Ruled Free Forms

Simon Flöry, Yukie Nagai

Vienna University of Technology

Florin Isvoranu

Evolute GmbH, Vienna

Helmut Pottmann

King Abdullah University of Science and Technology

Johannes Wallner

Graz University of Technology

Abstract. *The geometry of ruled surfaces is highly relevant for freeform architectural design, owing to the fact that any negatively curved freeform shape can be approximated by a smooth union of ruled surface strips. This information is important for the efficient manufacturing of both skins and underconstructions. The present paper discusses how to algorithmically solve this approximation problem. In particular we emphasize the special class of conoidal ruled surfaces which suggest new and simple ways of surface generation.*

1 Introduction

This paper reports on a study of piecewise-ruled surfaces and their use in freeform architecture. The notion "piecewise-ruled" means a watertight union of ruled strips, each of which is defined by two curves in space joined by straight line segments. The basic geometric problem to be solved is how to approximate a given freeform surface by a piecewise-ruled surface with the additional requirement that the latter is itself as smooth as possible. The importance of this geometric question comes from manufacturing processes [Flöry and Pottmann 2010]: a ruled strip is comparatively easy to make, owing to its straight defining elements. Therefore freeform skins or underconstructions benefit enormously from the possibility of being manufactured as a piecewise-ruled surface. We also consider additional functional requirements: Strip boundaries might be geodesic, so they can be realized by elements made by bending [Pottmann et al. 2010], or planar (again, for simple manufacturing). Other properties having to do with aesthetics are constant strip width, or the angle between rulings and strip boundaries. We further show some interesting applications of the concept of *conoidal* ruled surfaces, see Figure 1.

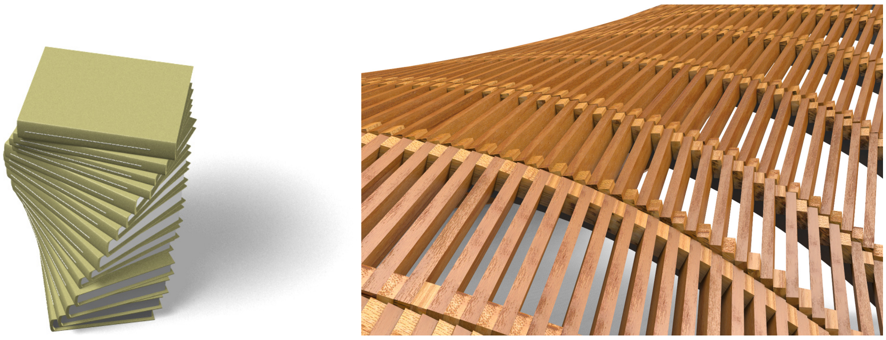

Figure 1: *Conoidal ruled surfaces*. The left hand image symbolically illustrates a conoidal surface, which by definition has rulings parallel to a certain fixed plane. Here spines of stacked books indicate the rulings, which are parallel to a horizontal reference plane. The right hand image illustrates the use of conoidal ruled strips in freeform architectural design: Each strip is made from wooden planks of simple geometry, which are stacked quite in the manner of the stack of books at left.

Previous work. The mathematical problem of approximating surfaces by ruled surfaces has been considered mostly in the context of approximation with B-spline surfaces and because of its importance for NC flank milling, see e.g. [Sprott and Ravani 2008], [Senatore et al. 2008], and [Flöry 2010, Ch. 5] and the references therein. Most work however considers neither aesthetics nor functional properties along boundaries. Work in connection with architectural geometry has been done by [Flöry and Pottmann 2010] and Pottmann *et al.* [2008], who approximate freeform shapes by piecewise-developable surfaces (which are piecewise-ruled). Since our focus is on surfaces which are as smooth as possible, which does not work with piecewise-developables, we do not consider them at all. From the viewpoint of differential geometry, smooth unions of ruled strips — see Figure 2 for an illustration of this phenomenon — have been considered by [Wallner 2012] and [Rörig and Huhnen-Venedey 2011].

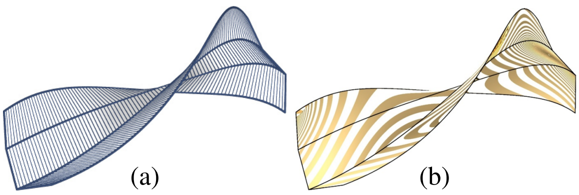

Figure 2: (a) Ruled strips can join smoothly even if their rulings do not. Smoothness is revealed in (b) by continuity of a reflected pattern of zebra stripes.

Geometric Basics. Having stated the problem — approximating a given surface Ψ by a smooth union Φ of ruled strips — we start with som geometric considerations closely related to it.

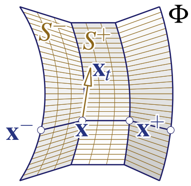

Figure 3: Kinks, resp. absence of kinks in piecewise-ruled surfaces. If strips S^- and S^+ are to join smoothly in the common point $\mathbf{x}$, then the unique tangent plane in $\mathbf{x}$ must contain both rulings $\mathbf{xx}^-$ and $\mathbf{xx}^+$, as well as the tangent vector $\mathbf{x}_t$ of the boundary curve.

Consider the symbolic illustration of a piecewise-ruled surface Φ by Figure 3. It is smooth if each strip is smooth by itself, and in addition two strips S^-, S^+, which lie to either side of their common boundary, have a common tangent plane there. With the notation introduced by Figure 3, we formulate the *smoothness condition:* Vectors $\mathbf{x}^+ - \mathbf{x}$, $\mathbf{x}^- - \mathbf{x}$ and $\mathbf{x}_t$ must lie in the tangent plane at $\mathbf{x}$. If this piecewise-ruled surface is to approximate a smooth surface Ψ, then the plane spanned by $\mathbf{x}, \mathbf{x}^+, \mathbf{x}^-$ approximates Ψ's tangent plane. The polyline of rulings, of which $\mathbf{x}^-, \mathbf{x}, \mathbf{x}^+$ is a 2-edge segment, corresponds to a curve in Ψ. The property that three successive points span a plane *tangent* to Ψ implies that such a polyline must follow an asymptotic curve (which is defined by osculating planes being tangent to Ψ, cf. [do Carmo 1976]). Figures 4 and 5a,b illustrate how to find such curves in principle, first short segments and then longer ones.

2 Algorithms

We have now established that a smooth piecewise-ruled surface Φ, which is to approximate a given surface Ψ, must have rulings which follow Ψ's asymptotic curves. Note that this is possible only on negatively curved surfaces (cf. Fig. 4).

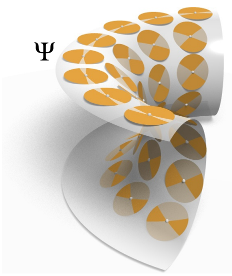

Figure 4: Asymptotic curves in a surface Ψ have the defining property that their osculating planes (planes spanned by three successive points which converge together) are tangent to Ψ. It is not difficult to find infinitesimally small segments of such curves, by intersecting Φ with its own tangent planes. One can see that only in the negatively curved areas of Ψ this intersection is meaningful.

We are going to find Φ by a two-stage process: Firstly, a *level set method* suggests strip boundaries on Ψ. This step already considers additional requirements imposed on the final result (strip width, conoidal property, . . .).

Secondly, we find ruled strips from the boundaries obtained in the first step, and apply a global approximation procedure to create a watertight Φ, always considering proximity of Φ, Ψ and the geometric side conditions which might be important for an individual application. The nonlinear nature of optimization implies that without proper initialization (i.e., without the first step) it does not usually succeed.

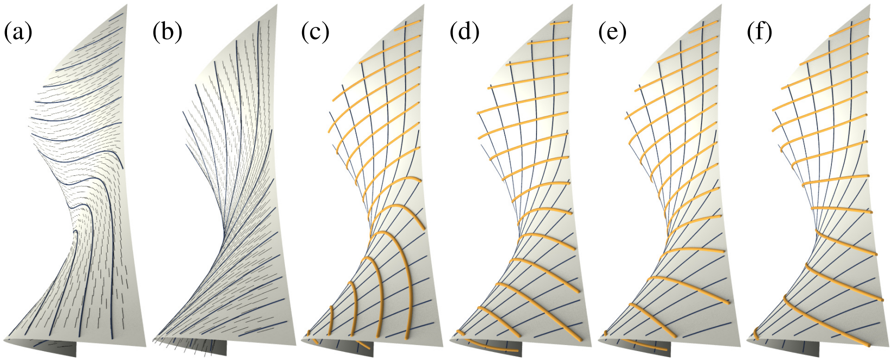

Figure 5: (a) Consistently selecting one of the two asymptotic directions per vertex yields a smooth vector field "$\mathbf{a}(\mathbf{v})$". Asymptotic curves (blue) are integral curves of that vector field. (b) shows the other possible field of asymptotic directions. (c) Once a family of asymptotic lines is selected, suggested strip boundaries are computed as transversals of that family. Different properties may be imposed: boundaries can be orthogonal to asymptotic curves (c), or at constant distance from each other (d), or geodesics (e), or planar (f).

2.1 Initialization

Computing a field of asymptotic directions. The implementation we have developed accepts the given surface Ψ as a mesh and extracts in each vertex two asymptotic directions (using the package CGAL, which works by a local fitting method, cf. [Yang and Lee 1999]). By local comparisons we distribute this collection of directions into two families of asymptotic directions and choose one of them for further use. We obtain a field of unit vectors "$\mathbf{a}(\mathbf{v})$" which are attached to the vertices "$\mathbf{v}$" of the mesh. An example is shown by Figure 5a,b.

Initialize strip boundaries. The next step is to suggest strip boundaries. They should be transverse to the asymptotic directions already obtained, and different kinds of conditions may be imposed on them (intersecting at right angles, being at constant distance from each other, being themselves planar, or being shortest paths in the surface, see Figure 5).

Our implementation uses a level set method analogous to the one of [Pottmann et al. 2010]: Strip boundaries are expressed as level sets $\{f(\mathbf{x}) = \text{const}\}$ of a function f which is defined by its values in vertices. We use a least squares method to optimize f, where the target functional is a linear combination of terms penalizing deviation from various desired properties. For the properties mentioned above (orthogonality, equidistance, geodesic property) we use $\sum_{\mathbf{v}} \|\nabla f(\mathbf{v}) - \mathbf{a}(\mathbf{v})\|^2$, $\sum_{\mathbf{v}}(\|\nabla f\| - 1)^2$, and $\sum_{\mathbf{v}}(\operatorname{div} \frac{\nabla f}{\|\nabla f\|})^2$, respectively. In all cases, the Laplacian term $\sum_{\mathbf{v}}(\operatorname{div} \nabla f)^2$ is added with low weight for regularization. Figures 5c–5f show examples. *Planar* strip boundaries are more easily found: intersect Ψ with suitably chosen planes (see Figures 5f and 6).

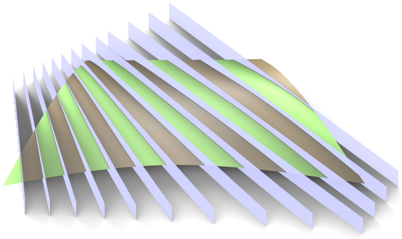

Figure 6: In order to suggest *planar* strip boundaries on a reference surface Ψ, one intersects Ψ with a sequence of planes, making sure that the cuts are transverse to a preselected family of asymptotic lines (not shown here, but see Figure 5b).

Initialization of strip boundaries in the conoidal case. In order to check if a ruled strip is conoidal (see Figure 1) we use unit vectors $\mathbf{r}$ indicating the directions of rulings. There is the following equivalence: The strip is conoidal, i.e., rulings are parallel to a fixed plane $\iff$ the set of all $\mathbf{r}$'s is a great circle in the unit sphere. The fact that great circles are geodesic curves of the unit sphere leads to the following initialization procedure:

Each vertex "$\mathbf{v}$" of the mesh Ψ is equipped with a unit vector "$\mathbf{a}(\mathbf{v})$" indicating an asymptotic direction. The latter constitute the vertices of a mesh Ψ^* contained in the unit sphere, which has the same combinatorics as Ψ. Like above, the unknown strip boundaries are found as level sets of a function f. The 1-1 correspondence between meshes Ψ, Ψ^* transfers the function f and its level sets to the mesh Ψ^*. The conoidal property now requires that we enforce the latter level sets to be geodesic. It is possible to manually choose those geodesics, but we prefer an algorithmic and automated solution: In addition to the functionals already described above we therefore use, with a high weight, the term $\sum_{\mathbf{v}}(\mathrm{div}^* \frac{\nabla^* f}{\|\nabla^* f\|})^2$ where the * symbol indicates that gradient and divergence are taken with respect to the metric of Ψ^* instead of Ψ. This geometric trick allows us to incorporate the conoidal constraint into our implementation with very little additional effort (currently there is the limitation that our implementation works only if Ψ^* does not overfold itself). Figure 7 shows an example.

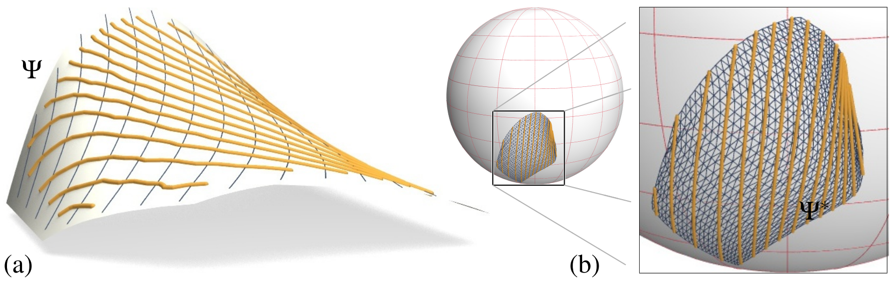

Figure 7: Initializing conoidal strips. (a) Suggested strip boundaries in the mesh Ψ (yellow). (b) If the mesh Ψ is mapped to the unit sphere via the asymptotic directions, then strip boundaries are mapped to geodesics in the image mesh Ψ^* (i.e., to great circles). Note that Ψ^* is covered by a family of great circles which are in general position to each other, quite unlike the system of meridians of the sphere.

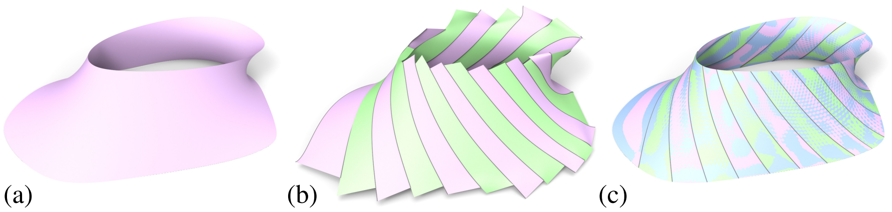

Figure 8: *Ruled strip optimization*. From left: (a) shows a detail of the *Cagliari museum project* by Zaha Hadid Architects, slightly remodelled for smoothness. On this reference surface of cylinder topology, strip boundaries orthogonal to asymptotic directions have been suggested. (b) shows the resulting optimized ruled strips which completely cover the reference surface. In (c) these strips have been trimmed along the original reference boundaries. Approximation quality can be read off the interference pattern with the reference surface, which has been overlaid on top of the result.

2.2 Ruled strip optimization

Strip boundaries having been suggested, we now turn to the optimization of the actual strips. Analogous to existing work on developable strips [Pottmann et al. 2008], we describe each strip as a degree (3,1) "cubic ruled" B-spline surface, with control points $\{\mathbf{b}_{ij}\}_{i=0,\dots,n}^{j=0,1}$. It is not difficult to initialize those control points: we simply choose $\mathbf{b}_{00},\dots,\mathbf{b}_{n0}$ along one suggested strip boundary and find the corresponding points $\mathbf{b}_{01},\dots,\mathbf{b}_{n1}$ on the opposite boundary by observing that $\mathbf{b}_{i0}-\mathbf{b}_{i1}$ follows asymptotic directions.

We then set up an optimization problem whose variables are the control points of all strips. The target functional is a linear combination of the following quadratic terms which are initialized in a nonlinear data-dependent way before a round of optimization is performed:

- To penalize deviation from the reference surface, we follow [Pottmann et al. 2008] and minimize $\sum_{\mathbf{x}} \operatorname{dist}(\mathbf{x}, T(\mathbf{x}))^2$, where the sum is taken over a sample of points $\mathbf{x}$ on the B-spline strips, and the plane $T(\mathbf{x})$ is found by closest point projection of the initial value of $\mathbf{x}$ onto Ψ and taking the tangent plane there.
- Similarly we penalize gaps between strips, with the sum taken over a sample of boundary points of strips, and $T(\mathbf{x})$ is a straight line found by closest point projection onto the opposite boundary and taking the tangent there.
- For fairness we use the thin plate energy (with a low weight).
- Note that overall smoothness (i.e., existence of tangent planes) is not formulated as an optimization target; we were content with taking care of this property during the initialization phase. It turned out that more was not necessary.

Implementation of constrained optimization. The optimization problem is solved as a sequence of quadratic programs, using the Armijo rule for step size control.

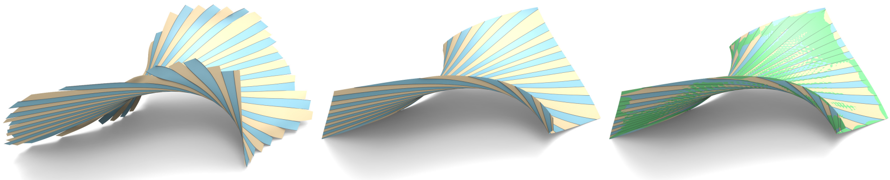

Figure 9: Optimizing ruled strips with geodesic boundaries on part of a so-called “Wohlgemuth-Thayer” minimal surface. From left: optimization result, trimmed result, and illustration of approximation power by overlaying the reference surface. Strip boundaries being geodesic, they can be manufactured by bending wooden panels.

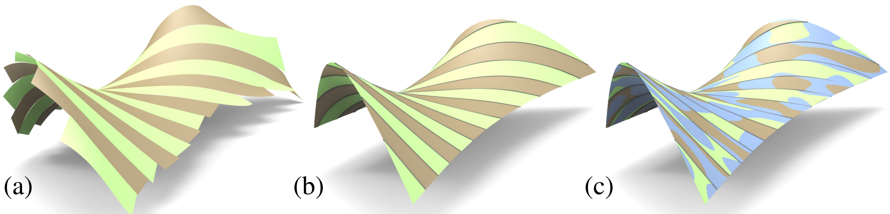

Figure 10: After suggesting strip boundaries as planar cuts through the reference surface (see Figure 6), ruled strips are optimized under the side condition that boundaries are constrained to their respective planes. The result is a watertight union of strips covering the reference surface (a). Subfigure (b) shows the trimmed result. Approximation quality is illustrated by overlaying the reference surface (blue) with the result of optimization in (c).

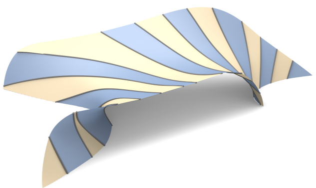

Figure 11: On a detail of the *Cagliari museum project* by Zaha Hadid Architects, strips of constant width have been initialized, but global optimization does not achieve this property. This limitation is due to the fact that strip width is large compared to the size of features of the design surface.

We employed the OOQP package [Gertz and Wright 2003] as a black box. Our academic implementation requires to manually weight the individual terms which contribute to the target functional. Experience shows that optimization converges in about 10 iterations. Side conditions (keeping boundary curves in previously selected planes, keeping strip endpoints fixed, keeping end ruling directions fixed, ...) are linear and thus easy to enforce as hard constraints.

Figure 10 shows a result of optimization with the side condition of planar boundaries. The conoidal condition is treated in a similar way: it is easy to see that a degree $(n, 1)$ B-spline strip defined by control points $\{\mathbf{b}_{ij}\}_{i=0,\dots,n}^{j=0,1}$ is conoidal if and only if all vectors $\mathbf{b}_{i0} - \mathbf{b}_{i1}$ are orthogonal to some common vector. Our experience shows that we can take that vector from the result of the initialization phase and keep it fixed during the optimization phase. This makes the conoidal condition linear. It is likewise enforced as a hard constraint. See Figure 13 for an example.

3 Results

Limitations. The approximation procedure described in this paper has some limitations – this is only to be expected if the feature size of the reference surface is of the same magnitude as the strip width. Figure 11 shows an example where the constant width of strips which was achieved during the initialization phase did not survive the second optimization stage. Our experience shows that the conoidal property (which we consider the most important in the context of this paper) does not seem to be a great restriction — see the comparison provided by Figure 12. Problems arose mostly in enforcing watertightness. We should remark that using a common boundary for adjacent strips will of course overcome that problem, but it will also cost valuable degrees of freedom.

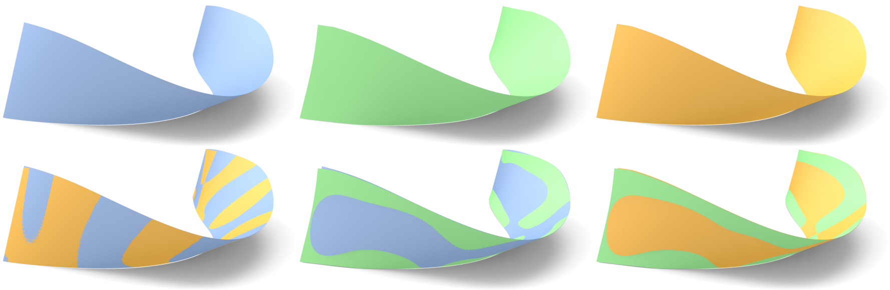

Figure 12: A reference surface (blue) which is an unmodified part of the *Cagliari museum* project by Zaha Hadid architects has been manually initialized and has subsequently been approximated with a single conoidal ruled surface (green) and with a general ruled surface (orange). The bottom row shows these three surfaces superimposed on each other in order to visualize approximation quality.

Applications, especially in timber constructions. The most obvious application of a ruled surface is that its shape is manufacturable by moving a straight object along the rulings. This might be a hot wire cutting a mold, but it might also refer to all kinds of auxiliary constructions and substructures where straight beams of whatever material are used. The interested reader is referred to [Flöry and Pottmann 2010] for an overview. In this paper we want to draw attention to the possibility of timber constructions based on conoidal ruled surfaces. The principle has already been shown by Figure 1. For more details see Figures 13 and 14.

Conclusion. We have demonstrated how to approximate freeform shapes by a smooth union of ruled strips, using a procedure consisting of differential geometry analysis, initialization by a level set method, and subsequent optimization. It turns out that several kinds of side conditions can be imposed on this approximation. As a new application of ruled surfaces in freeform architectural design we have shown the relevance of *conoidal* ruled strips for timber constructions from simple elements.

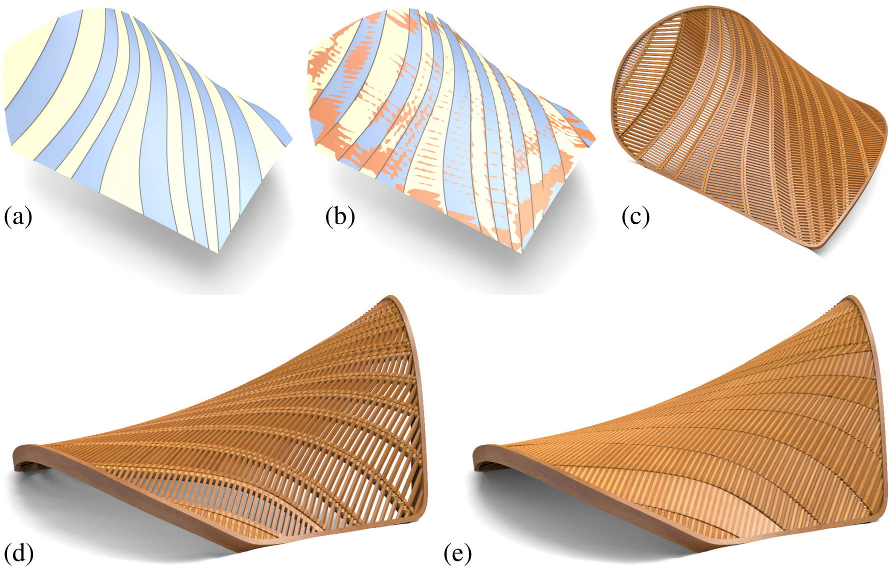

Figure 13: *Conoidal strip optimization.* A reference surface, which is part of the one used by Figures 5 and 10, has been treated by an initialization procedure (Figure 7). Afterwards conoidal strips which cover the reference surface are computed. Subfigures (a)–(c) respectively show the trimmed optimization result, a visualization of approximation quality, and a visualization of the rulings of the individual strips. The second row shows two ways to exploit the conoidal property which means that rulings may be realized by wooden slabs of constant thickness which are cut in a simple manner from standard parts.

Acknowledgments

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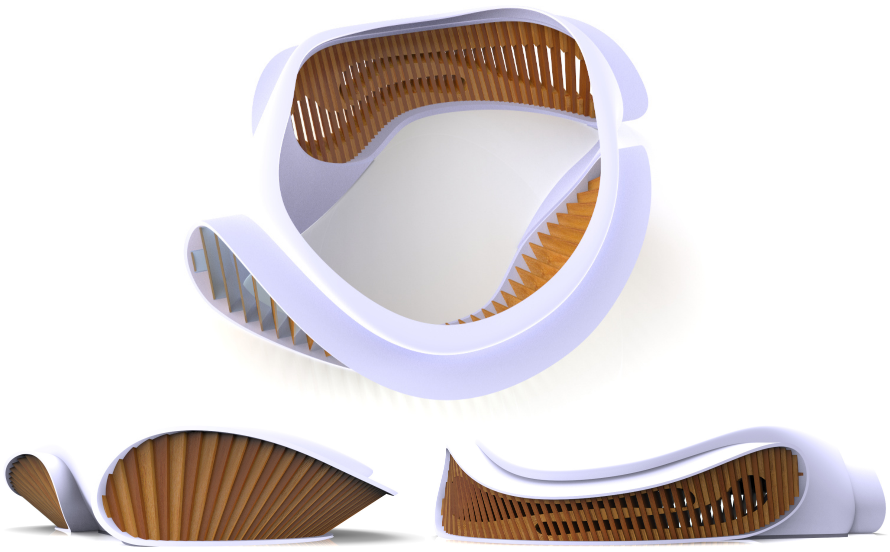

Figure 14: The conoidal ruled strip of Figure 12 serves as the geometric basis of this lamella construction with wooden panels. All panels are parallel to a fixed plane, which is not only a bonus for manufacturing, but has interesting applications with regard to shading. The image at left shows a top view, while the two images below show side views from different directions.

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