{\|\Delta x\|^2}{\nu\nu_1} \quad \text{and} \quad \tau = \frac{\|\partial x\|^2}{\nu^2}$$

and show $\partial\sigma = 0$, $\Delta\tau = 0$. This implies isothermicity, by Cor. 10.

From the circular property (see Figure 1) we see that there is a mirror reflection which transforms the circle $\mathscr{C}$ into itself and which exchanges the points and vectors

$$x \longleftrightarrow x_1, \quad [\partial x]_0 \longleftrightarrow [\partial x_1]_0, \quad \text{where } [v]_0 = \frac{v}{\|v\|}.$$

From this symmetry we conclude the following elementary geometric relation between normalized derivatives in points x and x_1:

$$\langle [\partial x]_0 + [\partial x_1]_0, \Delta x\rangle = 0. \tag{8}$$

We consider Equation (3) which characterizes dualizability, and multiply it with $[\partial x]_0 + [\partial x_1]_0$. Using the angle $\gamma = \cos\sphericalangle(\partial x, \partial x_1)$ which is different from π, this yields in succession

$$\begin{aligned}
&\Big(\frac{\langle\partial x, \partial x_1\rangle}{\|\partial x_1\|} + \|\partial x\|\Big)(\nu_1^2 + \nu\nu_1) = \Big(\frac{\langle\partial x, \partial x_1\rangle}{\|\partial x\|} + \|\partial x_1\|\Big)(\nu^2 + \nu\nu_1)\\
\iff\ & \|\partial x\|(1+\cos\gamma)\nu_1(\nu_1+\nu) = \|\partial x_1\|(1+\cos\gamma)\nu(\nu+\nu_1)\\
\iff\ & \frac{\|\partial x\|}{\nu} = \frac{\|\partial x_1\|}{\nu_1} \iff \Delta\tau = 0. \qquad (9)
\end{aligned}$$

We employ Equations (8) and (9) to compute

$$\begin{aligned}
&\langle\Delta\partial x, \Delta x\rangle = \langle\|\partial x_1\|\,[\partial x_1]_0, \Delta x\rangle - \langle\|\partial x\|\,[\partial x]_0, \Delta x\rangle\\
&= \frac{\nu_1}{\nu}\|\partial x\|\langle[\partial x_1]_0, \Delta x\rangle - \frac{\nu}{\nu_1}\|\partial x_1\|\langle[\partial x]_0, \Delta x\rangle\\
&= -\frac{\nu_1}{\nu}\|\partial x\|\langle[\partial x]_0, \Delta x\rangle + \frac{\nu}{\nu_1}\|\partial x_1\|\langle[\partial x_1]_0, \Delta x\rangle = \langle\frac{\nu}{\nu_1}\partial x_1 - \frac{\nu_1}{\nu}\partial x, \Delta x\rangle.
\end{aligned}$$

We multiply (3) with Δx and plug in the expression above. We get

$$0 = 2\langle\Delta\partial x, \Delta x\rangle - \Big(\frac{\partial\nu}{\nu} + \frac{\partial\nu_1}{\nu_1}\Big)\|\Delta x\|^2.$$

This equals $\nu\nu_1\partial\sigma$ by construction, so we have $\partial\sigma = 0$ and the proof of "dualizability $\implies$ isothermicity" is complete.

To show the converse, we assume that x is isothermic. We are going to show that the function ν required by Definition 4 is the one of (7). We use the functions σ, τ introduced in Cor. 10 and rewrite the derivatives of the dual x^* in the form

$$\partial x^* = -\frac{1}{\nu^2}\partial x = -\tau\frac{\partial x}{\|\partial x\|^2}, \quad \Delta x^* = \frac{1}{\nu\nu_1}\Delta x = \sigma\frac{\Delta x}{\|\Delta x\|^2} \tag{10}$$

and check consistency. Using (8),

$$\begin{aligned}\partial\Delta x^* &= \frac{\sigma}{\|\Delta x\|^4}\Big(\|\Delta x\|^2\partial\Delta x - 2\langle\Delta x, \partial\Delta x\rangle\Delta x\Big)\\ &= \frac{\sigma}{\|\Delta x\|^4}\Big(\|\Delta x\|^2\partial\Delta x - 2\Delta x\langle\Delta x, \partial x\rangle(-\frac{\|\partial x_1\|}{\|\partial x\|} - 1)\Big).\end{aligned}$$

The reflection mentioned above not only implies (8), but also causes Δx to be parallel to the difference of normalized tangent vectors:

$$\Delta x = \lambda\Delta\left[\partial x\right]_0, \text{ with } \lambda = -\frac{\|\Delta x\|}{2\langle[\partial x]_0, [\Delta x]_0\rangle}.$$

We substitute one occurrence of Δx in our expression for $\partial\Delta x^*$ and get

$$\begin{aligned}\partial\Delta x^* &= \frac{\sigma}{\|\Delta x\|^2}\Big(\Delta\partial x - \Delta\left[\partial x\right]_0(\|\partial x_1\| + \|\partial x\|)\Big)\\ &= \frac{\sigma}{\|\Delta x\|^2}(\|\partial x_1\|\left[\partial x\right]_0 - \|\partial x\|\left[\partial x_1\right]_0) \overset{(7)}{=} \frac{\partial x}{\nu^2} - \frac{\partial x_1}{\nu_1^2} = \Delta\partial x^*.\end{aligned}$$

Here we have used the expressions of (7). This establishes the integrability condition and thus existence of x^*. The surface x^* is circular because corresponding vectors ∂x, ∂x^* are parallel, and so are $\Delta x, \Delta x^*$. □

Remark 12. The formulae in this proof are similar of those concerning the Ribaucour transform found e.g. in [BS09a, p. 18]. We refrain from a systematic discussion.

5. Semidiscrete minimal surfaces

In imitation of the respective well known property of smooth minimal surfaces we define:

Definition 13. *If $x : \mathbb{Z}\times\mathbb{R} \to S^2$ is a semidiscrete isothermic surface inscribed in the unit sphere, then its Christoffel dual x^* is called a minimal surface.*

An example is shown by Figure 2. It turns out that the semidiscrete analogue of the discrete curvature theory of [BPW10] yields vanishing mean curvature for the semidiscrete minimal surfaces defined here: If we consider

$$a\,dt := \frac{1}{2}\det(\partial x + \partial x_1, \Delta x)\,dt$$

as the area of of the infinitesimal face $x(k,t)$, $x_1(k,t)$, $x_1(k,t+dt)$, $x(k,t+dt)$, then the mixed area of that face and the corresponding face in x^* has to be

$$A\,dt = \frac{1}{4}\big(\det(\partial x + \partial x_1, \Delta x^*) + \det(\partial x^* + \partial x_1^*, \Delta x)\big)\,dt.$$

Here the determinant is to be taken w.r.t. some consistently oriented normal vector. According to [BPW10], the mean curvature of x w.r.t. the Gauss image x^* has to defined by letting

$$H = -A/a.$$

When computing the mean curvature of x^* with respect to x as a Gauss image, we take an appropriately modified a^* for the denominator. We get the following result:

Proposition 14. *If x and x^* is a dual pair of isothermic surfaces, where x is considered as the Gauss image of x^*, then the mean curvature of x^* vanishes.*

Proof. We verify that the infinitesimal mixed area A as defined above vanishes. We first express the norms of vectors in terms of the functions ν, σ, τ of (7):

$$\|\Delta x^*\|^2 = \frac{\sigma}{\nu\nu_1}, \quad \|\partial x^*\|^2 = \frac{\tau}{\nu^2}.$$

We employ the angle $\gamma = \sphericalangle(\Delta x, \partial x) = \pi - \sphericalangle(\Delta x, \partial x_1)$ and get

$$4A = \sin\gamma\Big(-\big(\tau\nu^2\frac{\sigma}{\nu\nu_1}\big)^{1/2} - \big(\tau\nu_1^2\frac{\sigma}{\nu\nu_1}\big)^{1/2} + \big(\frac{\tau}{\nu^2}\sigma\nu\nu_1\big)^{1/2} + \big(\frac{\tau}{\nu_1^2}\sigma\nu\nu_1\big)^{1/2}\Big) = 0.$$

Here we have used that $\tau_1 = \tau$. $\square$

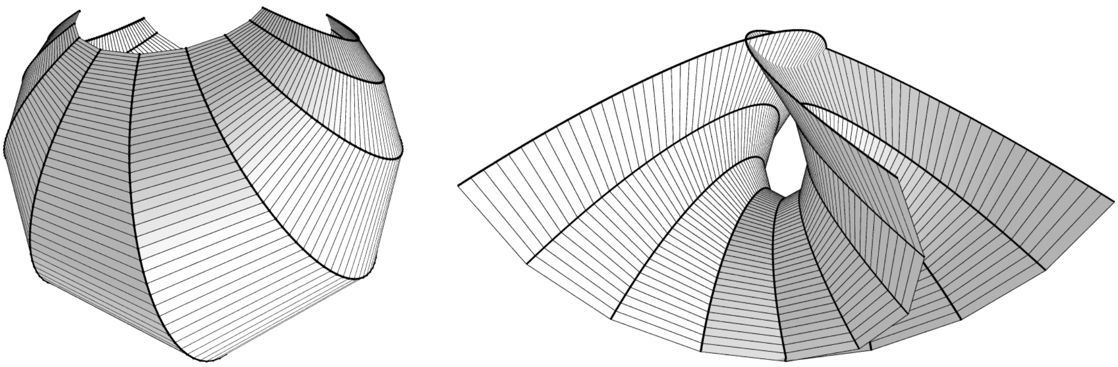

Figure 2: *Left:* A semidiscrete isothermic surface x inscribed in the unit sphere. *Right:* its Christoffel dual x^*, which is a semidiscrete minimal surface.

6. Incidence-geometric characterizations of Koenigs surfaces

In the category of discrete 'conjugate' nets $x : \mathbb{Z}^2 \to \mathbb{R}^3$ with planar faces, dualizability is characterized by the condition that the intersection points of diagonals

$$h = (x \vee x_{12}) \cap (x_1 \vee x_2).$$

itself constitute a conjugate net [BS09b]. The notation employed here is that the index i indicates a shift in the i-th integer parameter of the discrete surface in question.

The following semidiscrete version is due to H. Pottmann: We visualize a semidiscrete surface as the limit of a discrete surface whose elementary quadrilaterals become thinner and thinner — see Figure 3. The intersection of infinitesimally neighbouring rulings is known: It is the regression point

$$\Big(x \vee x_1\Big) \cap \Big((x + dt\,\partial x) \vee (x + dt\,\partial x_1)\Big) = r,$$

where r is given by Equation (5) (see Figure 4). The intersection point of the diagonals then converges to a point h such that x, x_1, h, r form a harmonic quadruple. This leads to the following definition of a semidiscrete Koenigs net:

For a conjugate semidiscrete surface x consider the regression points r and the surface $h : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^3$ defined by the cross-ratio condition

$$\mathrm{cr}(x, x_1, h, r) = -1. \tag{11}$$

Then x is said to be Koenigs $\iff$ h is conjugate. This condition does not make sense if x maps into a plane. In order to incorporate this case also, we give a slightly different definition:

Definition 15. *The conjugate semidiscrete surface x has property H $\iff$ the tangents*

$$x + \mathbb{R}\partial x, \quad h + \mathbb{R}\partial h, \quad h_{\bar{1}} + \mathbb{R}\partial h_{\bar{1}},$$

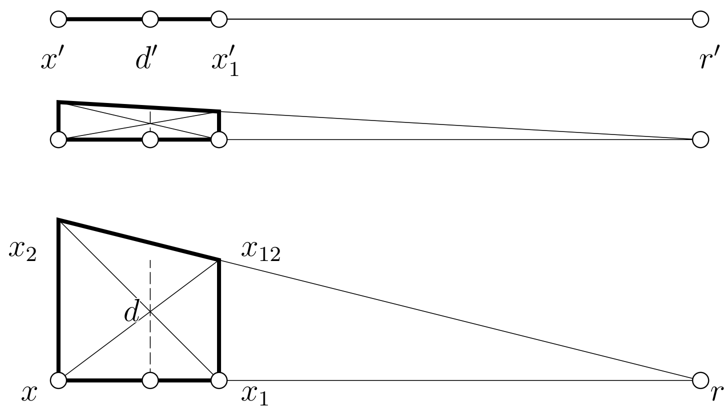

Figure 3: Behaviour of intersection points as quadrilaterals degenerate. Points r, d approach limits r', d' such that x, x_1, d, r constitute a harmonic quadruple. In this visualization of a semidiscrete surface x' as limit of a discrete surface x, the point r converges to the regression point of the ruling $x' \vee x'_1$.

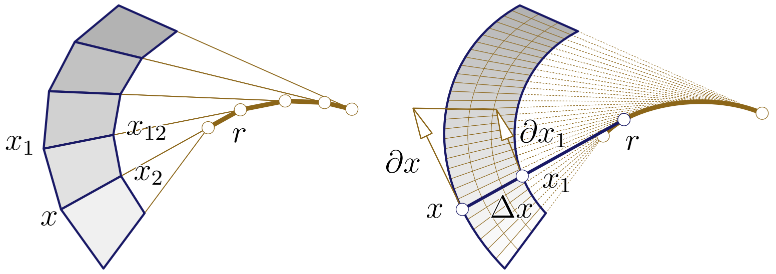

FIGURE 4: *Left:* A conjugate net $x : \mathbb{Z} \times \{0,1\} \to \mathbb{R}^3$ with regression points r. *Right:* A semidiscrete conjugate surface $x : \{0,1\} \times \mathbb{R} \to \mathbb{R}^3$ with regression curve r.

lie in a pencil (i.e., intersect or are parallel). Here h is defined by (11) *as the harmonic conjugate of the regression curves.*

THEOREM 16. *Every dualizable (conjugate semidiscrete) surface x has property H.*

PROOF. The regression curve of Equation (5) has the form $r = x + \alpha\Delta x$. Since $\operatorname{cr}(0, 1, \alpha, \frac{\alpha}{2\alpha-1}) = -1$, the harmonic conjugate h is given by $x + \frac{\alpha}{2\alpha-1}\Delta x$. The surface x is dualizable, so we may use (4) when computing α:

$$r = x + \frac{\nu}{\nu - \nu_1}\Delta x, \quad h = x + \frac{\nu}{\nu + \nu_1}\Delta x. \tag{12}$$

Equation (3), which reads $(\nu\nu_1 + \nu^2)\partial x_1 - \nu\partial\nu_1\Delta x = (\nu\nu_1 + \nu_1^2)\partial x + \nu_1\partial\nu\Delta x$ holds. We use this in our computation of the derivative ∂h:

$$\begin{aligned} \partial h &= \partial x + \frac{\nu_1\partial\nu - \nu\partial\nu_1}{(\nu + \nu_1)^2}\Delta x + \frac{\nu}{\nu + \nu_1}(\partial x_1 - \partial x) \\ &= \frac{1}{(\nu + \nu_1)^2}\Big((\nu\nu_1 + \nu_1^2)\partial x + (\nu_1\partial\nu - \nu\partial\nu_1)\Delta x + (\nu^2 + \nu\nu_1)\partial x_1\Big) \\ &\overset{(3)}{=} \frac{2\nu_1}{(\nu + \nu_1)^2}\Big((\nu + \nu_1)\partial x + \partial\nu\Delta x\Big). \end{aligned} \tag{13}$$

The tangents of x and h intersect in the common point

$$(x + \mathbb{R}\partial x) \cap (h + \mathbb{R}\partial h) = x - \frac{\nu}{\partial\nu}\partial x,$$

since that expression also equals $h - \frac{\nu(\nu+\nu_1)}{2\nu_1\partial\nu}\partial h$. The important fact here is that the intersection point of tangents does not involve x_1 but only x. The parameter transform $x(k, t) \to x(-k, t)$ causes the changes $\nu(k, t) \to \nu(-k, t)$ and $h(k, t) \to h(-1 - k, t)$, so tangents of $h_{\bar{1}}, x$ intersect in the same point $x - \frac{\nu}{\partial\nu}\partial x$. This establishes property H. □

THEOREM 17. *Any circular surface with property H can be re-parametrized to become dualizable (i.e., isothermic).*

PROOF. We assume that tangents of the curves $x, h, h_{\bar{1}}$ intersect in a common point $x + \lambda\partial x$. We are going to show isothermicity. For that purpose we define

$$\nu = \|\partial x\|.$$

The surface x is circular, and so the angle condition (8) implies that the mirror reflection defined by $\Delta x \mapsto -\Delta x$ exchanges $\nu_1\partial$ with $\nu\partial x_1$ (these two vectors have equal length by construction). We use this information to simplify the expression for ∂h given by (13):

$$\nu\partial x_1 + \nu_1\partial x = 2\nu_1\partial x - 2\langle \nu_1\partial x, [\Delta x]_0\rangle\, [\Delta x]_0$$
$$\Longrightarrow \partial h = \frac{\nu_1\partial\nu - \nu\partial\nu_1}{(\nu+\nu_1)^2}\Delta x + \frac{1}{\nu+\nu_1}(\nu\partial x_1 + \nu_1\partial x) = a\Delta x + \frac{2\nu_1}{\nu+\nu_1}\partial x. \tag{14}$$

By property H and the definition of λ, vectors ∂h and $x+\lambda\partial x - h$ are proportional. This immediately implies the ratios

$$a\Delta x : (x-h) = \frac{2\nu_1}{\nu+\nu_1} : \lambda \tag{15}$$

We insert the known expression for a and get the expression for λ given below. Re-indexing yields an analogous expression for λ_1, where $x_1 + \lambda_1\partial x_1$ is the point where tangents to x_1, h, h_1 meet:

$$\frac{2}{\lambda} = \frac{\partial\nu_1}{\nu_1} - \frac{\partial\nu}{\nu} + \frac{2(\nu+\nu_1)\langle\nu_1\partial x, \Delta x\rangle}{\nu\nu_1\|\Delta x\|^2},$$
$$\frac{2}{\lambda_1} = \frac{\partial\nu}{\nu} - \frac{\partial\nu_1}{\nu_1} - \frac{2(\nu+\nu_1)\langle\nu\partial x_1, \Delta x\rangle}{\nu\nu_1\|\Delta x\|^2}.$$

The mirror reflection employed above yields equality of the scalar products which occur in these two formulas. By taking differences we obtain

$$\frac{1}{\lambda} - \frac{1}{\lambda_1} = \frac{\partial\nu_1}{\nu_1} - \frac{\partial\nu}{\nu} \quad \text{i.e.,} \ -\Delta\frac{1}{\lambda} = \Delta\partial\log\nu. \tag{16}$$

A parameter transform $x' = x\circ\psi$ with $\psi(k,t) = (k, \psi_2(t))$ changes ν, λ via $\nu' = \partial\psi_2(\nu\circ\psi)$ and $\lambda' = \frac{1}{\partial\psi_2}(\lambda\circ\psi)$. It follows that $(\lambda\nu)\circ\psi = \lambda'\nu'$. Select $k = k_0$ and choose $\psi_2(t)$ as solution of an ordinary differential equation:

$$\partial\frac{1}{\nu'} = \frac{1}{(\lambda\nu)\circ\psi} \implies -\frac{\partial\nu'}{(\nu')^2} = \frac{1}{\nu'\lambda'} \implies \frac{1}{\lambda'} = -\partial\log\nu' \quad \text{for } k = k_0.$$

By (16), this parameter transformation yields $1/\lambda' = -\partial\log\nu'$ for *all* values of k. We drop the prime and denote the transformed surface by x. By back-substitution of $1/\lambda = \partial\nu/\nu$ in the proportion (15) we get the value of a:

$$a = \frac{2\nu_1\partial\nu}{(\nu+\nu_1)^2}.$$

The two expressions for ∂h, namely the one given by (13) and the other one involving a in (14), yield

$$\frac{\nu_1\partial\nu - \nu\partial\nu_1}{(\nu+\nu_1)^2}\Delta x + \frac{\nu\partial x_1 + \nu_1\partial x}{\nu+\nu_1} = \frac{2\nu_1\partial\nu}{(\nu+\nu_1)^2}\Delta x + \frac{2\nu_1}{\nu+\nu_1}\partial x$$

This easily expands to (3). The surface x is therefore dualizable, and by Theorem 11 it is isothermic. □

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