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On offsets and curvatures for discrete and semidiscrete surfaces

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Abstract This paper studies semidiscrete surfaces from the viewpoint of parallelity, offsets, and curvatures. We show how various relevant classes of surfaces are defined by means of an appropriate notion of infinitesimal quadrilateral, how offset surfaces behave in the semidiscrete case, and how to extend and apply the mixed-area based curvature theory which has been developed for polyhedral surfaces.

Keywords Semidiscrete surfaces · Curvatures

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A semidiscrete surface $x(i, u)$ is a mapping from $\mathbb{Z} \times \mathbb{R}$ to some vector space, i.e., a bivariate function of one discrete and one continuous variable. Such mixed continuous-discrete objects classically occur in the transformation theory of surfaces. For instance, a pair $x(u, v)$ and $x^+(u, v)$ of surfaces is seen as a semidiscrete mapping defined in $\{0, 1\} \times \mathbb{R}^2$. The viewpoint of smooth parameterized surfaces as limits of discrete nets has been systematically exploited (see [2] and the references therein) and directly leads to semidiscrete objects, if limits do not apply to all variables, only to some of them. In this way the theory of smooth surfaces, their transformations, and the permutability of their transformations appear as limit cases of a discrete master theory of discrete nets and integrable systems.

This paper is concerned with semidiscrete objects of very simple type. They fit the larger theory if they are considered as a transformation sequence of smooth curves, but they are interesting in their own right and in fact they have turned up in geometry processing applications [9].

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Even if in many senses semidiscrete surfaces are limit cases of discrete ones and their properties are similar to both the discrete and continuous cases, they nevertheless deserve separate study [15]. Classes of surfaces already treated are the asymptotic surfaces of constant Gaussian curvature [16], the isothermic surfaces [7], and the conjugate surfaces and their circular and conical reductions, which are also relevant for applications [9].

This paper demonstrates how the concept of parallel surfaces (i.e., Combescure transforms) and offsets (i.e., parallel surfaces at constant distance) lead to a theory of curvatures. For smooth surfaces this topic is classical: Steiner's formula

$$dA(x^\tau) = (1 - 2H\tau + K\tau^2)\, dA(x),$$

on the surface area element of an offset at distance τ belongs here. In the context of discrete surfaces, this relation between curvature and areas was utilized first by [11,12], in order to define the mean and Gaussian curvature of faces in circular meshes. A general discrete theory has been given by [8] and [1]. Besides applying this idea to the semidiscrete case, this paper also extends our knowledge of discrete curvatures, for instance by the formula

$$H(f) = -\frac{1}{2\,\mathrm{area}(f)} \sum_{e\in\partial f} \tan\frac{\alpha_e}{2} \times \mathrm{length}(e),$$

for the mean curvature of a face of a *conical* polyhedral surface, more details of which are found in the text below.

1 Smooth, Semidiscrete, and Discrete Surfaces

We define a *net* as a mapping $x : \mathbb{Z}^k \times \mathbb{R}^{m-k} \to \mathbb{R}^n$, which depends on discrete parameters $i_1, \dots, i_k$ and continuous parameters $u_{k+1}, \dots, u_m$. We do not insist on entire $\mathbb{Z}^k \times \mathbb{R}^{m-k}$ as the domain where x is defined, since our study concerns local properties. For the discrete parameter i_r we use the notation x_r for an index shift:

$$x_r(\dots, i_r, \dots) = x(\dots, i_r + 1, \dots),$$

and x_{-r} denotes the inverse shift. Differentiation with respect to i_r is done by the forward difference operator: $\delta_r x = x_r - r$. For the continuous parameters we use partial derivatives $\delta_j x = \frac{\partial x}{\partial u_j}$. For us the most important case is $k = m - k = 1$, i.e.,

$$x : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^3, \quad x = x(i,u). \tag{1}$$

The derivatives $\delta_1 x, \delta_2 x$ with respect to the one discrete and the one continuous parameter are written as

$$\varDelta x(i,u) = x(i+1,u) - x(i,u), \quad \dot{x}(i,u) = \frac{\partial x}{\partial u}(i,u).$$

A semidiscrete net $x(i, u)$ is visualized as piecewise-smooth surface, namely as the union of line segments

$$(1 - v)x(i, u) + vx(i + 1, u), \quad \text{where } v \in [0, 1]. \tag{2}$$

Recall that this strip is a developable surface, if and only if

$$\{\Delta x, \dot{x}, \dot{x}_1\} \quad \text{linearly dependent.}$$

1.1 Infinitesimal quadrilaterals

While the definition of certain classes of smooth surfaces (conjugate, circular, etc.) requires 2nd order derivatives, the analogous definition in the discrete category frequently involves only geometric properties of single faces. It turns out that for semidiscrete surfaces, there is a natural notion of *infinitesimal quadrilateral* which allows a similar approach. Motivated by the decomposition

$$\boxed{\begin{matrix} x & x_1 \\ x_2 & x_{12} \end{matrix}} = \boxed{\begin{matrix} x & x_1 \\ x & x_1 \end{matrix}} + \boxed{\begin{matrix} 0 & 0 \\ \delta_2 x & \delta_2 x_1 \end{matrix}}$$

of an elementary quadrilateral of a discrete surface $x : \mathbb{Z}^2 \to \mathbb{R}^n$ we define:

Definition 1.1. *An infinitesimal n-gon is a tangent vector in the affine space of n-gons; we use the notation $(P; V)$ for a tangent vector representing any smooth path $\tilde{P}(\tau)$ of n-gons with $\tilde{P}(0) = P$, $\frac{d}{d\tau}\tilde{P}(0) = V$. The elementary quadrilaterals $(P(i, u); V(i, u))$ of the semidiscrete surface $x(i, u)$ are represented by*

$$P = \boxed{\begin{matrix} x & x_1 \\ x & x_1 \end{matrix}}, \quad V = \boxed{\begin{matrix} 0 & 0 \\ \dot{x} & \dot{x}_1 \end{matrix}}. \tag{3}$$

Recall the notions of conjugate and circular discrete surfaces which are characterized by elementary quads being planar or possessing a circumcircle. Conical surfaces have the property that faces adjacent to a vertex always touch some common right circular cone [2,5]. The following definitions for semidiscrete surfaces are natural extensions and have in fact already been given by [9,7]:

Definition 1.2. *A semidiscrete surface $x(i, u)$ is regular/conjugate/circular/ conical $\iff$ the respective condition listed below is fulfilled for all i, u.*

1. *Regularity is linear independence of both $\{\dot{x}, \Delta x\}$ and $\{\dot{x}_{-1}, \Delta x\}$.*
2. *Conjugacy means regularity and linear dependence of $\{\Delta x, \dot{x}, \dot{x}_1\}$.*
3. *Circularity means that if in addition there is a circle passing through x and x_1 such that derivatives $\dot{x}$, $\dot{x}_1$ are tangent there.*
4. *The conical property is conjugacy and existence of a right circular cone with axis through x which touches the adjacent ruled strips along the rulings $x \vee x_1$ and $x \vee x_{-1}$.*

We use the symbol "$\vee$" for the line which joins two points. Note that conjugacy and circularity are properties of the infinitesimal quadrilateral $(P;V)$ *defined by* (3).

Corollary 1.3. *Circularity and conicality are equivalently expressed by existence of a Euclidean reflection in a plane which leaves a certain configuration of lines invariant. That configuration is given by lines*

$$L = x \vee x_1, \quad L' = x + \operatorname{span}(\dot{x}), \quad L'' = x_1 + \operatorname{span}(\dot{x}_1) \qquad \textit{(circular)}$$
$$L = x + \operatorname{span}(\dot{x}), \quad L' = x \vee x_1, \quad L'' = x \vee x_{-1}, \qquad \textit{(conical)}$$

respectively, and the reflection is to fix L *and to exchange* L', L''.

Proof. (cf. [9]): To show "$\Longleftarrow$" we assume existence of a reflective symmetry as stated. In the circular case, any circle passing through x, x_1 has its axis in the plane of symmetry. The unique circle tangent to L' in x will have L'' as a tangent in x_1, by reflective symmetry. This shows circularity. In the conical case, any cone containing L', L'' has its axis in the plane of symmetry. The cone tangent to the plane spanned by L, L' will have the plane spanned by L, L'' as tangent plane also. This shows conicality. The "$\Longrightarrow$" part is elementary. □

Remark 1.4. Conjugacy of surfaces is uniformly expressed by

$$\{\delta_1 x, \delta_2 x, \delta_{12} x\} \quad \text{linearly dependent},$$

where x may be discrete, semidiscrete, or smooth. This is in accordance with the definition that a conjugate discrete net is defined as a quad mesh with planar faces, i.e., a quad mesh which is a polyhedral surface.

Remark 1.5. Recall that smooth surfaces x, x^+ constitute a Jonas pair if we have linear dependence of $\{x^+ - x, \partial_k x, \partial_k x^+\}$ for all parameters i_k, throughout the domain of definition. They form a Darboux pair if similarly there is a circle passing through x^+, x such that the partial derivatives $\partial_k x, \partial_k x^+$ are tangent there (see e.g. [2]). It follows that for a conjugate semidiscrete surface, curves $x(i, \cdot)$ and $x(i+1, \cdot)$ are a Jonas pair. If x is circular, they constitute a Darboux pair.

1.2 **Parallelity of surfaces**

Combescure pairs x, x^+ of surfaces (i.e., surfaces where $\delta_i x$, $\delta_i x^+$ are parallel) turn up frequently in discrete differential geometry, ever since *parallel nets* were introduced by [4]. We are particularly interested in surfaces at constant distance which are discussed later. We prefer to speak of *parallel* surfaces, which is written as

$$x \parallel x^+.$$

For semidiscrete surfaces, parallelity means $\Delta x \parallel \Delta x^+$ and $\dot{x} \parallel \dot{x}^+$.

Proposition 1.6. *Assuming regular surfaces, the properties of* x *being conjugate, circular, or conical are invariant under parallelity.*

This follows directly from Cor. 1.3. For any x one can find a surface x^+ which is parallel by simply translating and scaling x; it turns out other, non-trivial, instances of parallelity occur only if x itself is conjugate:

Proposition 1.7. *Assume that x is regular, but not conjugate, i.e., we have linear dependence of $\{\dot{x}, \Delta x, \Delta\dot{x}\}$ only in a set of measure zero w.r.t. 1-dimensional Lebesgue measure in $\mathbb{R} \times \mathbb{Z}$. Then*

$$x^+ \parallel x \iff x^+, x \text{ homothetic.}$$

Proof. Let U be the ambient space which contains all surfaces. The statement is void unless $\dim U \geq 3$. We compute in the exterior algebra $\Lambda^2 U$ and show the statement by proving that the ratios $\lambda = \dot{x} : \dot{x}^+$ and $\gamma = \Delta x : \Delta x^+$ (well defined by parallelity and regularity) are constant and equal. Firstly,

$$\begin{aligned} 0 &= \partial_u(\Delta x \wedge \Delta x^+) = (\dot{x}_1 - \dot{x}) \wedge (\gamma \Delta x) + \Delta x \wedge (\lambda_1 \dot{x}_1 - \lambda \dot{x}) \\ &= (\gamma - \lambda)\Delta x \wedge \dot{x} - (\gamma - \lambda_1)\Delta x \wedge \dot{x}_1. \end{aligned}$$

Linear independence of $\Delta x, \dot{x}, \dot{x}_1$ implies linear independence of $\Delta x \wedge \dot{x}$, $\Delta x \wedge \dot{x}_1$, so we have $\gamma = \lambda$ and $\lambda = \lambda_1$. Secondly, differentiation of γ's defining equation together with $\lambda = \gamma$ yields

$$0 = \partial_u(\Delta x^+ - \gamma \Delta x) = \gamma \dot{x}_1 - \gamma \dot{x} - \dot{\gamma} \Delta x - \gamma \Delta \dot{x} = \dot{\gamma} \Delta x.$$

It follows that $\dot{\gamma} = 0$. The identities $\gamma = \lambda$, $\dot{\gamma} = 0$ and $\Delta\gamma = 0$ shown here hold everywhere except in a zero set, so we conclude $\gamma = \lambda = \text{const}$. □

1.3 **Parallel Infinitesimal Polygons.**

The curvature theory of discrete surfaces presented by [8,1] depends on the notion of parallel polygons, which are required to be contained in parallel planes. If U is an affine space over the reals, we define parallelity of n-gons $P = (p_0, \ldots, p_{n-1}), Q = (q_0, \ldots, q_{n-1}) \in U^n$ by

$$P \parallel Q \iff (p_{i+1} - p_i) \wedge (q_{i+1} - q_i) = 0 \quad \text{(indices modulo } n).$$

We extend the definition of parallelity to infinitesimal polygons in a way which serves our purposes when applied to the elementary quads of a semidiscrete surface. We first consider infinitesimal polygons $(P; V)$ and $(Q; W)$ being represented by n-gon paths

$$P + \tau V, \quad Q + \tau W.$$

For parallelity we require $P \parallel Q$; in addition each $\Lambda^2 U$-valued polynomial

$$(p_{i+1}(\tau) - p_i(\tau)) \wedge (q_{i+1}(\tau) - q_i(\tau)) \quad \text{(indices modulo } n)$$

in the indeterminate τ shall have a zero at $\tau = 0$ whose order is higher than would be the case for a generic element $V \times W \in T_P U^n \times T_Q U^n$.

In the discrete case, it is obvious that two surfaces have corresponding parallel edges if and only if corresponding faces are parallel (in the sense of parallel polygons). In the semidiscrete case this is no longer as obvious:

Lemma 1.8. *Parallelity of conjugate surfaces x, x^+ is equivalent to parallelity of elementary qadrilaterals*

$$\boxed{\begin{matrix} x & x_1 \\ x & x_1 \end{matrix}} + \tau \boxed{\begin{matrix} 0 & 0 \\ \dot x & \dot x_1 \end{matrix}}, \qquad \boxed{\begin{matrix} x^+ & x_1^+ \\ x^+ & x_1^+ \end{matrix}} + \tau \boxed{\begin{matrix} 0 & 0 \\ \dot x^+ & \dot x_1^+ \end{matrix}}.$$

The latter is expressed by the conditions

$$(x_1 - x) \parallel (x_1^+ - x^+), \quad \dot x_1 \parallel \dot x_1^+, \quad \dot x \parallel \dot x^+,$$
$$(\dot x - \dot x_1) \wedge (x^+ - x_1^+) + (x - x_1) \wedge (\dot x^+ - \dot x_1^+) = 0.$$

Proof. Parallelity of elementary quads firstly means that $(x_1 - x) \parallel (x_1^+ - x^+)$, and secondly that $\Lambda^2 U$-valued polynomials

$$(\tau \dot x) \wedge (\tau \dot x^+), \quad (\tau \dot x_1) \wedge (\tau \dot x_1^+),$$
$$(x - x_1 + \tau(\dot x - \dot x_1)) \wedge (x^+ - x_1^+ + \tau(\dot x^+ - \dot x_1^+))$$

have zeros of multiplicities 3, 3, 2, resp., for $\tau = 0$. This verifies the definition of parallelity of infinitesimal quadrilaterals.

Clearly parallelity of elementary quads implies parallelity of surfaces. For the converse we observe that the first three conditions express the definition of $x \parallel x^+$ while the last one is found by differentiating $\Delta x \wedge \Delta x^+ = 0$. □

1.4 **Offsets in three dimensions.**

In the elementary differential geometry of surfaces, an *offset* means a surface at constant distance to a given one. This requirement defines the offset uniquely, but for discrete surfaces there are different ways of defining offsets [11,12,5,8, 2]. In the semidiscrete category the situation is similar, as has been observed by [9]. We say that a parallel pair x, x^+ of surfaces is an offset pair at distance d, with *Gauss image*

$$s = \frac{1}{d}(x^+ - x),$$

if x, x^+ are at constant distance d from each other. This distance can be measured in different ways:

1. Case (V), *vertex offsets*: Distance of vertices is constant:
$$\mathrm{dist}(x, x^+) = d.$$
2. Case (E_1) *edge offsets, 1st kind*: Distance of rulings is constant:
$$\mathrm{dist}(x \vee x_1, x^+ \vee x_1^+) = d.$$
3. Case (E_2) *edge offsets, 2nd kind*: Distance of tangents is constant:
$$\mathrm{dist}(x + \mathrm{span}(\dot x),\ x^+ + \mathrm{span}(\dot x^+)) = d.$$

4. Case (F) *face offsets*: Distance of tangent planes is constant:

$$\mathrm{dist}(x + \mathrm{span}(\Delta x, \dot{x}),\ x^+ + \mathrm{span}(\Delta x^+, \dot{x}^+)) = d.$$

In the following discussion we restrict ourselves to $U = \mathbb{R}^3$ as ambient space. We will see that there are essentially only two cases of offsets, not four. Before we prove that, we state an obvious but important relation:

Proposition 1.9. *The surface x^+ is an offset of x at distance $d \iff s = \frac{1}{d}(x^+ - x)$ is an offset of the same type of the zero surface.*

Lemma 1.10. *Offset types* (E_1), (F) *are generically equivalent.*

Proof. We translate statements (E_1), (F) into equivalent statements (E_1'), (F') worded in terms of the Gauss image surface s, which is conjugate and parallel to both x, x^+:

1. case (E_1'): Rulings $s + \mathrm{span}(\Delta s) = s \vee s_1$ are tangent to the unit sphere.
2. case (F'): Tangent planes $s + \mathrm{span}(\dot{s}, \Delta s)$ are tangent to the unit sphere.

Assuming (F'), there is a normal vector field $n : \mathbb{Z} \times \mathbb{R} \to S^2$ such that

$$\langle s, n\rangle = \langle s_1, n\rangle = 1 \quad \text{and} \quad \langle \dot{s}, n\rangle = \langle \dot{s}_1, n\rangle = \langle \Delta s, n\rangle = 0$$

(we find n by normalizing $\dot{x} \times \Delta x$). We compute

$$0 = \partial_u \langle s, n\rangle = \langle \dot{s}, n\rangle + \langle s, \dot{n}\rangle = \langle s, \dot{n}\rangle \text{ and similarly } \langle s_1, \dot{n}\rangle = 0,$$
$$0 = \partial_u \langle n, n\rangle = 2\langle \dot{n}, n\rangle = 0.$$

It follows that each of s, s_1, n fulfills the two linear equations $\langle n, \cdot\,\rangle = 1$ and $\langle \dot{n}, \cdot\,\rangle = 0$. If $\dot{n} \neq 0$ (this is the genericity assumption) this implies that they are collinear and $n \in s \vee s_1$. Since the tangent plane touches the unit sphere in the point n, the line $s \vee s_1$ touches the unit sphere in the point n. This shows (E_1').

Assume now (E_1'). Consider the point of contact n of the ruling $s \vee s_1$ with the unit sphere. If $\dot{n}$, Δs are linearely independent (this is the genericity assumption), these two vectors, both orthogonal to n, span the tangent plane which obviously touches the unit sphere. This shows (F'). □

Lemma 1.11. *Offset cases (V) and (E_2) are generically equivalent.*

Proof. Polarity w.r.t. S^2 maps vertices "x", tangents "$x + \mathrm{span}(\dot{x})$", rulings "$x \vee x_1$" and tangent planes of a conjugate surface to the tangent planes, rulings, tangents, and vertices, respectively, of another conjugate surface. It follows that this polarity maps Gauss images of types (V) and (E_2) to Gauss images of types (F) and (E_1), resp., and vice versa. The equivalence (F) $\iff$ (E_1) shown above thus implies (V) $\iff$ (E_2). □

The following statement, which is analogous to the discrete case shown in [10], is stated already in [9].

Theorem 1.12. *Consider a semidiscrete conjugate surface x. A nontrivial vertex offset exists $\iff$ x is circular.*

Sketch of Proof. If x has a vertex offset, there exists a Gauss image s inscribed in S^2. By Prop. 1.6 it is sufficient to show that s is circular. This follows directly from conjugacy of s: The plane through s and s_1 which is spanned by vectors $\dot{s}, \Delta s, \dot{s}_1$ intersects S^2 in the desired circle.

For the converse we assume that x is circular. We construct a Gauss image surface $s : \mathbb{R} \times \mathbb{Z} \to S^2$ parallel to x.

Choose $s(i_0, u_0)$ arbitrarily on $S^2 \cap \dot{x}^\perp$. Since the straight line $s + \mathrm{span}(\Delta x)$, evaluated at (i_0, u_0) has exactly one other intersection point with the unit sphere, $s(i_0 + 1, u_0)$ is uniquely determined by parallelity. The circular condition ensures that also there, $s \in \dot{x}^\perp$. By induction we construct all values $s(i, u_0)$.

As to the continuous variable, $s(i_0, u)$ shall be the integral of the linear ODE

$$\dot{s} = -\frac{\langle s, \ddot{x}\rangle}{\langle \dot{x}, \dot{x}\rangle}\dot{x}.$$

By construction, $\partial_{uu}\langle s, s\rangle = 0$, and initial conditions at $u = u_0$ are such that $\langle s, s\rangle = \mathrm{const.} = 1$.

To compute arbitrary values $s(i, u)$ we can either apply the discrete construction to $s(i_0, u)$ or the continuous construction to $s(i, u_0)$. Consistency (i.e., integrability of this difference-differential equation) follows from the circular condition. We omit this calculation. □

1.5 **Support Functions.**

Our discussion of offsets in $\mathbb{R}^3$ leads us to consider support functions of surfaces, which is motivated by the well known concept of the same name in convex geometry. We start by defining a unit normal vector field n of a conjugate surface x, by the requirements

$$\langle \delta_1 x, n\rangle = \langle \delta_2 x, n\rangle = 0, \quad \|n\| = 1.$$

We can locally make the normal vector field unique by requiring

$$\det(\delta_1 x, \delta_2 x, n) > 0, \tag{4}$$

but this does not always serve our purposes; when we speak about a parallel pair x, x^+ with $\frac{1}{d}(x^+ - x) = s$, then we always consider a common normal vector field n for all three surfaces x, x^+, s, even if the handedness condition (4) is fulfilled only for, say, x.

Definition 1.13. *If n is a unit normal vector field of the conjugate surface s, then the associated support function is given by*

$$\sigma_{n,s} : \mathbb{R} \times \mathbb{Z} \to \mathbb{R}, \quad \sigma_{n,s} = \langle n, s\rangle.$$

Remark 1.14. Obviously, if $x^+ = x + \lambda s$, then $\sigma_{n,x^+} = \sigma_{n,x} + \lambda\sigma_{n,s}$. We illustrate this relation by means of a surface x^+ with normal vector field n tangentially circumscribed to the unit sphere. Since all tangent planes of x^+ are tangent to the unit sphere, $\sigma_{n,x^+} = 1$. The surface $x = x^+ - n$, endowed with the same normal vector field n, degenerates to $x = \text{const.} = 0$. This is consistent with $\sigma_{n,x^+} = \sigma_{n,x} + \sigma_{n,n} = 0 + 1 = 1 = \text{const.}$

Theorem 1.15. *Assume a vector field n with* $\det(n,\ \delta_1 n,\ \delta_2 n) \neq 0$. *Then there is a conjugate surface x whose support function w.r.t. n equals the given function σ, if and only if*

$$\det\begin{pmatrix} \sigma & \delta_1\sigma & \delta_2\sigma & \delta_2\delta_1\sigma \\ n & \delta_1 n & \delta_2 n & \delta_2\delta_1 n \end{pmatrix} = 0.$$

This statement applies to discrete, semidiscrete and smooth surfaces, with the appropriate meanings of δ_1, δ_2.

Proof. Define x by solving $\langle n, x\rangle = \sigma$, $\langle \delta_1 n, x\rangle = \delta_1\sigma$, $\langle \delta_2 n, x\rangle = \delta_2\sigma$. The vanishing determinant means that x also solves $\langle \delta_2\delta_1 n, x\rangle = \delta_2\delta_1\sigma$.

(i) In the discrete case a linear combination of equations gives the equivalent condition that the solution of $\langle n, x\rangle = \sigma$, $\langle n_1, x\rangle = \sigma_1$, $\langle n_2, x\rangle = \sigma_2$ also solves $\langle n_{12}, x\rangle = \sigma_{12}$. By an index shift, this is conjugacy.

(ii) In the smooth case, the definition of x means that the surface $x(u_1, u_2)$ is the envelope of tangent planes $\langle n, \cdot\,\rangle = \sigma$. Differentiation $0 = \delta_1(\langle \delta_2 n, x\rangle - \delta_2\sigma) = \langle \delta_2\delta_1 n, x\rangle + \langle \delta_2 n, \delta_1 x\rangle - \delta_1\delta_2\sigma$ shows that the determinant condition is equivalent to $\langle \delta_1 n, \delta_2 x\rangle = 0$, which is well known to express conjugacy.

(iii) The semidiscrete case is a mixture of (i) and (ii). A linear combination of equations defines x equivalently as solution of $\langle n, \cdot\rangle = \sigma$, $\langle \dot n, \cdot\rangle = \dot\sigma$, $\langle n_1, \cdot\rangle = \sigma_1$, and the determinant condition means x also solves $\langle \dot n_1, \cdot\rangle = \dot\sigma_1$. The first two equations define the ruling through x of the developable enveloped by planes $\langle n(i,t), \cdot\rangle = \sigma(i,t)$ as t is running and i fixed, while the last two equations define the corresponding ruling through x_1. Conjugacy means that x is contained in the latter ruling, which is just the determinant condition. ☐

Corollary 1.16. *Assume that conjugate surfaces x, x^+ have the same unit normal vector field n. If $\delta_1 n \neq 0$, $\delta_2 n \neq 0$, then x, x^+ are parallel.*

Thus there is a linear space of surfaces with given unit normal vector field which are conjugate apart from the condition of regularity.

Proof. The previous proof states that $\text{span}(\delta_1 x)$, $\text{span}(\delta_2 x)$ is determined by the normal vector field alone, e.g. in the smooth case by $\langle \delta_1 x, \delta_2 n\rangle = 0$. ☐

Corollary 1.17. *The conjugate surface x with unit normal vector field n has an offset at constant face-face distance* $\iff \det(\delta_1 n, \delta_2 n, \delta_2\delta_1 n) = 0$

Proof. In view of Prop. 1.9, existence of such offsets means that $\sigma = 1 = \text{const.}$ is an admissible support function. Now Theorem 1.15 immediately gives the result. ☐

Definition 1.18. *Parallel surfaces x, x^+ with normal vector fields $n = n^+$ and support functions σ, σ^+ are assigned the distance function $\sigma^+ - \sigma$.*

The next result was mentioned by [9]:

Theorem 1.19. *Consider a semidiscrete conjugate surface x and its unit normal vector field n. Provided we are in the generic case $\det(\dot{x}, \Delta x_{-1}, \Delta x) \neq 0$, we have the following equivalences:*

1. *x has a face offset $\iff$*
2. *n is circular $\iff$*
3. *x is conical.*

Proof. The proof uses the 'reflection' characterizations of circular and conical surfaces mentioned in Cor. 1.3.

1. $\implies$ 2. follows from Cor. 1.17: we have $\det(\Delta n, \dot{n}, \Delta\dot{n}) = 0$, i.e., n is conjugate. Being inscribed in S^2, n is circular.

2. $\implies$ 3.: The ruled surface strips associated with x are the envelopes of planes with normal vector n. Thus vectors $\Delta x, \Delta x_{-1}, \dot{x}$ are parallel to $n \times \dot{n}$, $n_{-1} \times \dot{n}_{-1}$, $n_{-1} \times n$, respectively, and the reflection required for n being circular immediately yields a reflection revealing x as conical.

3. $\implies$ 1.: Consider the oriented right circular cone with vertex x which touches the (oriented) adjacent ruled strips along rulings, and parallel translate it to become tangentially circumscribed to S^2. The vertex x thereby moves to a new position s; tangent planes are now tangent to S^2. It follows that the semidiscrete surface s, which by construction is parallel to x, is tangentially circumscribed to S^2. This implies that $\sigma = \text{const.} = 1$ is an admissible support function for x. □

2 Curvatures

We recapitulate how [8] and [1] introduce curvatures associated with the faces of a polyhedral surface. This is done via the classical Steiner formula, which relates the area element of an offset surface x^τ at distance τ from an original surface x via

$$x^\tau = x + \tau s \implies \frac{dA(x^\tau)}{dA(x)} = 1 - 2H\tau + K\tau^2, \tag{5}$$

locally around $\tau = 0$. Here s is the unit normal vector field (the Gauss map), and H, K are mean and Gaussian curvatures, respectively. In the framework of *relative* differential geometry this definition was generalized to a surface x and a Gauss map s which is not necessarily contained in the unit sphere, but such that x, s have the same unit normal vector in corresponding points [14].

For discrete polyhedral surfaces (and in particular for discrete conjugate nets), there is a similar construction which is used to define curvatures of circular surfaces in [11,12]. Suppose x, s are parallel, and s is regarded as the

Gauss image of x. Then the variation of surface area of faces as we travel through a 1-parameter family of offsets reads

$$x^\tau = x + \tau s \implies \frac{A(f^\tau)}{A(f)} = 1 - 2H(f)\tau + K(f)\tau^2,$$

where f and f^τ are corresponding faces of the surface x and its offset x^τ. The quantities $H(f)$ and $K(f)$ — mean and Gaussian curvatures of the face f — have been introduced by [8] and are studied by [1]. Their relation to mixed areas is explained in the next section.

2.1 **The oriented mixed area of polygons**

The above-mentioned curvature theory is based on the oriented area and oriented mixed area of polygons. We therefore first collect some definitions before we proceed to semidiscrete surfaces. The oriented area of an n-gon $P = (p_0, \dots, p_{n-1})$ in a two-dimensional vector space U, is given by Leibniz' sector formula:

$$A(P) = \frac{1}{2} \sum\nolimits_{0 \le i < n} [p_i, p_{i+1}].$$

Here and in the following indices in such sums are taken modulo n. We use the notation $[a, b]$ for a determinant form in U which defines the area, i.e., $[a, b] = \det(a, b, n)$, for some normal vector n of this plane, the choice of which is irrelevant for curvatures. The sector formula is invariant w.r.t. translation by a vector $t \in \mathbb{R}^3$, not necessarily contained in U;

$$\sum \det(p_i + t, p_{i+1} + t, n) = \sum \det(p_i, p_{i+1}, n) + \det(\sum p_i, t, n) \\ - \det(\sum p_{i+1}, t, n) = \sum \det(p_i, p_{i+1}, n).$$

This means that $A(P)$ can be extended to polygons lying in any affine subspace $t + U$. Apparently A is a quadratic form, whose associated symmetric bilinear form is denoted by the symbol $A(P, Q)$:

$$A(\lambda P + \mu Q) = \lambda^2 A(P) + 2\lambda\mu A(P, Q) + \mu^2 A(Q), \quad \text{where} \\ A(P, Q) = \frac{1}{4} \sum\nolimits_{0 \le i < n} [q_i, p_{i+1}] + [p_i, q_{i+1}]. \tag{6}$$

Note that in Equation (6) the sum of polygons is defined vertex-wise, and that $A(P, Q)$ does not, in general, equal the well known mixed area [13]. For boundaries of convex polygons which happen to be parallel, however, this is the case (as discussed by [1]). Thus $A(P, Q)$ is called the (oriented) mixed area of P, Q, provided $P \parallel Q$.

2.2 **Area and mixed area of infinitesimal quadrilaterals.**

Since infinitesimal quadrilaterals are tangent vectors, it is fitting that they serve as arguments of the respective differentials (i.e., linearizations) of area and mixed area. Using the symbol $d_x f(v) = df(x+tv)/dt|_{t=0}$ for such a differential would cause confusion with another symbol dA used in integration. We therefore employ the notation

$$\partial_v f(x) = \frac{df(x+\varepsilon v)}{d\varepsilon}\Big|_{\varepsilon=0}$$

for the derivative of f w.r.t. the tangent vector v. Observing $A(P+\varepsilon V) = A(P) + 2\varepsilon A(P,V) + \varepsilon^2 A(V)$ and an analogous relation for the mixed area of $P+\varepsilon V$ and $Q+\varepsilon W$. We see that

$$\partial_V A(P) = 2A(P,V), \quad \partial_{(V,W)} A(P,Q) = A(P,W) + A(V,Q).$$

For the special infinitesimal quadrilaterals according to Def. 1.1, we get

$$P = \boxed{\begin{matrix} x\ x_1 \\ x\ x_1 \end{matrix}}, \ V = \boxed{\begin{matrix} 0\ 0 \\ \dot x\ \dot x_1 \end{matrix}} \implies \partial_V A(P) = \frac{1}{2}[x_1 - x, \dot x_1 + \dot x].$$

The *area* of a semidiscrete surface is naturally defined as the surface area of the corresponding ruled strips (2) associated with it. In case x is conjugate, a normal vector field of x yields an orientation in the ruled strips and it makes sense to consider *signed* surface area (note that the area of infinitesimal quadrilaterals occurs here):

$$\mathrm{area}(x(D)) = \int_{(i,u)\in D} dA = \int_{(i,u)\in D} \frac{1}{2}[\varDelta x, \dot x + \dot x_1]\, du. \tag{7}$$

2.3 **Curvatures in the semidiscrete case**

Regarding the surface area of their offsets, semidiscrete surfaces behave in a way similar to their discrete and continuous counterparts.

Proposition 2.1. *Assume x is a conjugate semidiscrete surface, and s (considered to be the Gauss image of x) is parallel to x. Then the surface area of the offset family $x^\tau = x + \tau s$ reads*

$$\mathrm{area}(x^\tau(D)) = \int_{(i,u)\in D} (1 - 2H\tau + K^2\tau)\, dA,$$

where $dA(i,u) = \frac{1}{2}[\varDelta x, \dot x + \dot x_1]\, du$. This relation defines functions H (mean curvature) and K (Gaussian curvature). With the infinitesimal quadrilaterals

$$P + V d\varepsilon = \boxed{\begin{matrix} x\ x_1 \\ x\ x_1 \end{matrix}} + \boxed{\begin{matrix} 0\ 0 \\ \dot x\ \dot x_1 \end{matrix}} d\varepsilon, \quad Q + W d\varepsilon = \boxed{\begin{matrix} s\ s_1 \\ s\ s_1 \end{matrix}} + \boxed{\begin{matrix} 0\ 0 \\ \dot s\ \dot s_1 \end{matrix}} d\varepsilon,$$

we can express these curvatures as

$$H = -\frac{A(P,W) + A(V,Q)}{2A(P,V)}, \quad K = \frac{A(Q,W)}{A(P,V)}. \tag{8}$$

Proof. Equation (7) implies that the area element of $x + \tau s$ has the form $\partial_{V+\tau W} A(P + \tau Q)\, du = 2A(P + \tau Q, V + \tau W)\, du$. The rest is using bilinearity of the mixed area. □

Example 2.2. Expanding the previous definition of Gaussian curvature K and mean curvature H leads to the expressions

$$H = -\frac{[\Delta x, \dot{s} + \dot{s}_1] + [\Delta s, \dot{x} + \dot{x}_1]}{2[\Delta x, \dot{x} + \dot{x}_1]}, \quad K = \frac{[\Delta s, \dot{s} + \dot{s}_1]}{[\Delta x, \dot{x} + \dot{x}_1]}. \tag{9}$$

In terms of the infinitesimal area and mixed area of infinitesimal quadrilaterals given above, we also have the expressions

$$H = -\frac{\partial_{V,W} A(P, Q)}{\partial_V A(P)}, \quad K = \frac{\partial_W A(Q)}{\partial_V A(P)}.$$

This is a direct analogy to both the discrete and continuous cases. ♢

2.4 Curvatures from offset distances: discrete case

Here we derive a formula which expresses the mean curvature of a polyhedral offset pair in terms of edge lengths and dihedral angles. This is interesting because it can directly be compared with other notions of mean curvature derived via the Steiner formula: For a convex polyhedral surface (V, E, F) which bounds a convex set K, the area of an outer parallel body is given by

$$\text{area}(\partial(K + \varepsilon B)) = \sum\nolimits_{\text{faces } f} \text{area}(f) + \varepsilon \sum\nolimits_{\text{edges } e} \alpha_e \,\text{length}(e) + 4\pi\varepsilon^2.$$

Here α_e is the dihedral angle of the edge e. It is therefore natural to consider

$$H(e) = -\frac{1}{2}\alpha_e, \tag{10}$$

as the mean curvature density in the edge e.

Remark 2.3. The values $H(e)$ of (10) and $H(f)$ of Th. 2.4 are not immediately comparable. Total mean curvature "$\int_G H$" of a domain G in the surface under consideration would have to be defined as $\sum_{e \in E} H(e)\,\text{length}(e \cap G)$, or $\sum_{f \in F} H(f)\,\text{area}(f \cap G)$ (see e.g. [3] for Geometry Processing applications).

An expression in terms of angles similar to (10) is if we consider polyhedral surfaces which admit offsets at constant face-face distance:

Theorem 2.4. *For a conical mesh, i.e., a polyhedral surface which admits a face-face offset at constant distance, the mean curvature of a face is expressed in terms of the dihedral angles of edges by*

$$H(f) = -\frac{1}{2\,\text{area}(f)} \sum_{\textit{edges } e \subset f} \tan\frac{\alpha_e}{2}\,\text{length}(e).$$

The proof of Theorem 2.4 depends on the following more general result:

Lemma 2.5. *Assume that n-gons $P = (p_0, \ldots, p_{n-1})$ and $Q = (q_0, \ldots, q_{n-1})$ are corresponding faces in an offset pair of discrete surfaces, and that translating the plane of P by the vector $d \cdot n$, with n a unit normal vector, yields the plane of Q. Similarly corresponding edges $p_i p_{i+1}$ and $q_i q_{i+1}$ lie in parallel planes, carrying faces adjacent to P, Q, resp., whose relative position is given by normal vectors n_i and distances d_i. Then the mean curvature of the face P is expressed as*

$$H(P) = -\frac{1}{2A(P)} \sum_i \frac{d_i - d\cos\alpha_i}{\sin\alpha_i} \|\Delta p_i\|.$$

Here dihedral angles are computed by $\cos\alpha_i = \langle n, n_i\rangle$, $\sin\alpha_i = \langle n \times n_i, \frac{\Delta p_i}{\|\Delta p_i\|}\rangle$.

Proof. We consider the orthogonal projection of Q on the plane of P and measure the oriented distance ϕ_i of corresponding edges $p_i p_{i+1}$ and $q_i q_{i+1}$:

$$\phi_i = \frac{d_i - d\cos\alpha_i}{\sin\alpha_i}.$$

Here ϕ_i is positive, if (after projection) the edge of Q lies to the left of the corresponding edge of P — the plane being oriented by the normal vector n. The mixed area needed for mean curvature is then derived as

$$A(P, Q-P) = \frac{1}{2}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} A(P + \varepsilon(Q-P)) = \frac{1}{2}\sum \|\Delta p_i\|\phi_i. \qquad \square$$

Remark 2.6. Obviously the mean curvature can also be expressed as

$$H(P) = -\frac{1}{2A(P)} \sum_i \frac{d_i - d\langle n, n_i\rangle}{\det(\Delta p_i, n, n_i)} \|\Delta p_i\|^2.$$

Proof of Theorem 2.4. Let $d = d_i = 1$ and observe $\frac{1-\cos x}{\sin x} = \tan\frac{x}{2}$. $\square$

2.5 Curves from offset distances: semidiscrete case

Lemma 2.7 and Cor. 2.8 below attempt to carry over Lemma 2.5 and Theorem 2.4, respectively, to the semidiscrete case. Unfortunately the proof of Lemma 2.7 is rather long and not very instructive, and we refrain from printing it here.

Lemma 2.7. *Assume a pair x, x^+ of parallel conjugate surfaces, such that the nondegeneracy condition $\det(\ddot{x}, \dot{x}, \Delta x) \neq 0$ is satisfied. Then the mean curvature associated with the offset pair x, x^+ is expressed via the normal vector field n and the distance function d (cf. Def. 1.18) by*

$$H = \frac{-(\delta+\gamma)}{\det(\Delta x, \dot{x} + \dot{x}_1, n)}, \text{ where } \delta = \frac{d_{-1} - d\langle n, n_{-1}\rangle}{\det(\dot{x}, n, n_{-1})}\|\dot{x}\|^2 - \frac{d_1 - d\langle n, n_1\rangle}{\det(\dot{x}_1, n, n_1)}\|\dot{x}_1\|^2,$$

$$\gamma = \frac{2[\Delta\dot{x}, \dot{n}] + \det(\Delta x, \ddot{n}, n)}{[\Delta x, \dot{n}]^2}\dot{d}\|\Delta x\|^2 - \frac{(\ddot{d} + d\|\dot{n}\|^2)\|\Delta x\|^2 + 4\dot{d}\langle\Delta x, \Delta\dot{x}\rangle}{[\Delta x, \dot{n}]}.$$

Corollary 2.8. *Assume that the semidiscrete surface x has face offsets. Consider the angles β between successive normal vectors n, n_1 and the angular velocity ω of the normal vector field:*

$$\sin\beta = \det\Big(\frac{\dot x}{\|\dot x\|}, n, n_1\Big), \quad \omega = \det\Big(\frac{\Delta x}{\|\Delta x\|}, n, \dot n\Big),$$

Then $\langle n, n_1\rangle = \cos\beta$ and $\|\dot n\| = |\omega|$, and the mean curvature is expressed as

$$H = -\frac{1}{\det(\Delta x, \dot x + \dot x_1, n)}\left(\|\dot x\| \tan\frac{\beta_{-1}}{2} - \|\dot x_1\| \tan\frac{\beta}{2} + \omega\|\Delta x\|\right).$$

Proof. Letting $d = 1$ in Lemma 2.7 yields the result, if we observe the various expressions for β, ω given above, as well as $\langle \dot n, \dot n\rangle + \langle n, \ddot n\rangle = \partial_{uu}\langle n, n\rangle = 0$. Alternatively we may compute a limit of Theorem 2.4. □

3 Examples

This section deals with examples of rotational symmetry and then proceeds to the wider class of "trapezoidal" surfaces which do not possess a continuous analogue. It is interesting that surfaces of vanishing mean curvature are connected with the catenary, which for rotational surfaces of course is no surprise.

3.1 Semidiscrete surfaces of discrete rotational symmetry

Here we assume that the surface $x(i, u)$ and its Gauss image $s(i, u)$ are generated by rotating their respective planar meridian curves

$$x(0, u) = (\xi(u), \eta(u), 0), \quad s(0, u) = (\sigma(u), \tau(u), 0)$$

about the first coordinate axis, by the angle $i\alpha$. Obviously,

$$x \parallel s \iff \frac{\Delta s}{\Delta x} = \frac{\tau}{\eta} = \frac{\sigma}{\xi} \text{ and } \frac{\dot s}{\dot x} = \frac{\dot\tau}{\dot\eta} = \frac{\dot\sigma}{\dot\xi}$$

We use Ex. 2.2 to compute the mean and Gaussian curvatures (if $\eta, \dot\eta \neq 0$):

$$H = -\frac{\dot\eta\tau + \eta\dot\tau}{2\eta\dot\eta}, \quad K = \frac{\tau\dot\tau}{\eta\dot\eta}\ . \tag{11}$$

An immediate conseqence of these formulas is $K = 0 \iff \tau = \text{const.}$ (which leads to x, s as co-axial cylinders) and $H = 0 \iff \tau\eta = \text{const.}$

Lemma 3.1. *Assume a surface $x(i, u)$ is generated from the meridian $x(0, u) = (\xi(u), \eta(u), 0)$ by discrete rotation about the x_1 axis, with rotation angle α.*

If interpreted as a circular surface (with Gauss image s contained in the unit sphere, and having the same rotational symmetry), then curvatures coincide with that of the smooth surface of revolution having the same meridian.

If interpreted as a conical surface (with Gauss image tangentially circumscribed to the unit sphere, therefore having the same rotational symmetry), curvatures are those of the smooth surface with meridian $(\xi(u), \eta(u)\cos\frac{\alpha}{2}, 0)$.

Proof. If x is circular/conical, its Gauss image s is uniquely determined up to multiplication with -1 by parallelity of x, s, and by the requirement that s is inscribed/circumscribed to S^2.

In the circular case, continuous rotation of the respective meridians of x and s about the x_1 axis yields a smooth rotational surface and its Gauss image. It is easy to see that (11) computes mean and Gaussian curvatures of that smooth surface, since the normal curvatures of the meridians and of the pararallels are given by the ratios $-\dot{s} : \dot{x} = -\dot{\tau} : \dot{\eta}$ and $\tau : \eta$, respectively.

In the conical case, the affine mapping $(x_1, x_2, x_3) \mapsto (x_1, x_2 \cos\frac{\alpha}{2}, x_3 \cos\frac{\alpha}{2})$ makes s inscribed into S^2. The statement now follows from the fact that curvatures are affine invariants of the pair x, s. □

Lemma 3.1 implies that in the circular case x is minimal if the meridian has the form $(u, C\cosh\frac{u}{C}, 0)$, same as for smooth surfaces. In the conical case x is minimal if the meridian has the form $(u, \frac{C}{\cos(\alpha/2)}\cosh\frac{u}{C}, 0)$.

3.2 Semidiscrete surfaces of continuous rotational symmetry

We generate a surface $x(i,u)$ and its associated Gauss image $s(i,u)$ by rotating the respective "meridian" polylines

$$x(i,0) = (\xi_i, \eta_i, 0), \quad s(i,0) = (\sigma_i, \tau_i, 0)$$

about the first coordinate axis, the angle of rotation being equal to u. Parallelity $x \parallel s$ is equivalent to $\Delta\eta_i : \Delta\xi_i = \Delta\tau_i : \Delta\sigma_i$. It is not surprising that the formulas for the mean and Gaussian curvatures coincide with their discrete counterparts given by [1]:

$$H(i,u) = \frac{\eta_i\tau_i - \eta_{i+1}\tau_{i+1}}{\eta_{i+1}^2 - \eta_i^2}, \quad K(i,u) = \frac{\tau_i^2 - \tau_{i+1}^2}{\eta_{i+1}^2 - \eta_i^2}.$$

Example 3.2. The condition $H = 0$ implies $\eta_i\tau_i = \eta_{i+1}\tau_{i+1}$, which is equivalent to the 2nd row of the recursion

$$\begin{pmatrix}\xi_{i+1}\\ \eta_{i+1}\end{pmatrix} - \begin{pmatrix}\xi_i\\ \eta_i\end{pmatrix} = \frac{1}{\tau_{i+1}}\begin{pmatrix}0 & \sigma_i - \sigma_{i+1}\\ 0 & \tau_i - \tau_{i+1}\end{pmatrix}\begin{pmatrix}\xi_i\\ \eta_i\end{pmatrix}.$$

The first row follows from parallelity. We conclude that a *minimal* surface of continuous rotational symmetry is uniquely determined by its Gauss image up to scaling. The case of Gauss images inscribed in the unit sphere is analyzed by [6], especially the convergence of meridian polylines to the graph of the *cosh* function. If x is to have a face offset (conical case) we consider a Gauss image s which is tangentially circumscribed to the unit sphere. s is generated by a meridian polyline tangentially circumscribed to S^2, having the general form

$$(\sigma_i, \tau_i) = \frac{1}{\cos\frac{\alpha_i - \alpha_{i-1}}{2}}\Big(\cos\frac{\alpha_i + \alpha_{i-1}}{2}, \sin\frac{\alpha_i + \alpha_{i-1}}{2}\Big),$$

for some sequence α_i of angles. See Figure 1 for an example. ♢

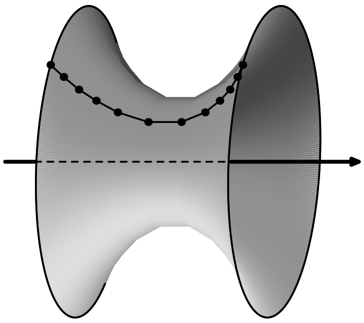

Fig. 1 A semidiscrete minimal surface with the conical property, defined by the angle sequence $\ldots, \frac{\pi}{8}, \frac{\pi}{7}, \frac{\pi}{6}, \frac{\pi}{5}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{4\pi}{5}, \frac{5\pi}{6}, \frac{6\pi}{7}, \frac{7\pi}{8}, \ldots$

3.3 Semidiscrete trapezoidal surfaces of the horizontal type

This class of surfaces contains those with discrete rotational symmetry. It is defined by the property that infinitesimal quadrilaterals are *trapezoids* in the sense that the direction of rulings $x \vee x_1$ depends only on the discrete parameter, but not on the continuous parameter (in fact, assuming conjugacy, it is sufficient to require that for each i, the rulings $x(i,u) \vee x(i+1,u)$ are parallel to a fixed plane). By parallelity, the Gauss image s must have the same property. We use the following representation of x and s:

$$\varDelta x(i,u) = \lambda(i,u)e(i), \quad \varDelta s(i,u) = \mu(i,u)e(i) \quad \text{with} \quad e(i) \in S^2,$$

where λ and μ are real-valued functions defined in $\mathbb{Z}\times\mathbb{R}$, and e is an S^2-valued function defined in the integers. Obviously $\varDelta s : \varDelta x = \mu : \lambda$.

Lemma 3.3. *For the trapezoidal surfaces x, s defined above, parallelity is equivalent to either of the following conditions:*

$$\varDelta(\dot\mu : \dot\lambda) = 0 \textit{ for all } i, \textit{ and } \dot s : \dot x = \dot\mu : \dot\lambda \textit{ for all } i. \tag{12}$$

$$\varDelta(\dot\mu : \dot\lambda) = 0 \textit{ for all } i, \textit{ and } \dot s : \dot x = \dot\mu : \dot\lambda \textit{ for some } i. \tag{13}$$

Proof. Assuming $x \parallel s$, we use the ratio $\nu = \dot s : \dot x$ and expand $\dot s_1 \wedge \dot x_1 = 0$:

$$0 = (\dot s + \dot\mu e) \times (\dot x + \dot\lambda e) = \dot s \times \dot\lambda e + \dot\mu e \times \dot x = (\nu\dot\lambda - \dot\mu)\dot x \times e,$$

i.e., $\dot s : \dot x = \nu = \dot\mu : \dot\lambda$. Since in this case obviously $\nu_1 = \dot s_1 : \dot x_1 = (\dot s + \dot\mu e) : (\dot x + \dot\lambda e) = \nu$, the ratio ν does not depend on the discrete variable, and we have shown that parallelity implies (12), and in turn (13).

Assume now (13) which in particular states that for some i, $\dot s$ and $\dot x$ are parallel. The previous computation shows that also $\dot x_1$ and $\dot x_1$ are parallel, with ratio $\dot\mu : \dot\lambda$. By induction, surfaces x, s are parallel and (12) holds. □

It is straightforward to compute the Gaussian and mean curvatures of the surface x w.r.t. its Gauss image s; we get

$$H = -\frac{1}{2}\Big(\frac{\mu}{\lambda} + \frac{\dot\mu}{\dot\lambda}\Big) = -\frac{\partial_u(\lambda\mu)}{\partial_u(\lambda^2)}, \quad K = \frac{\mu\dot\mu}{\lambda\dot\lambda}\,. \tag{14}$$

Similar to previous examples, $H = 0$ is equivalent to $\lambda\mu$ not depending on the continuous variable. We study the conical (face offset case) in more detail, since here the Gauss image s is defined by the surface x up to multiplication with -1.

Lemma 3.4. *For a trapezoidal surface x of the horizontal kind, the conical property is equivalent to each curve $x(i,\cdot)$ being planar, lying in the bisector plane $P(i) = P(i,u)$ of rulings $x \vee x_1$ and $x \vee x_{-1}$, which does not depend on the continuous variable.*

Proof. This is a direct consequence of Theorem 1.19: x has a face offset $\iff$ throughout the surface, the vector $\dot{x}(i,u)$ is contained in the bisector plane $P(i,u)$ of normalized vectors $\Delta x(i,u)$ and $-\Delta x(i-1,u)$. Since the direction of rulings does not depend on u, this is equivalent to what is stated. $\square$

Example 3.5 (*Minimal surfaces*)**.** We focus on a single strip bounded by curves $x(i,\cdot)$ and $x(i+1,\cdot)$ and assume a coordinate system with $0 \in P(i) \cap P(i+1)$. The Gauss image s is conical like x, but in addition it is tangentially circumscribed to S^2. Therefore, all planes carrying $s(i,\cdot)$ pass through the origin anyway, and we have $s(i,u) \in P(i)$, $s(i+1,u) \in P(i+1)$. Choosing the coordinate system such that $P(i)$ is spanned by its first two axes, we have $x(i,\cdot) = (\xi,\eta,0)$, $s(i,\cdot) = (\sigma,\tau,0)$, with real-valued functions ξ,η,σ,τ.

It is not difficult to check that the Gauss image s of x, up to multiplication with -1, is given by

$$s = \frac{\dot{x} \times e}{\|\dot{x} \times e\|} - \frac{\det(\dot{x} \times e, x, \dot{x})}{\|\dot{x} \times e\| \det(e, x, \dot{x})} e. \tag{15}$$

We wish to determine the shape of minimal surfaces among the 'horizontal trapezoidal' ones. For that we introduce coordinates $e(i) = (\alpha,\, \beta,\, \gamma)$ and explicitly compute the function τ from (15), simplifying it by a linear subsitution:

$$\tau = (\alpha\beta\dot{\eta} - (\beta^2+\gamma^2)\dot{\xi})\gamma^{-1}\left(\gamma^2(\dot{\xi}^2+\dot{\eta}^2) + (\beta\dot{\xi} - a\dot{\eta})^2\right)^{-\frac{1}{2}}$$

$$\xi = \phi + \frac{\alpha\beta}{\gamma}\psi,\ \eta = \frac{\beta^2+\gamma^2}{\gamma}\psi \implies \tau = -\dot{\phi}(\beta^2+\gamma^2)^{\frac{1}{2}}\gamma^{-1}(\dot{\phi}^2+\dot{\psi}^2)^{-\frac{1}{2}}.$$

By Equation (14), $H = (\beta^2+\gamma^2)^{-1/2}\left(\frac{\dot{\phi}}{\psi(\dot{\phi}^2+\dot{\psi}^2)^{1/2}} + \frac{\ddot{\phi}\dot{\psi} - \dot{\phi}\ddot{\psi}}{(\dot{\phi}^2+\dot{\psi}^2)^{3/2}}\right)$. It follows that $H = 0 \iff (\dot{\psi}^2+\dot{\phi}^2)\dot{\phi} + \psi(\ddot{\phi}\dot{\psi} - \dot{\phi}\ddot{\psi}) = 0$. This equation has the trivial solution $\phi = \text{const.}$, but otherwise it transforms to the equation $(\frac{d\psi}{d\phi})^2 + 1 - \psi\frac{d^2\psi}{d\phi^2} = 0$, whose solutions are $\psi = C\cosh((\phi - C^*)/C)$. Backsubstitution yields

$$H = 0 \iff \eta = \frac{C_1}{\gamma}\cosh\Big(\frac{(\beta^2+\gamma^2)\xi - \alpha\beta\eta - C_2}{C_1}\Big), \tag{16}$$

for some constants C_1 and C_2. Any parametric representation $\xi(u), \eta(u)$ which obeys this implicit relation yields $H = 0$. $\diamondsuit$

Remark 3.6. It is interesting that we have just found semidiscrete minimal surfaces which do not enjoy rotational symmetry and which are generated by a sequence of affinely-distorted catenary curves. It is tempting to try to achieve smooth minimal surfaces as limit shapes of a sequence of finer and finder semidiscrete surfaces $x^{(j)}$, $j \to \infty$. Such a limit unfortunately is always an ordinary catenoid, so we do not find cases of new smooth minimal surfaces in this way.

This can be seen as follows: Curves $x^{(j)}(i, \cdot)$ converge to planar principal curvature lines $x^\infty(u, \cdot)$ (because of the conical property [5]), which evolve with evolution velocity $\partial_v x^\infty$ orthogonal to the plane they are contained in (being conjugate to a principal curve). It follows that this evolution is isometric, and is actually generated by the rolling of a plane. All principal curves are congruent. In the coordinate system employed in the previous paragraph, the limit of vectors e (which is the direction of evolution) then reads $(0, 0, \pm 1)$, and the principal curve is explicitly given by $\eta = C_1 \cosh(\frac{\xi - C_2}{C_1})$. We already know that C_1 is constant (by the congruence property), so we see that the distance of the curve from the axis of rolling is constant. This concludes the argument.

3.4 Semidiscrete trapezoidal surfaces of the vertical type

This class of surfaces contains those with continuous rotational symmetry. Its defining property is that the infinitesimal quadrilaterals are trapezoids in the sense that the infinitesimal edges $\dot{x}\,dt$ and $\dot{x}_1\,dt$ are parallel, i.e., the direction of $\dot{x}$ does not depend on the discrete variable. By parallelity, any Gauss image s of x has the same property. We therefore have the representation

$$\dot{x}(i, u) = \lambda(i, u) e(u), \quad \dot{s}(i, u) = \mu(i, u) e(u), \quad \text{where} \quad e(u) \in S^2. \tag{17}$$

Lemma 3.7. *For semidiscrete surfaces x, s as defined by* (17)*,*

$$x \parallel s \iff \partial_u(\Delta\mu : \Delta\lambda) = 0 \text{ and } \Delta s : \Delta x = \Delta\mu : \Delta\lambda,$$

where the "$\Longrightarrow$" part is true in the generic case where the directions of $\Delta x, \Delta s$ on the one hand, and e on the other hand are linearely independent.

Proof. "$\Longleftarrow$" is clear. For "$\Longrightarrow$" assume $x \parallel s$ and differentiate $\Delta x \times \Delta s = 0$:

$$(\Delta\lambda \cdot e) \times \Delta s + \Delta x \times (\Delta\mu \cdot e) = (\Delta\mu\Delta x - \Delta\lambda\Delta s) \times e = 0.$$

The genericity assumption implies $\Delta\mu\Delta x - \Delta\lambda\Delta s = 0$, which is part of what we want to show. We differentiate a 2nd time and get

$$\Delta\dot{\mu}\Delta x + \Delta\mu\Delta\lambda e - \Delta\dot{\lambda}\Delta s - \Delta\lambda\Delta\mu e = \Delta\dot{\mu}\Delta x - \Delta\dot{\lambda}\Delta s = 0$$

Therefore, $\Delta\dot{\lambda} : \Delta\lambda - \Delta\dot{\mu} : \Delta\mu = 0$ which can also be written as

$$\partial_u(\log \Delta\lambda - \log \Delta\mu) = 0,$$

i.e., $\Delta\lambda : \Delta\mu$ does not depend on the continuous parameter. This is the second r.h.s. statement. □

We get the following formulas for mean and Gaussian curvature:

$$H = \frac{\mu\lambda - \mu_1\lambda_1}{\lambda_1^2 - \lambda^2}, \quad K = \frac{\mu_1^2 - \mu^2}{\lambda_1^2 - \lambda^2}. \tag{18}$$

Remark 3.8. Since in the context of this paper, curvatures of discrete surfaces are associated with the faces of that surface, the limit case of semidiscrete surfaces should have curvatures associated with edges. It is only by the regular combinatorics exhibited by surfaces defined in $\mathbb{Z} \times \mathbb{R}$ that we can consider curvatures being defined in $\mathbb{Z} \times \mathbb{R}$ too. However the combinatorial symmetries of Equation (18) directly reflect the fact that curvatures are associated with edges.

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